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Nonlinear contractions and fixed point theorems with modified ω-distance mappings in complete quasi metric spaces

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Abstract

Alegre and Marin [C. Alegre, J. Marin, Topol. Appl., **203** (2016), 32–41] introduced the concept of modified ω-distance mappings on a complete quasi metric space in which they studied some fixed point results. In this manuscript, we prove some fixed point results of nonlinear contraction conditions through modified ω-distance mapping on a complete quasi metric space in sense of Alegre and Marin.

Keywords: Quasi metric, fixed point theorem, nonlinear contraction, altering distance, modified ω-distance.
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1. Introduction

The Banach contraction principle is one of the main results in fixed point theory, which asserts that every contraction in a complete metric space has a unique fixed point. Subsequently, a large number of generalizations of Banach's contraction theorem is obtained by many authors. For more details we refer the readers to [1–4, 7–11, 16–19, 21].

A self-mapping T on a metric space (X, d) is called Kannan contraction if there is a $k \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leqslant k[d(x, Tx) + d(y, Ty)], \quad \forall x, y \in X.$$

Kannan [14] proved that every Kannan contraction in a complete metric space has a unique fixed point. It is worth mentioning that Kannans theorem is an important result since it characterizes the metric completeness (see [20]).

The concept of quasi metric spaces was introduced by Wilson [22].

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Definition 1.1 ([22])**.** Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a given function which satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) \leqslant d(x, z) + d(z, y)$ for any points $x, y, z \in X$.

Then d is called a quasi metric on X and the pair (X, d) is called a quasi metric space.

It is clear that every metric space is a quasi metric space, but the reverse is not necessarily true. A quasi metric d induces a metric d_m as follows:

$$d_m(x, y) = \max\{d(x, y), d(y, x)\}.$$

The convergence and completeness in a quasi-metric space are defined as follows.

Definition 1.2 ([13])**.** Let (X, d) be a quasi-metric space, (x_n) be a sequence in X, and $x \in X$. Then the sequence (x_n) converges to x if $\lim\limits_{n\to\infty} d(x_n, x) = \lim\limits_{n\to\infty} d(x, x_n) = 0$.

Definition 1.3 ([13])**.** Let (X, d) be a quasi-metric space and (x_n) be a sequence in X. Then

(1) We say that the sequence (x_n) is left-Cauchy if for every $\epsilon > 0$, there is a positive integer $N = N(\epsilon)$ such that $d(x_n, x_m) \leqslant \epsilon$ for all $n \geqslant m > N$.

(2) We say that the sequence (x_n) is right-Cauchy if for every $\epsilon > 0$ there is a positive integer $N = N(\epsilon)$ such that $d(x_n, x_m) \leqslant \epsilon$ for all $m \geqslant n > N$.

Definition 1.4 ([13])**.** Let (X, d) be a quasi-metric space and (x_n) be a sequence in X. We say that the sequence (x_n) is Cauchy if for every $\epsilon > 0$ there is positive integer $N = N(\epsilon)$ such that $d(x_n, x_m) \leqslant \epsilon$ for all $m, n > N$; that is (x_n) is a Cauchy sequence if and only if it is left and right Cauchy.

Definition 1.5 ([13])**.** Let (X, d) be a quasi-metric space. We say that

(1) (X, d) is left-complete if every left-Cauchy sequence in X is convergent;

(2) (X, d) is right-complete if every right-Cauchy sequence in X is convergent;

(3) (X, d) is complete if every Cauchy sequence in X is convergent.

A modified ω-distance (shortly $m\omega$-distance) mapping on quasi metric space is given by Alegre and Marin [5] as follows.

Definition 1.6 ([5])**.** A $m\omega$-distance on a quasi metric space (X, d) is a function $q : X \times X \to [0, \infty)$ satisfying the following conditions:

(W1) $q(x, y) \leqslant q(x, z) + q(z, y)$ for all $x, y, z \in X$;

(W2) $q(x, .) : X \to [0, \infty)$ is lower semi-continuous for all $x \in X$;

(mW3) for each $\epsilon > 0$ there is $\delta > 0$ such that if $q(y, x) \leqslant \delta$ and $q(x, z) \leqslant \delta$ then $d(y, z) \leqslant \epsilon$.

Definition 1.7 ([5])**.** A strong $m\omega$-distance on a quasi metric space (X, d) is a $m\omega$-distance $q : X \times X \to [0, \infty)$ satisfying the following condition:

(mW2) $q(., x) : X \to [0, \infty)$ is lower semi-continuous for all $x \in X$.

Remark 1.8 ([5]).

1. Every quasi metric d on X is an $m\omega$-distance on the quasi metric space (X, d).
2. In general, a quasi metric d on X need not to be a strong $m\omega$-distance on the quasi metric space (X, d).

For more details on $m\omega$-distance, we refer the readers to [5] and references therein.

Definition 1.9 ([15]). The function $\varphi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(1) φ is continuous and nondecreasing function;

(2) $\varphi(t) = 0$ if and only if $t = 0$.

Henceforth, we denote the class of all altering distance functions by Ψ.

Definition 1.10 ([12]). Let S be the class of all functions $\alpha : \mathbb{R}^+ \to [0,1)$ that satisfies the following implication

$$\alpha(t_n) \to 1 \Longrightarrow t_n \to 0.$$

Geraghty in [12] proved the following fixed point result.

Theorem 1.11. *Let (X,d) be a complete metric space and $T : X \to X$. Let $\alpha \in S$ such that*

$$d(Tx,Ty) \leqslant \alpha(d(x,y))\, d(x,y), \quad \forall x,y \in X.$$

Then T has a unique fixed point.

Recently Amini-Harandi and Emami characterized Geraghty's theorem in the setting of partially ordered metric spaces as follows.

Theorem 1.12 ([6]). *Let $(X,d,\preceq)$ be a partially ordered complete metric space. Let $T : X \to X$ be an increasing mapping such that there is $x_0 \in X$ with $x_0 \preceq Tx_0$. Suppose that there is $\alpha \in S$ such that*

$$d(Tx,Ty) \leqslant \alpha(d(x,y))\, d(x,y)$$

for all $x,y \in X$ with $x \succeq y$. Assume that either T is continuous or X is such that if an increasing sequence (x_n) converges to x, then $x_n \preceq x$ for each $n \geqslant 1$. Besides if for all $x,y \in X$, there exists $z \in X$ which is comparable to x and y, then T has a unique fixed point.

2. Main result

In this section, we present and prove some lemmas that will be used in the sequel.

Lemma 2.1. *Let (X,d) be a quasi metric space equipped with an $m\omega$-distance p. Let (x_n) be a sequence in X and (α_n), (β_n) be sequences in $[0,\infty)$ converging to zero and let (x_n) be a sequence in X. Then we have the following:*

(1) *If $p(x_n,x_m) \leqslant \alpha_n$ for any $m,n \in \mathbb{N}$ with $m \geqslant n$, then (x_n) is a right Cauchy sequence in (X,d).*

(2) *If $p(x_n,x_m) \leqslant \beta_m$ for any $m,n \in \mathbb{N}$ with $n \geqslant m$, then (x_n) is a left Cauchy sequence in (X,d).*

Proof.
(1) Assume that $p(x_n,x_m) \leqslant \alpha_n$, $m \geqslant n$. Then for each $\epsilon > 0$ we can find $N \in \mathbb{N}$ such that $p(x_n,x_{n+1}) \leqslant \alpha_n \leqslant \frac{\epsilon}{2}$ and $p(x_{n+1},x_m) \leqslant \alpha_{n+1} \leqslant \frac{\epsilon}{2}$, for all $m > n \geqslant N$. Thus by the definition of $m\omega$-distance we have $d(x_n,x_m) \leqslant \epsilon$ for all $m \geqslant n \geqslant N$. Hence (x_n) is right Cauchy sequence.

(2) Assume that $p(x_n,x_m) \leqslant \beta_m$, $n \geqslant m$. Then for each $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that $p(x_n,x_{n-1}) \leqslant \beta_{n-1} \leqslant \frac{\epsilon}{2}$ and $p(x_{n-1},x_m) \leqslant \beta_m \leqslant \frac{\epsilon}{2}$, for all $n > m \geqslant N$. Thus by the definition of $m\omega$-distance, we have $d(x_n,x_m) \leqslant \epsilon$, for all $n \geqslant m \geqslant N$. Hence (x_n) is left Cauchy sequence. □

Remark 2.2. The above lemma implies that if $\lim_{n,m\to\infty} p(x_n,x_m) = 0$, then (x_n) is a Cauchy sequence in (X,d).

Lemma 2.3. *Let (X, d) be a quasi metric space equipped with an $m\omega$-distance p. Let (x_n) be a sequence in X such that*

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{n\to\infty} p(x_{n+1}, x_n) = 0. \tag{2.1}$$

If (x_n) is not a Cauchy sequence, then there exist $\epsilon > 0$ and two sequences (n_k) and (m_k) of natural numbers such that

$$\lim_{k\to\infty} p(x_{n_k}, x_{m_k}) = \lim_{k\to\infty} p(x_{n_k+1}, x_{m_k+1}) = \epsilon.$$

Proof. Suppose that (x_n) is not a Cauchy sequence. Without loss of generality, we assume that (x_n) is not a right Cauchy sequence. Then there exist $\epsilon > 0$ and two subsequences (n_k) and (m_k) of the natural numbers such that

$$p(x_{n_k}, x_{m_k}) \geqslant \epsilon, \quad m_k \geqslant n_k, \tag{2.2}$$

where m_k is chosen as the smallest index satisfying (2.2). This means that

$$p(x_{n_k}, x_{m_k-1}) < \epsilon.$$

From (2.2), we have

$$\epsilon \leqslant p(x_{n_k}, x_{m_k}) \leqslant p(x_{n_k}, x_{m_k-1}) + p(x_{m_k-1}, x_{m_k}) < \epsilon + p(x_{m_k-1}, x_{m_k}).$$

Taking the limit as $k \to \infty$ and using (2.1), we get

$$\lim_{k\to\infty} p(x_{n_k}, x_{m_k}) = \epsilon. \tag{2.3}$$

Also,

$$\begin{aligned} p(x_{n_k}, x_{m_k}) - p(x_{n_k}, x_{n_k+1}) - p(x_{m_k+1}, x_{m_k}) &\leqslant p(x_{n_k+1}, x_{m_k+1}) \\ &\leqslant p(x_{n_k+1}, x_{n_k}) + p(x_{n_k}, x_{m_k}) + p(x_{m_k}, x_{m_k+1}). \end{aligned}$$

Let k go to infinity and using (2.1) and (2.3), we reach

$$\lim_{k\to\infty} p(x_{n_k+1}, x_{m_k+1}) = \epsilon.$$

□

In order to facilitate our work, we introduce the following definition.

Definition 2.4. Let (X, d) be a quasi metric space equipped with $m\omega$-distance p. A self-mapping $T : X \to X$ is called (φ, α)-Geraghty contraction if there exist $\varphi \in \Psi$ and $\alpha \in S$ such that

$$\varphi p(Tx, Ty) \leqslant \alpha(p(x, y))\ \varphi p(x, y), \quad \forall x, y \in X.$$

In the next theorem, we prove a fixed point result of Geraghty's type contraction condition in a complete quasi metric space through modified ω-distance mappings.

Theorem 2.5. *Let (X, d) be a complete quasi metric space and p be an $m\omega$-distance on X and $T : X \to X$ be a (φ, α)-Geraghty mapping. Assume that one of the following conditions holds true:*

(1) *If $u \neq Tu$, then* $\inf\{p(x, u) + p(Tx, u) : x \in X\} > 0$.

(2) T *is continuous.*

Then T has a unique fixed point.

Proof. Let $x_0 \in X$. Define a sequence $x_n = Tx_{n-1}$, $n \in \mathbb{N}$. Consider $n \in \mathbb{N}$. From the contractive condition, we have

$$\begin{aligned} \varphi p(x_n, x_{n+1}) &= \varphi p(Tx_{n-1}, Tx_n) \\ &\leqslant \alpha(p(x_{n-1}, x_n))\ \varphi p(x_{n-1}, x_n). \end{aligned} \tag{2.4}$$

Since $\alpha(t) < 1$ for all $t \geqslant 0$, then $\varphi p(x_n, x_{n+1}) < \varphi p(x_{n-1}, x_n)$. As φ is nondecreasing, we have

$p(x_n, x_{n+1}) < p(x_{n-1}, x_n)$. Thus the sequence $(p(x_n, x_{n+1}) : n \in \mathbb{N})$ is a nonnegative decreasing sequence. Hence there is $r \geqslant 0$ such that $\lim_{n\to\infty} p(x_n, x_{n+1}) = r$. Suppose that $r > 0$. By (2.4), we have

$$\frac{\varphi p(x_n, x_{n+1})}{\varphi p(x_{n-1}, x_n)} \leqslant \alpha(p(x_{n-1}, x_n)).$$

By taking the limit as $n \to \infty$ we get $\lim_{n\to\infty} \alpha(p(x_{n-1}, x_n)) = 1$. Since $\alpha \in S$, then $r = 0$ a contradiction. So,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

Similarly, we can show that

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0.$$

Now, our claim is to show that (x_n) is a Cauchy sequence in (X, d). Assume to the contrary that (x_n) is not a Cauchy sequence. Due to Lemma 2.3, there exist $\epsilon > 0$ and two sequences (n_k) and (m_k) of natural numbers such that

$$\lim_{k\to\infty} p(x_{n_k}, x_{m_k}) = \lim_{k\to\infty} p(x_{n_k+1}, x_{m_k+1}) = \epsilon.$$

Substitute $x = x_{n_k}$ and $y = x_{m_k}$ in the contractive condition, we obtain that

$$\begin{aligned} \varphi p(x_{n_k+1}, x_{m_k+1}) &= \varphi p(Tx_{n_k}, Tx_{m_k}) \\ &\leqslant \alpha(p(x_{n_k}, x_{m_k}))\ \varphi p(x_{n_k}, x_{m_k}). \end{aligned}$$

Therefore,

$$\frac{\varphi p(x_{n_k+1}, x_{m_k+1})}{\varphi p(x_{n_k}, x_{m_k})} \leqslant \alpha(p(x_{n_k}, x_{m_k})).$$

Taking the limit as $k \to \infty$, we deduce $\lim_{k\to\infty} \alpha(p(x_{n_k}, x_{m_k})) = 1$ and so $\lim_{k\to\infty} p(x_{n_k}, x_{m_k}) = 0$ a contradiction since $\epsilon > 0$. Hence (x_n) is a Cauchy sequence. Thus there is $u \in X$ such that (x_n) converges to u. Since $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, then for a given $\epsilon > 0$ there is $k \in \mathbb{N}$ such that $p(x_n, x_m) \leqslant \frac{\epsilon}{2}$, for all $n, m \geqslant k$. By the lower semi continuity of p we have

$$p(x_n, u) \leqslant \lim_{l\to\infty} \inf p(x_n, x_l) \leqslant \frac{\epsilon}{2}, \quad \forall n \geqslant k.$$

Now, assume that (1) holds true if $u \neq Tu$, then

$$\begin{aligned} \inf\{p(x, u) + p(Tx, u) : x \in X\} &\leqslant \inf\{p(x_n, u) + p(Tx_n, u) : n \in \mathbb{N}\} \\ &= \inf\{p(x_n, u) + p(x_{n+1}, u) : n \in \mathbb{N}\} \\ &\leqslant \epsilon \end{aligned}$$

for all $\epsilon > 0$ a contradiction. Hence $Tu = u$.

If (2) holds, then the continuity of T implies that $Tu = u$.

To prove the uniqueness, assume that there is $v \in X$ such that $Tv = v$. By the contractive condition, we have

$$\begin{aligned} \varphi p(u, v) &= \varphi p(Tu, Tv) \\ &\leqslant \alpha(p(u, v))\ \varphi p(u, v). \end{aligned}$$

As $\alpha \in S$, we have $\varphi p(u, v) = 0$ and so $p(u, v) = 0$.

Also,

$$\begin{aligned}\varphi p(u,u) &= \varphi p(Tu,Tu)\\ &\leqslant \alpha(p(u,u))\ \varphi p(u,u).\end{aligned}$$

Following the same argument, we obtain $p(u,u) = 0$. Therefore by (mW3) of the definition of $m\omega$-distance, we get $u = v$. □

Theorem 2.6. *Let (X,d) be a complete quasi metric space and p be a strong $m\omega$-distance on X and $T : X \to X$ be a (φ,α)-Geraghty mapping. Then T has a unique fixed point.*

Proof. Following the proof of Theorem 2.5 step by step, we can show that $\lim\limits_{n,m\to\infty} p(x_n,x_m) = 0$. So (x_n) is a Cauchy sequence in the complete quasi metric space (X,d). Thus there is $u \in X$ such that (x_n) converges to u.

Given $\epsilon > 0$. Since $\lim\limits_{n,m\to\infty} p(x_n,x_m) = 0$, then there is $N \in \mathbb{N}$ such that $p(x_n,x_m) \leqslant \epsilon$, for all $n,m \geqslant N$. So, by the lower semi continuity of p (W2) and (mW2), we have

$$p(x_n,u) \leqslant \lim_{j\to\infty}\inf p(x_n,x_j) \leqslant \epsilon, \quad \forall n \geqslant N,$$

$$p(u,x_n) \leqslant \lim_{l\to\infty}\inf p(x_l,x_n) \leqslant \epsilon, \quad \forall n \geqslant N.$$

Now, the contraction condition yields:

$$\begin{aligned}\varphi p(Tu,x_{n+1}) &= \varphi p(Tu,Tx_n)\\ &\leqslant \alpha(p(u,x_n))\ \varphi p(u,x_n).\end{aligned}$$

Since $\alpha(t) < 1$ for all $t \geqslant 0$ and φ is nondecreasing, then $p(Tu,x_{n+1}) < p(u,x_n)$. Hence, $p(Tu,x_{n+1}) < \epsilon$, for all $n \geqslant N$. Thus, by (mW3) of the definition of $m\omega$-distance, we have $d(Tu,u) = 0$ and so $Tu = u$. The proof of the uniqueness is the same as of the proof of Theorem 2.5. □

Definition 2.7. Let Φ be the set of all continuous functions $\phi : [0,\infty) \to [0,\infty)$ such that the following properties are satisfied:

(1) $\phi(t) < t$, $\forall t \in (0,\infty)$;
(2) $\phi(t) = 0$ if and only if $t = 0$.

Definition 2.8. Let (X,d) be a quasi metric space equipped with $m\omega$-distance p. A self-mapping $T : X \to X$ is called (λ,ϕ)-Kannan contraction if there are $\lambda \in [0,\frac{1}{2})$ and $\phi \in \Phi$ such that

$$p(Tx,Ty) \leqslant \lambda\ [\phi p(x,Tx) + \phi p(y,Ty)], \quad \forall x,y \in X.$$

Next, we move on to study some fixed point results of (λ,ϕ)-Kannan type contractions.

Theorem 2.9. *Let (X,d) be a complete quasi metric space and p be an $m\omega$-distance on X and $T : X \to X$ be (λ,ϕ)-Kannan contraction. Assume that one of the following conditions holds true:*

(1) *If $u \neq Tu$, then* $\inf\{p(x,u) + p(Tx,u) : x \in X\} > 0$.
(2) *T is continuous.*

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ and define a sequence $x_n = Tx_{n-1},\ n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. From the contractive condition, we have

$$\begin{aligned}p(x_n,x_{n+1}) &= p(Tx_{n-1},Tx_n)\\ &\leqslant \lambda\ [\phi p(x_{n-1},x_n) + \phi p(x_n,x_{n+1})]\\ &< \lambda[p(x_{n-1},x_n) + p(x_n,x_{n+1})].\end{aligned} \tag{2.5}$$

Thus,

$$p(x_n, x_{n+1}) < \frac{\lambda}{1-\lambda} p(x_{n-1}, x_n).$$

Since $\frac{\lambda}{1-\lambda} < 1$, the sequence $(p(x_n, x_{n+1}) : n \in \mathbb{N})$ is a nonnegative decreasing sequence. Hence there is $r \geqslant 0$ such that $\lim_{n\to\infty} p(x_n, x_{n+1}) = r$. From (2.5), we have

$$p(x_n, x_{n+1}) \leqslant \lambda\,[\phi p(x_{n-1}, x_n) + \phi p(x_n, x_{n+1})].$$

By taking the limit as $n \to \infty$ we get $r \leqslant \lambda[\phi(r) + \phi(r)] = 2\lambda\phi(r) < \phi(r)$. Thus $\phi(r) = 0$. Therefore,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

Also, we obtain

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0.$$

Due to Lemma 2.3, if (x_n) is not a Cauchy sequence, then there exist $\epsilon > 0$ and two sequences (n_k) and (m_k) of natural numbers such that

$$\lim_{k\to\infty} p(x_{n_k}, x_{m_k}) = \lim_{k\to\infty} p(x_{n_k+1}, x_{m_k+1}) = \epsilon.$$

By substituting $x = x_{n_k}$ and $y = x_{m_k}$ in the contractive condition, we obtain that

$$\begin{aligned} p(x_{n_k+1}, x_{m_k+1}) &= p(Tx_{n_k}, Tx_{m_k}) \\ &\leqslant \lambda\,[\phi p(x_{n_k}, x_{n_k+1}) + \phi p(x_{m_k}, x_{m_k+1})]. \end{aligned}$$

Taking the limit as $k \to \infty$, we get $\epsilon \leqslant \lambda[0+0] = 0$ a contradiction, because $\epsilon > 0$. Hence (x_n) is a Cauchy sequence. Therefore, there is $z \in X$ such that (x_n) converges to z.

Since $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, then for each $\epsilon > 0$ there is $N \in \mathbb{N}$ such that

$$p(x_n, x_m) \leqslant \epsilon, \quad \forall n, m \geqslant N.$$

By the lower semi continuity of p, we have

$$p(x_n, z) \leqslant \lim_{l\to\infty} \inf p(x_n, x_l) \leqslant \epsilon, \quad \forall n \geqslant N.$$

Now, assume that (1) holds true if $z \neq Tz$, then

$$\begin{aligned} \inf\{p(x,z) + p(Tx,z) : x \in X\} &\leqslant \inf\{p(x_n, z) + p(Tx_n, z) : n \in \mathbb{N}\} \\ &= \inf\{p(x_n, z) + p(x_{n+1}, z) : n \in \mathbb{N}\} \\ &\leqslant 2\epsilon \end{aligned}$$

for each $\epsilon > 0$ a contradiction. Hence $Tz = z$.

Also, if (2) holds, then the continuity of T implies that $Tz = z$.

To prove the uniqueness, first we show that for $x \in X$ if $Tx = x$, then $p(x, x) = 0$.

Assume $p(x, x) > 0$. The contractive condition yields

$$\begin{aligned} p(x,x) = p(Tx,Tx) &\leqslant \lambda[\phi p(x,x) + \phi p(x,x)] \\ &< \lambda[p(x,x) + p(x,x)] \\ &< p(x,x), \end{aligned}$$

a contradiction. Thus $p(x, x) = 0$.

Now, assume that there is $\nu \in X$ such that $T\nu = \nu$. By the contractive condition, we have

$$\begin{aligned} p(\nu, z) = p(T\nu, Tz) &\leqslant \lambda[\phi p(\nu, T\nu) + \phi p(z, Tz)] \\ &= \lambda[\phi p(\nu,\nu) + \phi p(z,z)] \\ &= 0. \end{aligned}$$

Therefore, by (mW3) of the definition of $m\omega$-distance, we have $u = \nu$. □

If we consider a strong $m\omega$-distance instead of $m\omega$-distance in Theorem 2.9, then conditions (1), (2) can be dropped.

Theorem 2.10. *Let (X, d) be a complete quasi metric space and p be a strong $m\omega$-distance on X. Assume that $T : X \to X$ is a (λ, φ)-Kannan contraction. Then T has a unique fixed point.*

Proof. Following the proof of Theorem 2.9 step by step, we can show that $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. So (x_n) is a Cauchy sequence in the complete quasi metric space (X, d). Thus there is $z \in X$ such that (x_n) converges to z.

Given $\epsilon > 0$. Since $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, then there is $N_1 \in \mathbb{N}$ such that $p(x_n, x_m) \leqslant \frac{\epsilon}{2}$, for all $n, m \geqslant N_1$. Thus,

$$p(x_n, x_{n+1}) \leqslant \frac{\epsilon}{2}, \quad \forall n \geqslant N_1. \tag{2.6}$$

Also, by the lower semi continuity of p, (mW2), we have

$$p(z, x_n) \leqslant \lim_{l\to\infty} \inf p(x_l, x_n) \leqslant \frac{\epsilon}{2}, \quad \forall n \geqslant N_1. \tag{2.7}$$

By the triangle inequality, we get

$$\begin{aligned} p(z, Tz) &\leqslant p(z, x_{n+1}) + p(x_{n+1}, Tz) \\ &\leqslant p(z, x_{n+1}) + \lambda[\varphi p(x_n, x_{n+1}) + \varphi p(z, Tz)] \\ &< p(z, x_{n+1}) + \lambda p(x_n, x_{n+1}) + \lambda p(z, Tz). \end{aligned}$$

Hence

$$p(z, Tz) < \frac{1}{1-\lambda} p(z, x_{n+1}) + \frac{\lambda}{1-\lambda} p(x_n, x_{n+1}).$$

Now, the contraction condition yields:

$$\begin{aligned} p(x_{n+1}, Tz) = p(Tx_n, Tz) &\leqslant \lambda\, [\varphi p(x_n, x_{n+1}) + \varphi p(z, Tz)] \\ &< \lambda p(x_n, x_{n+1}) + \lambda p(z, Tz) \\ &< \lambda p(x_n, x_{n+1}) + \frac{\lambda}{1-\lambda} p(z, x_{n+1}) + \frac{\lambda^2}{1-\lambda} p(x_n, x_{n+1}) \\ &= \frac{\lambda}{1-\lambda}\, [p(x_n, x_{n+1}) + p(z, x_{n+1})] \\ &< p(x_n, x_{n+1}) + p(z, x_{n+1}). \end{aligned}$$

Hence, by (2.6) and (2.7), we have

$$p(x_{n+1}, Tz) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \forall n \geqslant N_1.$$

Therefore, by (mW3) of the definition of $m\omega$-distance, we have $d(z, Tz) = 0$ and so $z = Tz$. The proof of the uniqueness is the same as in the proof of Theorem 2.9. □

Let $\varphi : [0, \infty) \to [0, \infty)$ be defined by $\varphi(t) = kt,\ k \in (0, 1)$ and use Theorems 2.9-2.10. Then we get the following results.

Corollary 2.11. *Let (X, d) be a complete quasi metric space and p be an $m\omega$-distance on X. Let $k \in (0, \frac{1}{2})$ and $T : X \to X$ be a self-mapping such that*

$$p(Tx, Ty) \leqslant k[p(x, Tx) + p(y, Ty)], \quad \forall x, y \in X.$$

Also, assume that one of the following conditions holds true:

(1) *If* $u \neq Tu$, *then* $\inf\{p(x,u)+p(Tx,u) : x \in X\} > 0$.

(2) T *is continuous.*

Then T *has a unique fixed point.*

Corollary 2.12. *Let* (X, d) *be a complete quasi metric space and* p *be a strong* $m\omega$*-distance on* X. *Let* $k \in (0, \frac{1}{2})$ *and* $T : X \to X$ *be a self-mapping satisfying the following condition*

$$p(Tx, Ty) \leqslant k[p(x, Tx) + p(y, Ty)], \quad \forall x, y \in X.$$

Then T *has a unique fixed point.*

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