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Some sharp continued fraction inequalities for the Euler-Mascheroni constant

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Abstract
The aim of this paper is to establish some new continued fraction inequalities for the Euler-Mascheroni constant by multiple-correction method.

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1 Introduction

Euler introduced a constant, then later this constant was called 'Euler's constant' as the limit of the sequence

$$\gamma(n) := \sum_{m=1}^{n} \frac{1}{m} - \ln n. \tag{1.1}$$

It is also known as the Euler-Mascheroni constant. There are many famous unsolved problems about the nature of this constant (see *e.g.* Dence and Dence [1], Havil [2] and Lagarias [3]). For example, it is a long-standing open problem if it is a rational number. A good part of its mystery comes from the fact that the known algorithms converging to γ are not very fast, at least, when they are compared to similar algorithms for π and e.

The sequence $(\gamma(n))_{n\in\mathbb{N}}$ converges very slowly toward γ, like $(2n)^{-1}$, by Young (see [4]). Up to now, many authors are preoccupied to improve its rate of convergence, see *e.g.* [1, 4–13] and the references therein. We list some main results as follows:

$$\sum_{m=1}^{n} \frac{1}{m} - \ln\left(n+\frac{1}{2}\right) = \gamma + O\left(n^{-2}\right) \quad \text{(DeTemple [9])},$$

$$\sum_{m=1}^{n} \frac{1}{m} - \ln \frac{n^3+\frac{3}{2}n^2+\frac{227}{240}+\frac{107}{480}}{n^2+n+\frac{97}{240}} = \gamma + O\left(n^{-6}\right) \quad \text{(Mortici [4])},$$

$$\sum_{m=1}^{n} \frac{1}{m} - \ln\left(1+\frac{1}{2n}+\frac{1}{24n^2}-\frac{1}{48n^3}+\frac{23}{5{,}760n^4}\right)$$
$$= \gamma + O\left(n^{-5}\right) \quad \text{(Chen and Mortici [6])}.$$

Recently, Mortici and Chen [5] provided a very interesting sequence

$$\begin{aligned}\nu(n) &= \sum_{m=1}^{n} \frac{1}{m} - \frac{1}{2}\ln\left(n^2+n+\frac{1}{3}\right) \\ &\quad - \left(\frac{-\frac{1}{180}}{(n^2+n+\frac{1}{3})^2} + \frac{\frac{8}{2{,}835}}{(n^2+n+\frac{1}{3})^3} + + \frac{\frac{5}{1{,}512}}{(n^2+n+\frac{1}{3})^4} + \frac{\frac{592}{93{,}555}}{(n^2+n+\frac{1}{3})^5}\right)\end{aligned}$$

and proved

$$\lim_{n\to\infty} n^{12}\bigl(\nu(n)-\gamma\bigr) = -\frac{796{,}801}{43{,}783{,}740}. \tag{1.2}$$

Hence the rate of convergence of the sequence $(\nu(n))_{n\in\mathbb{N}}$ is n^{-12}.

Very recently, by inserting the continued fraction term in (1.1), Lu [11] introduced a class of sequences $(r_k(n))_{n\in\mathbb{N}}$ and showed

$$\frac{1}{72(n+1)^3} < \gamma - r_2(n) < \frac{1}{72n^3}, \tag{1.3}$$

$$\frac{1}{120(n+1)^4} < r_3(n) - \gamma < \frac{1}{120(n-1)^4}. \tag{1.4}$$

It is their works that motivate our study. In this paper, starting from the sequence $\nu(n)$, based one the works of Mortici, Chen and Lu, we provide some new classes of convergent sequences with faster rate of convergence for the Euler-Mascheroni constant as follows.

Theorem 1 *For the Euler-Mascheroni constant, we have the following convergent sequence*:

$$\begin{aligned}r_k(n) &= \sum_{m=1}^{n} \frac{1}{m} - \frac{1}{2}\ln\left(n^2+n+\frac{1}{3}\right) \\ &\quad - \frac{a_1}{(n+\frac{1}{2})^4 + b_2(n+\frac{1}{2})^2 + b_0 +} \frac{a_2}{(n+\frac{1}{2})^2 + k_2 +} \ddots,\end{aligned}$$

where

$$\begin{aligned}
&b_2 = \frac{85}{126}, \qquad b_0 = -\frac{18{,}287}{63{,}504};\\
&a_1 = -\frac{1}{180}, \qquad a_2 = \frac{1{,}830{,}112}{2{,}750{,}517}, \qquad a_3 = -\frac{36{,}637{,}398{,}233{,}630{,}775}{9{,}056{,}534{,}057{,}598{,}976},\\
&a_4 = -\frac{16{,}486{,}938{,}208{,}076{,}386{,}182{,}240{,}455{,}433{,}101{,}197{,}312}{1{,}114{,}333{,}428{,}433{,}648{,}110{,}295{,}680{,}727{,}490{,}570{,}625},\\
&a_5 = -\frac{202\ldots665}{538\ldots376}; \qquad \ldots,\\
&k_2 = \frac{19{,}949{,}142{,}781}{5{,}995{,}446{,}912}, \qquad k_3 = \frac{83{,}116{,}192{,}006{,}963{,}596{,}639{,}745{,}097}{13{,}306{,}111{,}671{,}966{,}702{,}431{,}356{,}800},\\
&k_4 = \frac{3{,}223{,}193{,}895{,}482{,}188{,}285{,}536{,}076{,}617{,}166{,}535{,}131{,}877{,}815{,}854{,}443}{314{,}899{,}884{,}866{,}916{,}635{,}406{,}301{,}198{,}738{,}123{,}952{,}960{,}626{,}300{,}800},\\
&k_5 = \frac{155\ldots06831}{102\ldots440}, \qquad \ldots.
\end{aligned}$$

For $1 \le l \le 5$, *we have*

$$\begin{aligned}
&\lim_{n\to\infty} n^{11}\big(r_1(n)-\gamma\big) = \frac{457{,}528}{123{,}773{,}265} := C_1, \\
&\lim_{n\to\infty} n^{15}\big(r_2(n)-\gamma\big) = \frac{16{,}615{,}600{,}105{,}955}{1{,}111{,}124{,}185{,}307{,}136} := C_2, \\
&\lim_{n\to\infty} n^{19}\big(r_3(n)-\gamma\big) = \frac{10{,}827{,}769{,}830{,}530{,}486{,}830{,}966{,}003{,}768}{48{,}939{,}635{,}836{,}927{,}856{,}157{,}843{,}580{,}875} := C_3, \\
&\lim_{n\to\infty} n^{23}\big(r_4(n)-\gamma\big) \\
&\quad = \frac{10{,}081{,}089{,}508{,}192{,}698{,}621{,}180{,}096{,}244{,}326{,}317{,}115{,}783{,}678{,}789{,}502{,}403}{1{,}210{,}912{,}402{,}904{,}902{,}219{,}665{,}371{,}334{,}928{,}392{,}072{,}785{,}413{,}652{,}121{,}600} \\
&\quad := C_4, \\
&\lim_{n\to\infty} n^{27}\big(r_5(n)-\gamma\big) = \frac{587\ldots212}{890\ldots625} := C_5.
\end{aligned} \tag{1.5}$$

Furthermore, for $r_2(n)$ and $r_3(n)$, we also have the following inequalities.

Theorem 2 *Let* $r_2(n)$, $r_3(n)$, C_2 *and* C_3 *be defined in Theorem* 1, *then*

$$C_2\frac{1}{(n+\frac{3}{2})^{14}} < r_2(n)-\gamma < C_2\frac{1}{n^{14}}, \tag{1.6}$$

$$C_3\frac{1}{(n+\frac{3}{2})^{18}} < r_3(n)-\gamma < C_3\frac{1}{n^{18}}. \tag{1.7}$$

Remark 1 In fact, Theorem 2 implies that $r_2(n)$ and $r_3(n)$ are strictly increasing functions of n. Certainly, it has similar inequalities for $r_l(n)$ $(4 \le k \le 5)$, we omit these details. It should also be noted that (1.4) cannot deduce the monotony of $r_3(n)$.

Remark 2 It is worth pointing out that Theorem 2 provides sharp bounds and faster rate of convergence for harmonic sequence, which are superior to Theorems 3 and 4 in Mortici and Chen [5].

2 The proof of Theorem 1

The following lemma gives a method for measuring the rate of convergence. This lemma was first used by Mortici [14–18] for constructing asymptotic expansions, or to accelerate some convergences.

Lemma 1 *If the sequence* $(x_n)_{n\in\mathbb{N}}$ *is convergent to zero and there exists the limit*

$$\lim_{n\to+\infty} n^s(x_n - x_{n+1}) = l \in [-\infty, +\infty] \tag{2.1}$$

with $s > 1$, *then there exists the limit*

$$\lim_{n\to+\infty} n^{s-1}x_n = \frac{l}{s-1}. \tag{2.2}$$

In the sequel, we always assume $n \ge 2$.

Based on our previous works [19–22], we will apply *multiple-correction method* to study faster convergence problem for constants of Euler-Mascheroni. In this paper, we always assume that the following conditions hold.

Condition 1 The initial-correction function $\eta_0(n)$ satisfies

$$\lim_{n\to\infty}\big(\nu(n)-\eta_0(n)\big)=0,$$
$$\lim_{n\to\infty} n^{l_0}\big(\nu(n)-\nu(n+1)-\eta_0(n)+\eta_0(n+1)\big)=C_0\neq 0,$$

with some a positive integer $l \geq 2$.

Condition 2 The kth correction function $\eta_k(n)$ has the form of $-\frac{C_{k-1}}{\Phi_k(l_{k-1};n)}$, where

$$\lim_{n\to\infty} n^{l_{k-1}}\left(\nu(n)-\nu(n+1)-\sum_{j=0}^{k-1}\big(\eta_j(n)-\eta_j(n+1)\big)\right)=C_{k-1}\neq 0.$$

Condition 3 The function $\nu(x)$ satisfies $\nu(x)\in C^{\infty}[1,+\infty)$.

Step 1 (The initial-correction) We choose $\eta_0(n)=0$, and let

$$r_0(n):=\sum_{m=1}^{n}\frac{1}{m}-\frac{1}{2}\ln\big(n^2+bn+c\big)-\eta_0(n). \tag{2.3}$$

Developing expression (2.3) into power series expansion in $\frac{1}{n}$, we obtain

$$r_0(n)-r_0(n+1)=\frac{1-b}{2}\frac{1}{n^2}+\frac{3b^2+3b-6c-4}{6}\frac{1}{n^3}+O\left(\frac{1}{n^4}\right). \tag{2.4}$$

By Lemma 1, we have

(i) If $b\neq 1$ and $c\neq\frac{1}{3}$, then the rate of convergence of $(r_0(n)-\gamma)_{n\in\mathbb{N}}$ is n^{-1} since

$$\lim_{n\to\infty} n\big(r_0(n)-\gamma\big)=\frac{1-b}{2}\neq 0.$$

(ii) If $b=1$ and $c=\frac{1}{3}$, from (2.4) we have

$$r_0(n)-r_0(n+1)=-\frac{1}{45}\frac{1}{n^5}+O\left(\frac{1}{n^6}\right).$$

Hence the rate of convergence of $(r_1(n)-\gamma)_{n\in\mathbb{N}}$ is n^{-5} since

$$\lim_{n\to\infty} n^4\big(r_0(n)-\gamma\big)=-\frac{1}{180}.$$

Step 2 (The first-correction) We let

$$\eta_1(n)=\frac{a_1}{(n+\frac{1}{2})^4+b_2(n+\frac{1}{2})^2+b_0} \tag{2.5}$$

and define

$$r_1(n) := \sum_{m=1}^{n} \frac{1}{m} - \frac{1}{2} \ln\left(n^2 + n + \frac{1}{3}\right) - \eta_0(n) - \eta_1(n). \tag{2.6}$$

By the same method as above, we find $a_1 = -\frac{1}{180}$, $b_2 = \frac{85}{126}$, $b_0 = -\frac{18,287}{63,504}$.

Applying Lemma 1 again, one has

$$\lim_{n\to\infty} n^{11}\bigl(r_1(n) - r_1(n+1)\bigr) = \frac{915,056}{24,754,653}, \tag{2.7}$$

$$\lim_{n\to\infty} n^{10}\bigl(r_1(n) - \gamma\bigr) = \frac{457,528}{123,773,265}. \tag{2.8}$$

Step 3 (The second-correction) Similarly, we set the second-correction function in the form of

$$\eta_2(n) = \frac{a_2}{(n+\frac{1}{2})^2 + k_2} \tag{2.9}$$

and define

$$\begin{aligned} r_2(n) :=& \sum_{m=1}^{n} \frac{1}{m} - \frac{1}{2} \ln\left(n^2 + n + \frac{1}{3}\right) \\ &- \frac{a_1}{(n+\frac{1}{2})^4 + b_2(n+\frac{1}{2})^2 + b_0 +} \frac{a_2}{(n+\frac{1}{2})^2 + k_2}. \end{aligned} \tag{2.10}$$

By the same method as above, we find $a_2 = \frac{1,830,112}{2,750,517}$, $k_2 = \frac{19,949,142,781}{5,995,446,912}$.

Applying Lemma 1 again, one has

$$\lim_{n\to\infty} n^{15}\bigl(r_2(n) - r_2(n+1)\bigr) = \frac{16,615,600,105,955}{79,366,013,236,224}, \tag{2.11}$$

$$\lim_{n\to\infty} n^{14}\bigl(r_2(n) - \gamma\bigr) = \frac{16,615,600,105,955}{1,111,124,185,307,136}. \tag{2.12}$$

Repeat the above approach to determine a_3 to a_5 step by step. However, the computations become very difficult to compute a_l and k_l, $l > 5$. In this paper we will use the *Mathematica* software to manipulate symbolic computations.

This completes the proof of Theorem 1.

3 The proof of Theorem 2

The following lemma plays an important role in the proof of our inequalities, which is a direct consequence of the Hermite-Hadamard inequality.

Lemma 2 *Let $f''(x)$ be a continuous function. If $f''(x) > 0$, then*

$$\int_a^{a+1} f(x)\,dx > f(a + 1/2). \tag{3.1}$$

In the sequel, the notation $P_k(x)$ means a polynomial of degree k in x with all of its nonzero coefficients positive, which may be different at each occurrence.

Let us begin to prove Theorem 2. Note $r_2(\infty) = \gamma$, it is easy to see

$$r_2(n) - \gamma = \sum_{m=n}^{\infty}\bigl(r_2(m) - r_2(m+1)\bigr) = \sum_{m=n}^{\infty} f(m), \tag{3.2}$$

where

$$\begin{aligned} f(m) &= -\frac{1}{m+1} + \frac{1}{2}\ln\frac{(m+1)^2 + (m+1) + \frac{1}{3}}{m^2 + m + \frac{1}{3}} \\ &\quad + \frac{a_1}{(m+\frac{3}{2})^4 + b_2(m+\frac{3}{2})^2 + b_0 +} \frac{a_2}{(m+\frac{3}{2})^2 + k_2} \\ &\quad - \frac{a_1}{(m+\frac{1}{2})^4 + b_2(m+\frac{1}{2})^2 + b_0 +} \frac{a_2}{(m+\frac{1}{2})^2 + k_2}. \end{aligned}$$

Let $D_1 = \frac{83{,}078{,}000{,}529{,}775}{26{,}455{,}337{,}745{,}408}$. By using the *Mathematica* software, we have

$$\begin{aligned} &f'(x) + D_1\frac{1}{(x+\frac{3}{2})^{16}} \\ &\quad = -\frac{P_{29}(x)}{904{,}235{,}176{,}845(1+x)^2(3+2x)^{16}(1+3x+3x^2)(7+9x+3x^2)P_6^{(1)^2}(x)P_6^{(2)^2}(x)} > 0 \end{aligned}$$

and

$$\begin{aligned} &f'(x) + D_1\frac{1}{(x+\frac{1}{2})^{16}} \\ &\quad = \frac{P_{29}(x)}{904{,}235{,}176{,}845(1+x)^2(1+2x)^{16}(1+3x+3x^2)(7+9x+3x^2)P_8^{(1)^2}(x)P_8^{(2)^2}(x)} > 0. \end{aligned}$$

Hence, we get the following inequalities for $x \geq 1$:

$$D_1\frac{1}{(x+\frac{3}{2})^{16}} < -f'(x) < D_1\frac{1}{(x+\frac{1}{2})^{16}}. \tag{3.3}$$

Applying $f(\infty) = 0$, (3.3) and Lemma 2, we get

$$\begin{aligned} f(m) &= -\int_m^{\infty} f'(x)\,dx \leq D_1\int_m^{\infty}\left(x + \frac{1}{2}\right)^{-16} dx \\ &= \frac{D_1}{15}\left(m + \frac{1}{2}\right)^{-15} \leq \frac{D_1}{15}\int_m^{m+1} x^{-15}\,dx. \end{aligned} \tag{3.4}$$

From (3.1) and (3.4) we obtain

$$\begin{aligned} r_2(n) - \gamma &\leq \sum_{m=n}^{\infty}\frac{D_1}{15}\int_m^{m+1} x^{-15}\,dx \\ &= \frac{D_1}{15}\int_n^{\infty} x^{-15}\,dx = \frac{D_1}{210}\frac{1}{n^{14}} = C_2\frac{1}{n^{14}}. \end{aligned} \tag{3.5}$$

Similarly, we also have

$$\begin{aligned} f(m) &= -\int_m^\infty f'(x)\,dx \ge D_1 \int_m^\infty \left(x+\frac{3}{2}\right)^{-16} dx \\ &= \frac{D_1}{15}\left(m+\frac{3}{2}\right)^{-15} \ge \frac{D_1}{19}\int_{m+\frac{3}{2}}^{m+\frac{5}{2}} x^{-15}\,dx \end{aligned}$$

and

$$\begin{aligned} r_2(n)-\gamma &\ge \sum_{m=n}^\infty \frac{D_1}{15}\int_{m+\frac{3}{2}}^{m+\frac{5}{2}} x^{-15}\,dx \\ &= \frac{D_1}{15}\int_{n+\frac{3}{2}}^\infty x^{-15}\,dx = \frac{D_1}{210}\frac{1}{(n+\frac{3}{2})^{14}} = C_2\frac{1}{(n+\frac{3}{2})^{14}}. \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) completes the proof of (1.6).

Note $r_3(\infty)=\gamma$, it is easy to see

$$r_3(n)-\gamma = \sum_{m=n}^\infty \bigl(r_3(m)-r_3(m+1)\bigr) = \sum_{m=n}^\infty g(m), \tag{3.7}$$

where

$$\begin{aligned} g(m) &= \frac{1}{m+1} - \frac{1}{2}\ln\frac{(m+1)^2+(m+1)+\frac{1}{3}}{m^2+m+\frac{1}{3}} \\ &\quad - \frac{a_1}{(m+\frac{3}{2})^4+b_2(m+\frac{3}{2})^2+b_0+\frac{a_2}{(m+\frac{3}{2})^2+k_2+\frac{a_3}{(m+\frac{3}{2})^2+k_3}}} \\ &\quad + \frac{a_1}{(m+\frac{1}{2})^4+b_2(m+\frac{1}{2})^2+b_0+\frac{a_2}{(m+\frac{1}{2})^2+k_2+\frac{a_3}{(m+\frac{1}{2})^2+k_3}}}. \end{aligned}$$

Let $D_2 = \frac{21{,}655{,}539{,}661{,}060{,}973{,}661{,}932{,}007{,}536}{286{,}196{,}700{,}800{,}747{,}696{,}829{,}494{,}625}$. By using the *Mathematica* software, we have

$$\begin{aligned} & g'(x) + D_2\frac{1}{(x+\frac{3}{2})^{20}} \\ &\quad = -\frac{P_{37}(x)}{286\ldots625(1+x)^2(3+2x)^{20}(1+3x+3x^2)(7+9x+3x^2){P_8^{(1)}}^2(x){P_8^{(2)}}^2(x)} < 0 \end{aligned}$$

and

$$\begin{aligned} & g'(x) + D_2\frac{1}{(x+\frac{1}{2})^{20}} \\ &\quad = \frac{P_{37}(x)}{286\ldots625(1+x)^2(1+2x)^{20}(1+3x+3x^2)(7+9x+3x^2){P_8^{(1)}}^2(x){P_8^{(2)}}^2(x)} > 0. \end{aligned}$$

Hence, we get the following inequalities for $x \ge 1$:

$$D_2\frac{1}{(x+\frac{3}{2})^{20}} < -g'(x) < D_2\frac{1}{(x+\frac{1}{2})^{20}}. \tag{3.8}$$

Applying $g(\infty) = 0$, (3.8) and Lemma 2, we get

$$\begin{aligned} g(m) &= -\int_m^\infty g'(x)\,dx \le D_2 \int_m^\infty \left(x + \frac{1}{2}\right)^{-20} dx \\ &= \frac{D_2}{19}\left(m + \frac{1}{2}\right)^{-19} \le \frac{D_2}{19}\int_m^{m+1} x^{-19}\,dx. \end{aligned} \tag{3.9}$$

From (3.1) and (3.4) we obtain

$$\begin{aligned} r_3(n) - \gamma &\le \sum_{m=n}^\infty \frac{D_2}{19}\int_m^{m+1} x^{-19}\,dx \\ &= \frac{D_2}{19}\int_n^\infty x^{-19}\,dx = \frac{D_2}{342}\frac{1}{n^{18}} = C_3\frac{1}{n^{18}}. \end{aligned} \tag{3.10}$$

Similarly, we also have

$$\begin{aligned} g(m) &= -\int_m^\infty g'(x)\,dx \ge D_2 \int_m^\infty \left(x + \frac{3}{2}\right)^{-20} dx \\ &= \frac{D_2}{19}\left(m + \frac{3}{2}\right)^{-19} \ge \frac{D_2}{19}\int_{m+\frac{3}{2}}^{m+\frac{5}{2}} x^{-19}\,dx \end{aligned}$$

and

$$\begin{aligned} r_3(n) - \gamma &\ge \sum_{m=n}^\infty \frac{D_2}{19}\int_{m+\frac{3}{2}}^{m+\frac{5}{2}} x^{-19}\,dx \\ &= \frac{D_2}{19}\int_{n+\frac{3}{2}}^\infty x^{-19}\,dx = \frac{D_2}{342}\frac{1}{(n+\frac{3}{2})^{18}} = C_3\frac{1}{(n+\frac{3}{2})^{18}}. \end{aligned} \tag{3.11}$$

Combining (3.5) and (3.6) completes the proof of (1.7).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

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