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Bishop spherical images of a spacelike curve in Minkowski 3-space

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In this work, we introduce Bishop spherical images of a spacelike curve in terms of Bishop trihedra in Minkowski 3-space. With this aim, we translate Bishop frame vectors to the center of Lorentzian sphere and give some characterizations. Additionally, we illustrate an example of our main results.

Key words: Classical differential geometry, spacelike curve, spherical image, Minkowski space.

INTRODUCTION

Frenet-Serret frame of a regular curve in the Euclidean space opened a door to treatment of classical differential geometry of the curves. For instance, spherical images of a regular curve in the Euclidean space are obtained by translating of Frenet-Serret vector fields to the center of the unit sphere, for details (Do Carmo, 1976).

With the discovery of an alternative moving frame by Bishop (1975), the researchers aimed to study some of classical topics in terms of this new frame. It was different, because Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative (Bükcü and Karacan, 2009). Similar to Euclidean space, recently Yılmaz et al. (Yılmaz S, Turgut M and Özyılmaz E: Dokuz Eylül University, Personal Communication) investigated Bishop spherical images of a regular curve in the Euclidean space. They defined the tangent, M_1 and M_2 Bishop spherical images and presented some characterizations. By the spirit of this paper, we extend these approaches to the Minkowski 3-space. Firstly, we considered a spacelike curve with a spacelike principal normal and translate Bishop frame vectors to the center Lorentzian sphere. By this way, we obtained some characterizations. Besides, we illustrate an example of our main results. Finally, the case of a spacelike curve with a timelike principal normal is similar.

PRELIMINARIES

To meet the requirements in the next sections, here the basic elements of the theory of the curves in the space E_1^3 are briefly presented (A more complete elementary treatment can be found in Lopez (2008), Ali and Turgut (2010).

The Minkowski three dimensional space E_1^3 is a real vector space R^3 endowed with the standard flat Lorentzian metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since g is an indefinite metric. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be arbitrary vectors in E_1^3, the Lorentzian cross product of u and v defined by

$$u \times v = -\begin{vmatrix} -i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Recall that a vector $v \in E_1^3$ can have one of three Lorentzian characters: it can be spacelike if $g(v,v) > 0$ or $v = 0$; timelike if $g(v,v) < 0$ and null (lightlike) if $g(v,v) = 0$ for $v \neq 0$. Similarly, an arbitrary curve $\delta = \delta(s)$ in E_1^3 can locally be spacelike, timelike or null (lightlike) if all of its velocity vectors δ' are respectively spacelike, timelike or null (lightlike), for every $s \in I \subset R$. The pseudo-norm of an arbitrary vector $a \in E_1^3$ is given by

$$\|a\| = \sqrt{|g(a,a)|}.$$

The curve $\delta = \delta(s)$ is called a unit speed curve if velocity vector δ' is unit i.e., $\|\delta'\| = 1$. For vectors $v, w \in E_1^3$ it is said to be orthogonal if and only if $g(v,w) = 0$.

Denote by $\{T, N, B\}$ the moving Frenet-Serret frame along the curve $\delta = \delta(s)$ in the space E_1^3. For a unit speed spacelike curve with spacelike principal normal, with first and second curvature (torsion), $\kappa(s)$ and $\tau(s)$ the following Frenet-Serret formulae are given in Pekmen and Pasali (1999), Petrovic-Torgasev and Sucurovic (2000):

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where $g(T,T) = g(N,N) = 1$, $g(B,B) = -1$ and $g(T,N) = g(T,B) = g(N,B) = 0$.
Here the first and second curvature defined by $\kappa = \kappa(s) = \|T'(s)\|$ and $\tau(s) = -g(N, B')$.

The Lorentzian sphere S_1^2 of radius $r > 0$ and with the center in the origin of the space E_1^3 is defined by

$$S_1^2 = \{p = (p_1, p_2, p_3) \in E_1^3 : g(p,p) = r^2\}.$$

The Bishop frame is due to Bishop (1975). Recently, this special frame is extended to Minkowski-Lorentz spaces and other related areas. There is an extensive literature on this subject, for instance, Bükcü and Karacan (2007a, b), Bükcü and Karacan (2009), Karacan and Bükcü (2008), Karacan et al. (2008), Yılmaz (2009). For a spacelike curve with a spacelike principal normal, the derivative equations of Bishop frame is defined in Özdemir and Ergin (2008) as:

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix},$$

where $g(T,T) = g(N_1,N_1) = 1$, $g(N_2,N_2) = -1$ and $g(T,N_1) = g(T,N_2) = g(N_1,N_2) = 0$.
Here we shall call the set $\{T, N_1, N_2\}$ as Bishop trihedra and k_1 and k_2 as Bishop curvatures. The relation matrix may be expressed as Özdemir and Ergin (2008)

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta(s) & \sinh\theta(s) \\ 0 & \sinh\theta(s) & \cosh\theta(s) \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{1}$$

where $\theta(s) = \operatorname{arctanh}\dfrac{k_2}{k_1}$, $\tau(s) = \dfrac{d\theta(s)}{ds}$ and $\kappa(s) = \sqrt{|k_1^2 - k_2^2|}$. Here, Bishop curvatures are defined by the equations

$$\begin{cases} k_1 = \kappa \cosh\theta(s) \\ k_2 = \kappa \sinh\theta(s) \end{cases}.$$

In the rest of the paper, we shall suppose $k_1 \neq k_2$. It is well-known that for a spacelike curve with non-vanishing curvatures, the following propositions hold.

Proposition

Let $\xi = \xi(s)$ be a spacelike curve with curvatures κ and τ. The curve ξ lies on the Lorentzian sphere if and only if

$$\frac{d}{ds}\left[\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\kappa}\right)\right] = \frac{\tau}{\kappa},$$ Pekmen and Pasali (1999).

Proposition

Let $\xi = \xi(s)$ be a spacelike curve with curvatures κ and τ. The curve ξ is a general helix if and only if

$$\frac{\kappa}{\tau} = constant,$$ Barros et al. (2001).

MAIN RESULTS

In this section, first we adapt to the definitions of Euclidean one expressed as in Yılmaz et al. to Minkowski 3-space.

Tangent Bishop spherical image of a spacelike curve

Definition

Let $\xi = \xi(s)$ be a spacelike curve in Minkowsi 3-space E_1^3. If we translate of the first vector field of Bishop trihedra to the center O of the unit Lorentzian sphere S_1^2, we have a spherical image $\delta = \delta(s_\delta)$. This is curve is called Tangent Bishop Spherical Image in E_1^3.

Let $\delta = \delta(s_\delta)$ be tangent Bishop spherical image of a spacelike curve $\xi = \xi(s)$. First we differentiate

$$\delta' = \frac{d\delta}{ds_\delta}\frac{ds_\delta}{ds} = k_1 N_1 + k_2 N_2.$$

Here, we shall denote differentiation according to s by a dash, and differentiation according to s_δ by a dot. By the Bishop frame, we have the tangent vector of

$\delta = \delta(s_\delta)$ as

$$T_\delta = \frac{k_1 N_1 + k_2 N_2}{\sqrt{\left|k_1^2 - k_2^2\right|}},$$

where

$$\frac{ds_\delta}{ds} = \sqrt{\left|k_1^2 - k_2^2\right|} = \kappa(s).$$

To determine the first curvature κ_δ of $\delta = \delta(s_\delta)$, we express

$$\dot{T} = -T_1 + \frac{k_1^2}{\left(k_1^2 - k_2^2\right)^2}\left(\frac{k_2}{k_1}\right)'(k_2 N_1 + k_1 N_2).$$

So we arrive at

$$\kappa_\delta = \sqrt{\left|1 - \frac{k_1^4}{\left(k_1^2 - k_2^2\right)^3}\left(\frac{k_2}{k_1}\right)'^2\right|}. \qquad (2)$$

Therefore, we have the principal normal of the curve $\delta = \delta(s_\delta)$

$$N_\delta = \frac{1}{\kappa_\delta}\left\{-T_1 + \frac{k_1^2}{\left(k_1^2 - k_2^2\right)^2}\left(\frac{k_2}{k_1}\right)'(k_2 N_1 + k_1 N_2)\right\}.$$

In terms of Lorentzian cross product $T_\delta \times N_\delta$, we obtain binormal vector field as follows:

$$B_\delta = \frac{1}{\kappa_\delta\sqrt{\left|k_1^2 - k_2^2\right|}}\left\{-\frac{k_1^2}{\left(k_1^2 - k_2^2\right)}\left(\frac{k_2}{k_1}\right)'T_1 + k_2 N_1 + k_1 N_2\right\}.$$

Using obtained equations and the formulae of the torsion, we have the torsion of the tangent spherical image

$$\tau_\delta = \frac{\left\{\begin{array}{l}\left(k_1 k_2' + k_2\left(k_2^2 - k_1^2\right)\right)\left[k_1'' - k_1\left(k_1^2 - k_2^2\right)\right] - \\ k_1 k_1'\left[k_2'' - k_2\left(k_1^2 - k_2^2\right)\right] - k_1'\left(k_2 k_2' - k_1 k_1'\right)\end{array}\right\}}{\left[k_1^2\left(\dfrac{k_2}{k_1}\right)'\right]^2 + \left(k_1^2 - k_2^2\right)^2}. \qquad (3)$$

Corollary 1

Let $\delta = \delta(s_\delta)$ be tangent Bishop spherical image of a spacelike curve $\xi = \xi(s)$. If ratio of Bishop curvatures of $\xi = \xi(s)$ is constant ($\frac{k_2}{k_1} = constant$), then the tangent Bishop spherical image $\delta = \delta(s_\delta)$ is a circle in the osculating plane.

Proof: Let $\delta = \delta(s_\delta)$ be tangent Bishop spherical image of a spacelike curve $\xi = \xi(s)$ and Bishop curvatures of $\xi = \xi(s)$ satisfy $\frac{k_2}{k_1} = constant$. Substituting this to equations (2) and (3), we have $\kappa_\delta = constant$, and $\tau_\delta = 0$, respectively. Since, the tangent image is a circle in the osculating plane.

Remark 1

Considering $\theta_\delta = \int_0^{s_\delta} \tau_\delta ds_\delta$ and using transformation matrix, one can obtain the Bishop trihedra $\{T_\delta, N_{1\delta}, N_{2\delta}\}$ of the curve $\delta = \delta(s_\delta)$.

N_1 Bishop spherical image of a spacelike curve

Definition

Let $\xi = \xi(s)$ be a spacelike curve in Minkowsi 3-space E_1^3. If we translate of the second vector field of Bishop trihedra to the center O of the unit Lorentzian sphere S_1^2, we have a spherical image $\phi = \phi(s_\phi)$. This is curve is called *N_1 Bishop Spherical Image* in E_1^3.

Let $\phi = \phi(s_\phi)$ be N_1 Bishop Spherical Image of the spacelike curve $\xi = \xi(s)$ in E_1^3. We shall follow same procedure to determine relations among Bishop and Frenet-Serret invariants. Let us first form

$$\phi' = \frac{d\phi}{ds_\phi}\frac{ds_\phi}{ds} = -k_1 T.$$

Since, we have

$$T_\phi = T \tag{4}$$

$$\frac{ds_\phi}{ds} = -k_1. \tag{5}$$

One can differentiate (4)

$$T_\phi' = \dot{T}_\phi \frac{ds_\phi}{ds} = k_1 N_1 + k_2 N_2,$$

in another words

$$\dot{T}_\phi = -N_1 - \frac{k_2}{k_1} N_2.$$

Using the formula above, we have the first curvature and the principal normal vector

$$\kappa_\phi = \left\|\dot{T}_\phi\right\| = \sqrt{\left|1 - \left(\frac{k_2}{k_1}\right)^2\right|} \tag{6}$$

and

$$N_\phi = -\frac{N_1}{\kappa_\phi} - \frac{k_2}{k_1 \kappa_\phi} N_2.$$

The cross product of $T_\phi \times N_\phi$ gives us the binormal vector of N_1 Bishop Spherical Image of the spacelike curve $\xi = \xi(s)$ as

$$B_\phi = -\left(\frac{k_2 N_1 + k_1 N_2}{\sqrt{\left|k_1^2 - k_2^2\right|}}\right).$$

Besides, using the formulae of the torsion, we express

$$\tau_\phi = \frac{k_1 \left(\frac{k_2}{k_1}\right)'}{\left(k_1^2 - k_2^2\right)} \tag{7}$$

or

$$\tau_\phi = \frac{1}{k_1}\frac{d}{ds}\left(\tanh^{-1}\left(\frac{k_2}{k_1}\right)\right).$$

Corollary 2

Let $\phi = \phi(s_\phi)$ be N_1 Bishop Spherical Image of the spacelike curve $\xi = \xi(s)$ in E_1^3. If ratio of Bishop curvatures of $\xi = \xi(s)$ is constant ($\frac{k_2}{k_1} = constant$, i.e.), then the N_1 Bishop spherical image $\phi = \phi(s_\phi)$ is a circle in the osculating plane.

Theorem 1

Let $\phi = \phi(s_\phi)$ be N_1 Bishop Spherical Image of the spacelike curve $\xi = \xi(s)$ in E_1^3. There exists a relation among Frenet-Serret invariants of $\phi = \phi(s_\phi)$ and Bishop curvatures of $\xi = \xi(s)$ as follows:

$$\frac{k_2}{k_1} = \int_0^{s_\phi} \kappa_\phi^2 \tau_\phi ds_\phi. \tag{8}$$

Proof: Let $\phi = \phi(s_\phi)$ be N_1 Bishop Spherical Image of the spacelike curve $\xi = \xi(s)$. Then, the equations (5) and (6) hold. Substituting these equations to (7) and using the chain rule, we get

$$\tau_\phi = \frac{k_1 \frac{d}{ds_\phi}\left(\frac{k_2}{k_1}\right)' \frac{ds_\phi}{ds}}{\left(k_1^2 - k_2^2\right)}.$$

Finally integrating both sides, we have (8).
In the light of the propositions 2.1 and 2.2, we express the following theorems without proofs.

Theorem 2

Let $\phi=\phi(s_\phi)$ be N_1 Bishop Spherical Image of the spacelike curve $\xi=\xi(s)$ in Minkowski space E_1^3. If ϕ is a general helix, then Bishop curvatures of $\xi=\xi(s)$ satisfy

$$\frac{k_1^2\left(\frac{k_2}{k_1}\right)'}{\sqrt{\left|k_1^2-k_2^2\right|\left(k_1^2-k_2^2\right)}}=constant.$$

We suppose that $\phi=\phi(s_\phi)$ is a Lorentzian spherical curve. So it is safe to report that:

Theorem 3

Let $\phi=\phi(s_\phi)$ be N_1 Bishop Spherical Image of the spacelike curve $\xi=\xi(s)$ in Minkowski space E_1^3. Then, Bishop curvatures of $\xi=\xi(s)$ satisfy the differential equation

$$\frac{d}{ds}\left\{\frac{k_1^2-k_2^2}{k_1\left(\frac{k_2}{k_1}\right)'}\frac{d}{ds}\left[\frac{k_1}{\sqrt{\left|k_1^2-k_2^2\right|}}\right]\right\}-\frac{k_1\left(\frac{k_2}{k_1}\right)'}{\sqrt{\left|k_1^2-k_2^2\right|\left(k_1^2-k_2^2\right)}}=0.$$

Remark 2

Considering $\theta_\phi=\int_0^{s_\phi}\tau_\phi ds_\phi$ and using transformation matrix, one can obtain the Bishop trihedra $\{T_\phi, N_{1\phi}, N_{2\phi}\}$ of the curve $\phi=\phi(s_\phi)$.

Here, one question may come to mind about the obtained tangent spherical image, since, Frenet-Serret and Bishop frames have a common tangent vector field. Images of such tangent images are the same as we shall demonstrate in section 4. But, here we are concerned with the tangent Bishop spherical image's Frenet-Serret apparatus according to Bishop invariants.

N_2 Bishop spherical image of a spacelike curve

Definition

Let $\xi=\xi(s)$ be a spacelike curve in Minkowsi 3-space E_1^3. If we translate the third vector field of Bishop trihedra to the center O of the unit Lorentzian sphere S_1^2, we have a spherical image $\psi=\psi(s_\psi)$. This is curve is called *N_2* Bishop Spherical Image in E_1^3.

Let $\psi=\psi(s_\psi)$ be N_2 Bishop Spherical Image of the spacelike curve $\xi=\xi(s)$ in E_1^3. We shall follow same procedure to determine relations among Bishop and Frenet-Serret invariants. We can write that

$$\psi'=\frac{d\psi}{ds_\psi}\frac{ds_\psi}{ds}=k_2T.$$

Using the scalar and vector parts of the above equation, we express

$$T_\psi=T \tag{9}$$

and

$$\frac{ds_\psi}{ds}=k_2. \tag{10}$$

Differentiating of the formulae (9), we easily have

$$T'_\psi=\dot{T}_\psi\frac{ds_\psi}{ds}=k_1N_1+k_2N_2.$$

Dividing both sides with k_2, we have

$$\dot{T}_\psi=\frac{k_1}{k_2}N_1+N_2.$$

So, we obtain

$$\kappa_\psi=\left\|\dot{T}_\psi\right\|=\sqrt{\left|\left(\frac{k_1}{k_2}\right)^2-1\right|} \tag{11}$$

and

$$N_\psi = \frac{k_1}{k_2 \kappa_\psi} N_1 + \frac{1}{\kappa_\psi} N_2 .$$

In terms of the cross product of $T_\psi \times N_\psi$, we have the binormal vector fields of the

$\psi = \psi(s_\psi)$ as

$$B_\psi = \left(\frac{k_2 N_1 + k_1 N_2}{\sqrt{\left|k_1^2 - k_2^2\right|}}\right).$$

Using the formulae of the torsion, we express

$$\tau_\psi = -\frac{k_2 \left(\frac{k_1}{k_2}\right)'}{\left(k_1^2 - k_2^2\right)} \tag{12}$$

or

$$\tau_\psi = \frac{-1}{k_2} \frac{d}{ds}\left(\tanh^{-1}\left(\frac{k_1}{k_2}\right)\right).$$

Corollary 3

Let $\psi = \psi(s_\psi)$ be N_2 Bishop Spherical Image of the spacelike curve $\xi = \xi(s)$ in E_1^3. If ratio of Bishop curvatures of $\xi = \xi(s)$ is constant ($\frac{k_1}{k_2} = constant$), then the N_2 Bishop spherical image $\psi = \psi(s_\psi)$ is a circle in the osculating plane.

Theorem 4

Let $\psi = \psi(s_\psi)$ be N_2 Bishop Spherical Image of the spacelike curve $\xi = \xi(s)$ in E_1^3. There exists a relation among Frenet-Serret invariants of $\psi = \psi(s_\psi)$ and Bishop curvatures of $\xi = \xi(s)$ as follows:

$$\frac{k_1}{k_2} + \int_0^{s_\psi} \kappa_\psi^2 \tau_\psi ds_\psi . \tag{13}$$

Proof: Similar to proof of the theorem 4, (13) can be obtained in terms of the equations (10), (11) and (12).

Similar to N_1 Bishop Spherical Image of a spacelike curve, we express the following theorems without proofs:

Theorem 5

Let $\psi = \psi(s_\psi)$ be N_2 Bishop Spherical Image of the spacelike curve $\xi = \xi(s)$ in Minkowski space E_1^3. If ψ is a general helix, then Bishop curvatures of $\xi = \xi(s)$ satisfy

$$\frac{k_2^2 \left(\frac{k_1}{k_2}\right)'}{\sqrt{\left|k_1^2 - k_2^2\right|}\left(k_1^2 - k_2^2\right)} = constant.$$

Theorem 6

Let $\psi = \psi(s_\psi)$ be N_2 Bishop Spherical Image of the spacelike curve $\xi = \xi(s)$ in Minkowski space E_1^3. Then, Bishop curvatures of $\xi = \xi(s)$ satisfy the differential equation

$$\frac{d}{ds}\left\{\frac{k_1^2 - k_2^2}{k_2\left(\frac{k_1}{k_2}\right)'} \frac{d}{ds}\left[\frac{k_1}{\sqrt{\left|k_1^2 - k_2^2\right|}}\right]\right\} + \frac{k_2 \left(\frac{k_1}{k_2}\right)'}{\sqrt{\left|k_1^2 - k_2^2\right|}\left(k_1^2 - k_2^2\right)} = 0.$$

Remark 3

Considering $\theta_\psi = \int_0^{s_\psi} \tau_\psi ds_\psi$ and using transformation matrix, one can obtain the Bishop trihedra $\{T_\psi, N_{1\psi}, N_{2\psi}\}$ of the curve $\psi = \psi(s_\psi)$.

Remark 4

In the case of a spacelike curve with timelike principal normal, similar characterizations can be easily obtained in following same procedure.

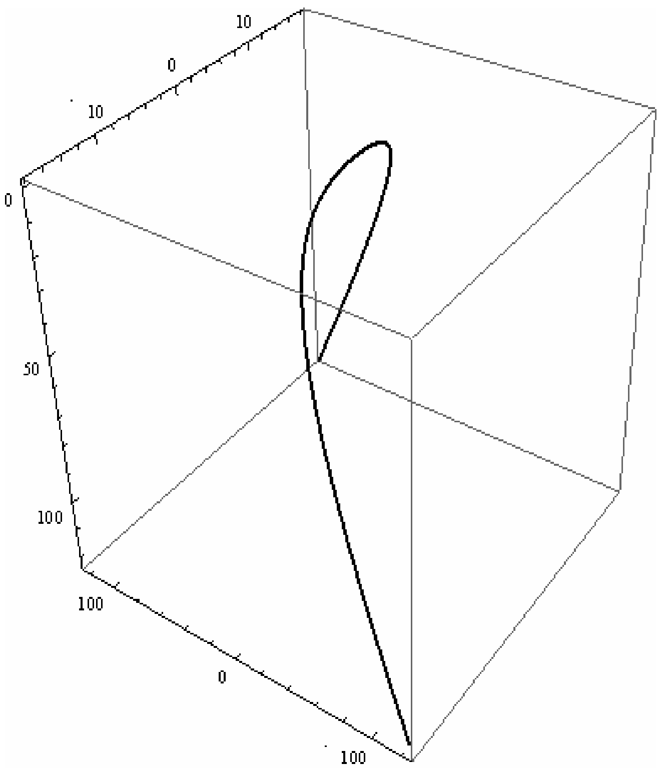

Figure 1. $\delta = \delta(s)$ with $a = 3, b = 4$ and $c = 5$.

EXAMPLE

In this section, we illustrated an example of Bishop spherical image of a spacelike curve in Minkowski 3-space.

Let us consider a spacelike curve:

$$\delta = \delta(s) = \left(a \cosh \frac{s}{c}, a \sinh \frac{s}{c}, \frac{bs}{c} \right),$$

where $c = \sqrt{a^2 + b^2}$. Let us choose $a = 3, b = 4$ and $c = 5$. We plot this curve in the Figure 1.

In this case, one can calculate its curvatures and Frenet-Serret trihedra as

$$\begin{cases} \kappa = \frac{3}{25}, \\ \tau = \frac{4}{25}, \\ T = \left(\frac{3}{5} \sinh \frac{s}{5}, \frac{3}{5} \cosh \frac{s}{5}, \frac{4}{5} \right), \\ N = \left(\cosh \frac{s}{5}, \sinh \frac{s}{5}, 0 \right), \\ B = \left(\frac{4}{5} \sinh \frac{s}{5}, \frac{4}{5} \cosh \frac{s}{5}, -\frac{3}{5} \right). \end{cases}$$

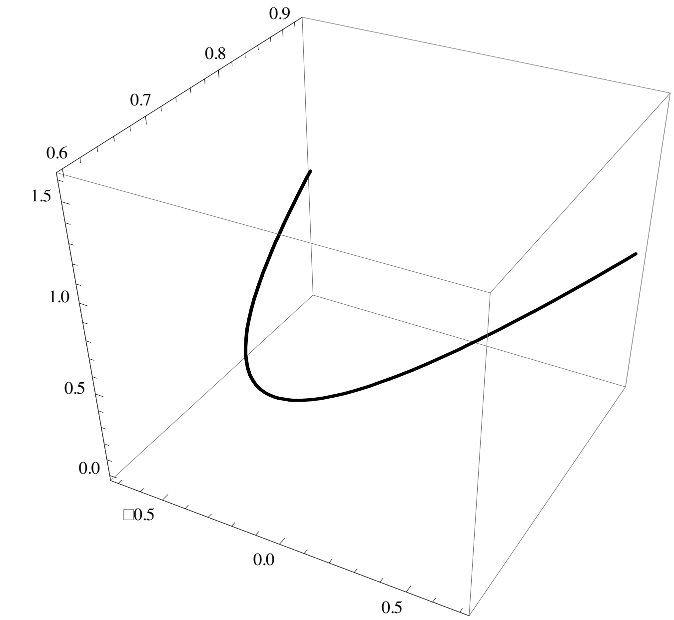

Figure 2. Tangent Bishop Spherical Image of $\delta = \delta(s)$.

Since we form the finite integral which is a key of the Bishop trihedra as follows:

$$\theta(s) = \int_0^s \frac{4}{25} ds = \frac{4s}{25}.$$

Therefore, we express the relation matrix by the aid of (1)

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh \frac{4s}{25} & \sinh \frac{4s}{25} \\ 0 & \sinh \frac{4s}{25} & \cosh \frac{4s}{25}) \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}.$$

We know that the tangent vector fields are common between Frenet-Serret and Bishop frames. So, by the method of Cramer, we have the other vector fields of Bishop trihedra:

$$N_1 = \left(\cosh \frac{4s}{25} \cosh \frac{s}{5} - \frac{4}{5} \sinh \frac{4s}{25} \sinh \frac{s}{5}, -\frac{4}{5} \cosh \frac{s}{5} \sinh \frac{4s}{25} + \cosh \frac{4s}{25} \sinh \frac{s}{5}, \frac{3}{5} \sinh \frac{4s}{25} \right)$$

And

$$N_2 = \left(-\cosh \frac{s}{5} \sinh \frac{4s}{25} + \frac{4}{5} \cosh \frac{4s}{25} \sinh \frac{s}{5}, \frac{4}{5} \cosh \frac{4s}{25} \cosh \frac{s}{5} - \sinh \frac{4s}{25} \sinh \frac{s}{5}, -\frac{3}{5} \cosh \frac{4s}{25} \right)$$

Now, we plot the tangent Bishop spherical image of $\delta = \delta(s)$ (Figure 2):

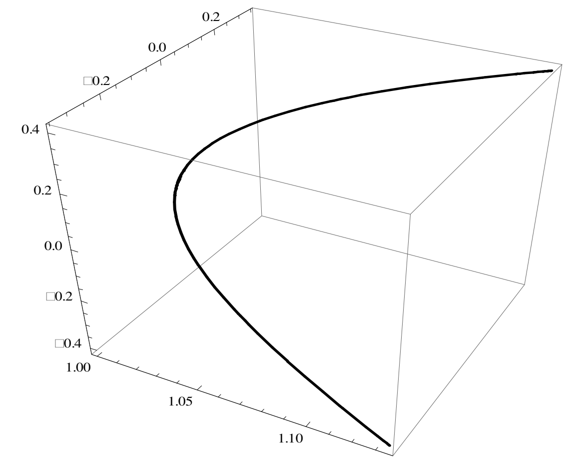

Figure 3. N_1 Bishop Spherical Image of $\delta = \delta(s)$.

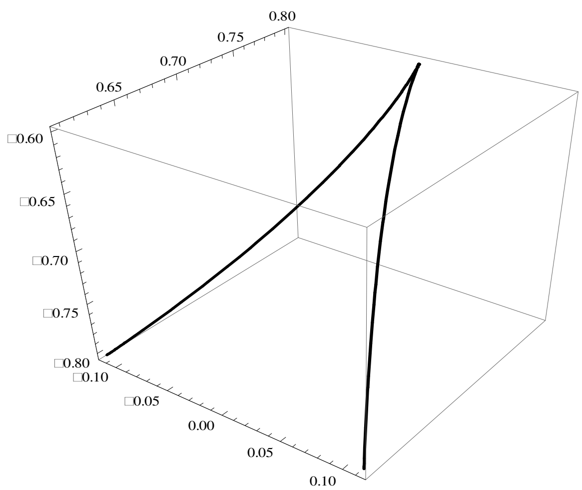

Figure 4:N_2 Bishop Spherical Image of $\delta = \delta(s)$

We have N_1 Bishop Spherical Image of $\delta = \delta(s)$. One can see N_2 Bishop spherical image in the Figure 3."

We have N_2 Bishop Spherical Image of $\delta = \delta(s)$. One can see N_2 Bishop spherical image in the Figure 4."

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