Ul-Haq and Noor *Journal of Inequalities and Applications* 2012, **2012**:309
http://www.journalofinequalitiesandapplications.com/content/2012/1/309

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A certain class of analytic functions and the growth rate of Hankel determinant

Wasim Ul-Haq[1*] and Khalida Inayat Noor[2]

*Correspondence:
wasim474@hotmail.com
[1]Department of Mathematics, Abdul Wali Khan University, Mardan, Pakistan
Full list of author information is available at the end of the article

Abstract

The growth rate of coefficients and Hankel determinant for any class of analytic functions is well known. In this present investigation, we define a class of analytic functions related to strongly close-to-convex functions. We investigate different interesting properties for this class including arc length, the growth rate of coefficients and the growth rate of Hankel determinant by using the method of Noonan and Thomas. Several well-known results appear as special cases from our results.

MSC: 30C45; 30C50

Keywords: analytic functions; close-to-convexity; Hankel determinant

1 Introduction

We denote by A the class of functions f which are analytic in the open unit disc $E = \{z : |z| < 1\}$ and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Also let us denote by $P_k(\rho)$ the class of functions h with $h(0) = 1$, which are analytic in E and satisfying

$$\int_0^{2\pi} \left| \mathrm{Re}\left\{ \frac{h(z) - \rho}{1 - \rho} \right\} \right| d\theta \le k\pi,$$

where $z = re^{i\theta}$, $k \ge 2$ and $0 \le \rho < 1$. This class has been investigated by Padmanabhan and Parvatham [1]. For $\rho = 0$, we obtain the class P_k, introduced by Pinchuk [2] and for $\rho = 0$, $k = 2$, we obtain the class P of functions with a positive real part.

The class T_k was first introduced and investigated by the second author [3] as a generalization of close-to-convexity. She studied its geometrical interpretation and various other interesting properties including the growth rate of coefficient differences and a radius of convexity problem. Recently, she [4] studied the class of analytic functions corresponding to strongly close-to-convex functions. She employed a modification to a method of Pommerenke [5] to investigate the growth rate of Hankel determinant problems regarding this class.

In this paper, we define a class of analytic functions related to strongly close-to-convex functions. We investigate different interesting properties including inclusion relations, arc length, the growth rate of coefficients and the growth rate of Hankel determinant by using a different method from that given in [4].

We now define the following classes of analytic functions.

Definition 1.1 Let $f \in A$ be locally univalent in E. Then, for $\eta \neq 0$ (complex), $0 \leq \rho < 1$, $f \in V_k(\eta, \rho)$ if and only if

$$\left(1 + \frac{1}{\eta}\frac{zf''(z)}{f'(z)}\right) \in P_k(\rho), \quad z \in E.$$

We note that for $\eta = 1$, we have the class $V_k(\rho)$ of bounded boundary rotations of order ρ introduced by Padmanabhan and Parvatham [1]. Also, $V_k(0) = V_k$, the class of functions of bounded boundary rotations and $V_2(\eta, 0) = C(\eta)$ denotes the class of convex functions of complex order.

Definition 1.2 Let $f \in A$. Then $f \in \widetilde{N}_k(\eta, \rho, \beta)$ if and only if, for $k \geq 2$, $\beta \geq 0$, there exists a function $g \in V_k(\eta, \rho)$ such that

$$\left|\arg \frac{f'(z)}{g'(z)}\right| \leq \frac{\beta\pi}{2}, \quad z \in E.$$

For $\eta = 1$, we have the class $\widetilde{T}_k(\rho, \beta)$ which was recently introduced and studied by Noor [4]. For $k = 2$, $\eta = 1$, $\rho = 0$, $\widetilde{N}_2(1, 0, \beta)$ is the class of strongly close-to-convex functions. Also, $\widetilde{N}_2(1, \rho, 0) = C(\rho)$ is the class of convex functions of order ρ. For $\eta = 1$, $\rho = 0$, $\beta = 1$, the class of $\widetilde{N}_k(\eta, \rho, \beta)$ reduces to the class T_k introduced by Noor [3].

We need the following results in our investigation.

Lemma 1.1 *A function $f \in V_k(\eta, \rho)$ if and only if*

(i) $f'(z) = [f_1'(z)]^{(1-\rho)\eta}, f_1(z) \in V_k$,

(ii) $f'(z) = [f_2'(z)]^{\eta}, f_1(z) \in V_k(\rho)$,

(iii) *there exist two normalized starlike functions $s_1(z)$ and $s_2(z)$ such that*

$$f'(z) = \left[\frac{(s_1(z)/z)^{(\frac{k}{4}+\frac{1}{2})}}{(s_2(z)/z)^{(\frac{k}{4}-\frac{1}{2})}}\right]^{(1-\rho)\eta}. \tag{1.2}$$

The above lemma is a special case of the result discussed in [6].

Lemma 1.2 [7] *Let $h \in P$ with $z = re^{i\theta}$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \left|h(z)\right|^2 d\theta \leq \frac{1 + 3r^2}{1 - r^2}.$$

Lemma 1.3 [8] *Let f be univalent and $0 \leq r < 1$. Then there exists a number z_1 with $|z_1| = r$, such that for all z, $|z| = r$, we have*

$$|z - z_1|\left|f(z)\right| \leq \frac{2r^2}{1 - r^2}.$$

2 Some properties of the class $\widetilde{N}_k(\eta, \rho, \beta)$

Following essentially the same method as given in [4], we can easily obtain the following result.

Theorem 2.1 *The function $f \in \widetilde{N}_k(\eta, \rho, \beta)$ if and only if*

$$f'(z) = \frac{(f_1(z))^{(\frac{k}{4}+\frac{1}{2})(1-\rho)\eta}}{(f_2(z))^{(\frac{k}{4}-\frac{1}{2})(1-\rho)\eta}},$$

where f_1 and f_2 are strongly close-to-convex functions of order β.

Theorem 2.2 *Let $f \in \widetilde{N}_k(\eta, \rho, \beta)$ in E. Then $f \in C_\eta$ for $|z| < r_0$, where*

$$r_0 = \frac{2|\eta|}{[(1-\rho)|\eta|k + 2\beta] + \sqrt{[(1-\rho)|\eta|k + 2\beta]^2 - 4(1-2\rho)|\eta|^2}}. \tag{2.1}$$

This result is sharp.

Proof We can write

$$f'(z) = g'(z)h^\beta(z), \quad g(z) \in V_k(\eta, \rho), h(z) \in P.$$

Using Lemma 1.1, we have

$$f'(z) = \left[\frac{(s_1(z)/z)^{(\frac{k}{4}+\frac{1}{2})}}{(s_2(z)/z)^{(\frac{k}{4}-\frac{1}{2})}}\right]^{(1-\rho)\eta} h^\beta(z), \tag{2.2}$$

where s_1 and s_2 are starlike functions. Logarithmic differentiation of (2.2) gives us

$$\frac{zf''(z)}{f'(z)} = (1-\rho)\eta\left[-1 + \left(\frac{k}{4}+\frac{1}{2}\right)\frac{zs_1'(z)}{s_1(z)} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{zs_2'(z)}{s_2(z)}\right] + \beta\frac{zh'(z)}{h(z)},$$

which implies that

$$1 + \frac{1}{\eta}\frac{zf''(z)}{f'(z)} = \rho + (1-\rho)\left[\left(\frac{k}{4}+\frac{1}{2}\right)\frac{zs_1'(z)}{s_1(z)} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{zs_2'(z)}{s_2(z)}\right] + \frac{\beta}{\eta}\frac{zh'(z)}{h(z)}.$$

Now using distortion results for the class P, we have

$$\begin{aligned}\operatorname{Re}\left(1 + \frac{1}{\eta}\frac{zf''(z)}{f'(z)}\right) &\geq \rho + (1-\rho)\left[\left(\frac{k}{4}+\frac{1}{2}\right)\frac{1-r}{1+r} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1+r}{1-r}\right] - \frac{\beta}{|\eta|}\frac{2r}{1-r^2} \\ &= \frac{\rho|\eta|(1-r^2) + (1-\rho)|\eta|[1-kr+r^2] - 2\beta r}{|\eta|(1-r^2)}.\end{aligned} \tag{2.3}$$

The right-hand side of (2.3) is positive for $|z| < r_0$, where r_0 is given by (2.1). The sharpness can be viewed from the function $f_0 \in \widetilde{N}_k(\eta, \rho, \beta)$, given by

$$f_0'(z) = \frac{(1+z)^{(\frac{k}{2}-1)(1-\rho)\eta+\beta}}{(1-z)^{(\frac{k}{2}+1)(1-\rho)\eta+\beta}}, \quad z \in E. \tag{2.4}$$

We note the following.

(i) For $\eta = 1$, we have the radius of convexity for the class $\widetilde{T}_k(\rho, \beta)$ studied by Noor [4].
(ii) For $\eta = 1$, $\rho = 0$, $\beta = 1$, we have the radius of convexity for the class T_k, proved by Noor [3].
(iii) For $\eta = 1$, $\rho = 0$, $\beta = 1$, $k = 2$, we have the radius of convexity for close-to-convex functions which is well known. □

We now discuss the arc length problem and the growth rate of coefficients for the class $\widetilde{N}_k(\eta, \rho, \beta)$.

Theorem 2.3 *Let $f \in \widetilde{N}_k(\eta, \rho, \beta)$, for $\operatorname{Re}\eta > 0$, $\beta \geq 0$, $0 \leq \rho < 1$ and $\frac{(k+2)(1-\rho)\operatorname{Re}\eta}{2-\beta} > 1$. Then*

$$L_r(f) \leq c(k, \eta, \rho, \beta)\left(\frac{1}{1-r}\right)^{(\frac{k}{2}+1)(1-\rho)\operatorname{Re}\eta+\beta-1},$$

where $c(k, \eta, \rho, \beta)$ is a constant depending only on k, η, ρ, β. The exponent $[(\frac{k}{2}+1)(1-\rho)\operatorname{Re}\eta + \beta - 1]$ is sharp.

Proof We have

$$L_r(f) = \int_0^{2\pi} \left|zf'(z)\right| d\theta, \quad z = re^{i\theta}.$$

Using Definition 1.1, Lemma 1.1(iii) and the distortion theorem for starlike functions, we have

$$\begin{aligned}
L_r(f) &= \int_0^{2\pi} \left|zg'(z)h^{\beta}(z)\right| d\theta, \quad g(z) \in V_k(\eta, \rho), h(z) \in P \\
&= \int_0^{2\pi} \left|z\frac{(s_1(z)/z)^{(\frac{k}{4}+\frac{1}{2})(1-\rho)\eta}}{(s_2(z)/z)^{(\frac{k}{4}-\frac{1}{2})(1-\rho)\eta}}\right|\left|h^{\beta}(z)\right| d\theta \\
&= \int_0^{2\pi} \left|z^{1-\eta(1-\rho)}\frac{(s_1(z))^{(\frac{k}{4}+\frac{1}{2})(1-\rho)\eta}}{(s_2(z))^{(\frac{k}{4}-\frac{1}{2})(1-\rho)\eta}}\right|\left|h^{\beta}(z)\right| d\theta \\
&\leq \frac{2^{(\frac{k}{2}-1)(1-\rho)\operatorname{Re}\eta}}{r^{(\frac{k}{4}+\frac{1}{2})(1-\rho)\operatorname{Re}\eta-1}} \int_0^{2\pi} \left|s_1(z)\right|^{(\frac{k}{4}+\frac{1}{2})(1-\rho)\operatorname{Re}\eta}\left|h(z)\right|^{\beta} d\theta \\
&\leq \frac{2^{(\frac{k}{2}-1)(1-\rho)\operatorname{Re}\eta}}{r^{(\frac{k}{4}+\frac{1}{2})(1-\rho)\operatorname{Re}\eta-1}} \int_0^{2\pi} \left(\left|s_1(z)\right|^{\frac{(\frac{k}{2}+1)(1-\rho)\operatorname{Re}\eta}{2-\beta}}\right)^{\frac{2-\beta}{2}}\left(\left|h(z)\right|^2\right)^{\frac{\beta}{2}} d\theta.
\end{aligned}$$

Using Holder's inequality with $p = \frac{2}{2-\beta}$, $q = \frac{2}{\beta}$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$\begin{aligned}
L_r(f) \leq{}& \frac{2^{(\frac{k}{2}-1)(1-\rho)\operatorname{Re}\eta}}{r^{(\frac{k}{4}+\frac{1}{2})(1-\rho)\operatorname{Re}\eta-1}}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|s_1(z)\right|^{\frac{(\frac{k}{2}+1)(1-\rho)\operatorname{Re}\eta}{2-\beta}} d\theta\right)^{\frac{2-\beta}{2}} \\
&\times \left(\frac{1}{2\pi}\int_0^{2\pi}\left|h(z)\right|^2 d\theta\right)^{\frac{\beta}{2}}.
\end{aligned}$$

Since $\frac{(k+2)(1-\rho)\operatorname{Re}\eta}{2-\beta} > 1$, therefore using subordination for starlike functions and Lemma 1.2, we have

$$\begin{aligned}
L_r(f) &\leq \frac{2^{(\frac{k}{2}-1)(1-\rho)\operatorname{Re}\eta}}{r^{(\frac{k}{4}+\frac{1}{2})(1-\rho)\operatorname{Re}\eta-1}}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|s_1(z)\right|^{\frac{(\frac{k}{2}+1)(1-\rho)\operatorname{Re}\eta}{2-\beta}}\,d\theta\right)^{\frac{2-\beta}{2}}\left(\frac{1+3r^2}{1-r^2}\right)^{\frac{\beta}{2}} \\
&\leq \frac{2^{(\frac{k}{2}-1)(1-\rho)\operatorname{Re}\eta+\frac{\beta}{2}}}{r^{(\frac{k}{4}+\frac{1}{2})(1-\rho)\operatorname{Re}\eta-1}}\left(\frac{1}{1-r}\right)^{\frac{\beta}{2}}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{r^{\frac{(\frac{k}{2}+1)(1-\rho)\operatorname{Re}\eta}{2-\beta}}}{|1+re^{i\theta}|^{\frac{(k+2)(1-\rho)\operatorname{Re}\eta}{2-\beta}}}\,d\theta\right)^{\frac{2-\beta}{2}} \\
&\leq \frac{2^{(\frac{k}{2}-1)(1-\rho)\operatorname{Re}\eta+\frac{\beta}{2}}}{r^{(\frac{k}{4}+\frac{1}{2})(1-\rho)\operatorname{Re}\eta-1-\frac{(\frac{k}{2}+1)(1-\rho)\operatorname{Re}\eta}{2-\beta}}}\left(\frac{1}{1-r}\right)^{\frac{\beta}{2}}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1+re^{i\theta}|^{\frac{(k+2)(1-\rho)\operatorname{Re}\eta}{2-\beta}}}\,d\theta\right)^{\frac{2-\beta}{2}} \\
&\leq c(k,\eta,\rho,\beta)\left(\frac{1}{1-r}\right)^{\frac{\beta}{2}}\left(\frac{1}{1-r}\right)^{(\frac{k}{2}+1)(1-\rho)\operatorname{Re}\eta+\beta-1}.
\end{aligned}$$

The function $F_0 \in \widetilde{N_k}(\eta,\rho,\beta)$ defined by

$$F_0'(z) = G_0'(z)h_0^{\beta}(z), \tag{2.5}$$

where

$$G_0'(z) = \frac{(1+z)^{(\frac{k}{2}-1)(1-\rho)\eta}}{(1-z)^{(\frac{k}{2}+1)(1-\rho)\eta}} \quad \text{and} \quad h_0(z) = \frac{1+z}{1-z},$$

shows that the exponent is sharp. □

By assigning different values to the parameters involved in the above theorem, we have the following interesting results.

Corollary 2.1 *Let $f \in \widetilde{T}_k(\rho,\beta)$. Then*

$$L_r(f) \leq c(k,\rho,\beta)\left(\frac{1}{1-r}\right)^{(\frac{k}{2}+1)(1-\rho)+\beta-1}.$$

Corollary 2.2 *Let $f \in T_k$. Then*

$$L_r(f) \leq c(k,\rho,\beta)\left(\frac{1}{1-r}\right)^{\frac{k}{2}+1}.$$

Coefficient growth problems The problem of growth rate and asymptotic behavior of coefficients is well known. In the upcoming results, we investigate these problems for a different set of classes by varying different parameters.

Theorem 2.4 *Let $f \in \widetilde{N_k}(\eta,\rho,\beta)$ and be of the form* (1.1). *Then, for $n > 3$, $k \geq 2$, $\operatorname{Re}\eta > 0$, $0 \leq \rho < 1$, $\beta \geq 0$, we have*

$$|a_n| \leq c(k,\eta,\rho,\beta)n^{(\frac{k}{2}+1)(1-\rho)\operatorname{Re}\eta+\beta-2},$$

where $c(k,\eta,\rho,\beta)$ is a constant depending only on k, η, ρ, β. The exponent $[(\frac{k}{2}+1)(1-\rho)\operatorname{Re}\eta+\beta-2]$ is sharp.

Proof With $z = re^{i\theta}$, Cauchy's theorem gives us

$$na_n = \frac{1}{2\pi r^n}\int_0^{2\pi}\left|zf'(z)\right|d\theta = \frac{1}{2\pi r^n}L_r(f), \quad z = re^{i\theta}.$$

Using Theorem 2.3 and putting $r = 1 - \frac{1}{n}$, we obtain the required result. The sharpness follows from the function F_0 defined by the relation (2.5). □

Corollary 2.3 *Let $f \in \widetilde{T}_k(\rho,\beta)$ and be of the form* (1.1). *Then, for $n > 3$, $k \geq 2$, we have*

$$|a_n| = \mathrm{O}(1)n^{(\frac{k}{2}+1)(1-\rho)+\beta-2}.$$

For $\rho = 0$, $\beta = 1$ in the above corollary, we have the growth rate of coefficients problem for functions in this class T_k and for $k = 2$, $\rho = 0$, $\beta = 1$ gives us the growth rate of coefficient estimates for close-to-convex functions, which is well known.

3 Hankel determinant problem

The Hankel determinant of a function f of the form (1.1) is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{3.1}$$

The growth rate of Hankel determinant $H_q(n)$ as $n \to \infty$, when f is a member of any class of analytic functions, is well known. Pommerenke [9] proved that when f is an areally mean p-valent function, then for $p \geq 1$,

$$H_q(n) = \mathrm{O}(1)n^{s\sqrt{q}-\frac{q}{2}} \quad \text{as } n \to \infty$$

and $s = 16p^{3/2}$ and where O(1) depends on p, q, and the function f. In particular, this shows that $H_q(n) \to 0$ as $n \to \infty$ for large q relative to p. In fact, for $p = 1$, $q = 2$

$$H_2(n) = \mathrm{O}(1)n^{\frac{1}{2}} \quad \text{as } n \to \infty.$$

The exponent $\frac{1}{2}$ is exact.

Noonan and Thomas [10] gave the exact growth rate of $H_q(n)$ for large p relative to q, and they proved that

$$H_q(n) = \mathrm{O}(1)\begin{cases} n^{2p-1}, & q = 1, p > \frac{1}{4}, \\ n^{2pq-q^2}, & q \geq 2, p \geq 2(q-1), \end{cases}$$

where O(1) depends upon p, q only and the exponent $2pq - q^2$ is best possible. Also, for univalent functions, Pommerenke [5] has proved that for $q \geq 2$,

$$H_q(n) < c(q)\, n^{-(\frac{1}{2}+\beta)q+\frac{3}{2}} \quad (n \to \infty),$$

where $\beta > \frac{1}{4000}$, which in particular shows that

$$H_2(n) < cn^{\frac{1}{2}-2\beta}.$$

Pommerenke [9] has shown that if f is starlike, then for $q \geq 1$,

$$H_q(n) = \mathrm{O}(1)n^{2-q} \quad (n \to \infty),$$

where O(1) depends upon q only and the exponent $2 - q$ is best possible. Noor [11] generalized this result for close-to-convex functions. We also refer to [4, 12–14].

Also, for $f \in V_k$, it is shown [15] that for $q \geq 1$, $n \to \infty$,

$$H_q(n) = \mathrm{O}(1) \begin{cases} n^{\frac{k}{2}-1}, & q = 1, \\ n^{\frac{kq}{2}-q^2}, & q \geq 2, k \geq 8q - 10, \end{cases}$$

where O(1) depends upon p, q and f only. The exponent $\frac{kq}{2} - q^2$ is best possible.

Following the notation of Noonan and Thomas [10], we define the following.

Definition 3.1 Let z_1 be a non-zero complex number. Then for $f(z)$, given by (1.1), we define

$$\Delta_j\big(n, z_1, f(z)\big) = \Delta_{j-1}\big(n, z_1, f(z)\big) - z_1\Delta_{j-1}\big(n+1, z_1, f(z)\big), \quad j \geq 1$$

with $\Delta_1(n, z_1, f(z)) = a_n$.

The following two lemmas are due to Noonan and Thomas [10] which are essential in our investigations.

Lemma 3.1 *Let $f \in A$ and let the Hankel determinant of f be defined by* (3.1). *Then, writing $\Delta_j = \Delta_j(n, z_1, f)$, we have*

$$H_q(n) = \begin{vmatrix} \Delta_{2q-2}(n) & \Delta_{2q-3}(n+1) & \cdots & \Delta_{q-1}(n+q-1) \\ \Delta_{2q-3}(n+1) & \Delta_{2q-4}(n+2) & \cdots & \Delta_{q-2}(n+q) \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{q-1}(n+q-1) & \Delta_{q-2}(n+q) & \cdots & \Delta_0(n+2q-2) \end{vmatrix}. \tag{3.2}$$

Lemma 3.2 *With $z_1 = \frac{n}{n+1}y$ and $v \geq 0$ any integer,*

$$\Delta_j\big(n+v, z_1, zf'\big) = \sum_{l=0}^{j} \binom{j}{l} \frac{y^l(v-(l-1)n)}{(n+1)^l} \Delta_{j-l}(n+v+l, y, f).$$

We also need the following remark given in [10].

Remark 3.1 Consider any determinant of the form

$$D = \begin{vmatrix} y_{2q-2} & y_{2q-3} & \cdots & y_{q-1} \\ y_{2q-3} & y_{2q-4} & \cdots & y_{q-2} \\ \vdots & \vdots & \vdots & \vdots \\ y_{q-1} & y_{q-2} & \cdots & y_0 \end{vmatrix},$$

with $1 \leq i, j \leq q$ and $\alpha_{i,j} = y_{2q-(i+j)}$, $D = \det(\alpha_{i,j})$. Thus,

$$D = \sum_{\nu_1 \in S_q} (\operatorname{sgn} \nu_1) \prod_{j=1}^{q} y_{2q-(\nu_1(j)+j)},$$

where S_q is the symmetric group on q elements, and $\operatorname{sgn} \nu_1$ is either +1 or –1. Thus, in the expansion of D, each summand has q factors, and the sum of the subscripts of the factors of each summand is $q^2 - q$.

Now, let n be given and $H_q(n)$ be as in Lemma 3.1, then each summand in the expansion of $H_q(n)$ is of the form

$$\prod_{i=1}^{q} \Delta_{\nu_1(i)}\bigl(n + 2q - 2 - \nu_1(i)\bigr),$$

where $\nu_1 \in S_q$ and

$$\sum_{i=1}^{q} \nu_1(i) = q^2 - q; \quad 0 \leq \nu_1(i) \leq 2q - 2.$$

We now prove the following.

Theorem 3.1 *Let $f \in \widetilde{N}_k(\eta, \rho, \beta)$ and let the Hankel determinant of f, for $q \geq 2$, $n \geq 1$, be defined by* (3.1). *Then, for $q \geq 2$ and $k > 4\frac{(q-1)}{(1-\rho)\operatorname{Re}\eta} - 2$, we have*

$$H_q(n) = \mathrm{O}(1) n^{[(\frac{k}{2}+1)(1-\rho)\operatorname{Re}\eta]q - q^2 - (n+1)q},$$

where O(1) *depends only on k, η, ρ, β and q.*

Proof Since $f \in \widetilde{N}_k(\eta, \rho, \beta)$, there exists $g \in V_k(\eta, \rho)$ such that

$$f'(z) = g(z) h^{\beta}(z) \in P, \quad z \in E.$$

Now, for $j \geq 1$, z_1 any non-zero complex number and $z = re^{i\theta}$, we consider for $F(z) = zf'(z)$,

$$\begin{aligned} &\bigl|\Delta_j(n, z_1, F)\bigr| \\ &\quad = \left| \frac{1}{2\pi r^{n+j}} \int_0^{2\pi} (z - z_1)^j F(z) e^{-i(n+j)\theta} \, d\theta \right| \end{aligned}$$

$$\begin{aligned} &\le \frac{1}{2\pi r^{n+j}} \int_0^{2\pi} |z-z_1|^j \left| z^{1-(1-\rho)\eta} \frac{(s_1(z)/z)^{(\frac{k}{4}+\frac{1}{2})(1-\rho)\eta}}{(s_2(z)/z)^{(\frac{k}{4}-\frac{1}{2})(1-\rho)\eta}} \right| \left| h^\beta(z) \right| d\theta \\ &\le \frac{1}{2\pi r^{n+j}} \int_0^{2\pi} |z-z_1|^j \left| s_1(z) \right|^j \frac{|s_1(z)|^{(\frac{k}{4}+\frac{1}{2})(1-\rho)\operatorname{Re}\eta - j}}{|s_2(z)|^{(\frac{k}{4}-\frac{1}{2})(1-\rho)\operatorname{Re}\eta}} \left| h(z) \right|^\beta d\theta, \end{aligned}$$

where we have used Lemma 1.1(iii). Using Lemma 1.3, we have

$$\left| \Delta_j(n, z_1, F) \right| \le \frac{1}{2\pi r^{n+j}} \left(\frac{2r^2}{1-r^2} \right)^j \int_0^{2\pi} \frac{|s_1(z)|^{(\frac{k}{4}+\frac{1}{2})(1-\rho)\operatorname{Re}\eta - j}}{|s_2(z)|^{(\frac{k}{4}-\frac{1}{2})(1-\rho)\operatorname{Re}\eta}} \left| h(z) \right|^\beta d\theta. \tag{3.3}$$

By employing distortion results for starlike functions and simplifying, we obtain, from (3.3),

$$\left| \Delta_j(n, z_1, F) \right| \le \frac{1}{2\pi} \frac{(2)^{(\frac{k}{2}-1)(1-\rho)\operatorname{Re}\eta}}{r^{(\frac{k}{4}-\frac{1}{2})(1-\rho)\operatorname{Re}\eta + n - j - 1}} \left(\frac{1}{1-r} \right)^j \int_0^{2\pi} \left| s_1(z) \right|^{(\frac{k}{4}+\frac{1}{2})(1-\rho)\operatorname{Re}\eta - j} \left| h(z) \right|^\beta d\theta.$$

Using Holder's inequality, with $p = \frac{2}{2-\beta}$, $q = \frac{2}{\beta}$, such that $\frac{1}{p} + \frac{1}{q} = 1$, we can write

$$\begin{aligned} \left| \Delta_j(n, z_1, F) \right| \le{}& \frac{(2)^{(\frac{k}{2}-1)(1-\rho)\operatorname{Re}\eta}}{r^{(\frac{k}{4}-\frac{1}{2})(1-\rho)\operatorname{Re}\eta + n - j - 1}} \left(\frac{1}{1-r} \right)^j \left(\frac{1}{2\pi} \int_0^{2\pi} \left| s_1(z) \right|^{\frac{(\frac{k}{2}+1)(1-\rho)\operatorname{Re}\eta - 2j}{2-\beta}} \right)^{\frac{2-\beta}{2}} \\ & \times \left(\frac{1}{2\pi} \int_0^{2\pi} \left| h(z) \right|^2 d\theta \right)^{\frac{\beta}{2}}. \end{aligned}$$

Proceeding in a similar way as in Theorem 2.3, we have

$$\begin{aligned} \left| \Delta_j(n, z_1, F) \right| \le{}& \frac{(2)^{(\frac{k}{2}-1)(1-\rho)\operatorname{Re}\eta + \frac{\beta}{2}}}{r^{n-1}} \left(\frac{1}{1-r} \right)^{\frac{\beta}{2}+j} \\ & \times \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1-re^{i\theta}|^{\frac{(k+2)(1-\rho)\operatorname{Re}\eta - 4j}{2-\beta}}} d\theta \right)^{\frac{2-\beta}{2}}. \end{aligned}$$

Subordination for starlike functions further yields

$$\left| \Delta_j(n, z_1, F) \right| = \mathrm{O}(1) \left(\frac{1}{1-r} \right)^{(\frac{k}{2}+1)(1-\rho)\operatorname{Re}\eta - j + \beta - 1},$$

where O(1) depends only on k, η, β and j.

Now, applying Lemma 3.2 and putting $z_1 = \frac{n}{n+1} e^{i\theta_m}$ $(n \to \infty)$, we have for $k \ge (\frac{4j}{(1-\rho)\operatorname{Re}\eta} - 2)$, $j \ge 1$,

$$\Delta_j\left(n, e^{i\theta_n}, f(z)\right) = \mathrm{O}(1) n^{(\frac{k}{2}+1)(1-\rho)\operatorname{Re}\eta - j + \beta - 1}.$$

We now estimate the growth rate of $H_q(n)$. For $q = 1$, $H_q(n) = a_n = \Delta_0(n)$ and from Theorem 2.4, it follows that

$$H_1(n) = \mathrm{O}(1) n^{(\frac{k}{2}+1)(1-\rho)\operatorname{Re}\eta + \beta - 2}.$$

For $q \geq 2$, we use Remark 3.1 together with Lemma 3.1, to have

$$H_q(n) = \mathrm{O}(1) n^{q[(\frac{k}{2}+1)(1-\rho)\operatorname{Re}\eta+\beta]-q^2}, \quad k > \left(\frac{4(q-1)}{(1-\rho)\operatorname{Re}\eta} - 2\right),$$

where O(1) depends only on k, η, ρ, β and q. □

By giving special values to the parameters involved in the above theorem, we obtain the following interesting results.

Corollary 3.1 *Let $f \in \widetilde{T}_k(\rho,\beta)$ and be defined as in* (1.1). *Then, for $q \geq 2$, $k > (\frac{4(q-1)}{1-\rho} - 2)$,*

$$H_q(n) = \mathrm{O}(1) n^{q[(\frac{k}{2}+1)(1-\rho)+\beta]-q^2} \quad (n \to \infty),$$

where O(1) *depends only on k, ρ, β and q.*

Noor [4] studied the above corollary with a different method.

Corollary 3.2 *Let $f \in T_k$ and be defined as in* (1.1). *Then, for $q \geq 2$, $k > (4q-6)$,*

$$H_q(n) = \mathrm{O}(1) n^{q[(\frac{k}{2}+2]-q^2} \quad (n \to \infty),$$

where O(1) *depends only on k and q.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
WH carried out all the calculations and drafted the manuscript. KIN provided results and ideas which were used in proofs of main theorems.

Author details
[1]Department of Mathematics, Abdul Wali Khan University, Mardan, Pakistan. [2]Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan.

Acknowledgements
The authors would like to thank the reviewers of this article for their valuable comments on the earlier version of this paper.

Received: 17 September 2012 Accepted: 7 December 2012 Published: 28 December 2012

References
1. Padmanabhan, KS, Parvatham, R: Properties of a class of functions with bounded boundary rotations. Ann. Pol. Math. **31**, 311-323 (1975)
2. Pinchuk, B: Functions of bounded boundary rotations. Isr. J. Math. **10**, 6-16 (1971)
3. Noor, KI: On a generalization of close-to-convexity. Int. J. Math. Math. Sci. **6**, 327-334 (1983)
4. Noor, KI: On certain analytic functions related with strongly close-to-convex functions. Appl. Math. Comput. **197**, 149-157 (2008)
5. Pommerenke, Ch: On hankel determinant of univalent functions. Mathematika **14**, 108-112 (1967)
6. Aouf, MK: On certain classes of p-valent functions. Int. J. Math. Math. Sci. **9**, 55-64 (1986)
7. Pommerenke, Ch: On close-to-convex analytic functions. Trans. Am. Math. Soc. **111**, 176-186 (1965)
8. Goluzin, GM: On distortion theorem and coefficients of univalent functions. Mat. Sb. **19**, 183-203 (1946)
9. Pommerenke, Ch: On the coefficients of Hankel determinant of univalent functions. J. Lond. Math. Soc. **41**, 111-122 (1966)
10. Noonan, JW, Thomas, DK: On the Hankel determinant of areally mean p-valent functions. Proc. Lond. Math. Soc. **25**(3), 503-524 (1972)
11. Noor, KI: On the Hankel determinant of close-to-convex univalent functions. Int. J. Math. Math. Sci. **3**, 447-481 (1980)

12. Arif, M, Noor, KI, Raza, M: Hankel determinant problem of a subclass of analytic functions. J. Inequal. Appl. **2012**, 22 (2012)
13. Noor, KI: On the Hankel determinant problem for strongly close-to-convex functions. J. Nat. Geom. **11**, 29-34 (1997)
14. Noor, KI, Al-Naggar, IA: Hankel determinant problem. J. Nat. Geom. **14**(2), 133-140 (1998)
15. Noor, KI: Hankel determinant problem for functions of bounded boundary rotations. Rev. Roum. Math. Pures Appl. **28**, 731-739 (1983)

doi:10.1186/1029-242X-2012-309
Cite this article as: Ul-Haq and Noor: A certain class of analytic functions and the growth rate of Hankel determinant. *Journal of Inequalities and Applications* 2012 **2012**:309.