Electronic Transactions on Numerical Analysis.
Volume 16, pp. 165-185, 2003.

ISSN 1068-9613.

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ANALYSIS OF TWO-DIMENSIONAL FETI-DP PRECONDITIONERS BY THE STANDARD ADDITIVE SCHWARZ FRAMEWORK*

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Abstract. FETI-DP preconditioners for two-dimensional elliptic boundary value problems with heterogeneous coefficients are analyzed by the standard additive Schwarz framework. It is shown that the condition number of the preconditioned system for both second order and fourth order problems is bounded by $C\big(1+\ln(H/h)\big)^2$, where H is the maximum of the diameters of the subdomains, h is the mesh size of a quasiuniform triangulation, and the positive constant C is independent of h, H, the number of subdomains and the coefficients of the boundary value problems on the subdomains. The sharpness of the bound for second order problems is also established.

Key words. FETI-DP, additive Schwarz, domain decomposition, heterogeneous coefficients.

AMS subject classifications. 65N55, 65N30.

1. Introduction. The Finite Element Tearing and Interconnecting (FETI) method [14, 20, 21, 19, 32, 18, 15, 34, 35, 1] is a nonoverlapping domain decomposition method that has been implemented for large scale engineering applications. In the FETI approach the system of finite element equations for the nodal variables (primal variables) is enlarged to a system where the nodal variables on any subdomain are independent of the nodal variables on the other subdomains, and the continuity of the finite element functions across the interface of the subdomains is enforced by Lagrange multipliers (dual variables). By eliminating the primal variables, a system of equations for the dual variables is obtained, which can then be solved by the conjugate gradient method. In the case of many subdomains, preconditioning is necessary for fast convergence and FETI preconditioners have been studied in [29, 31, 23, 24, 5, 4].

Recently, a new FETI approach for two-dimensional problems was introduced in [16, 17, 33], where the continuity of the finite element functions at the cross points is retained in the enlarged system and Lagrange multipliers are introduced only to enforce the continuity at the other nodes on the interface of the subdomains. After the primal variables associated with these other nodes and the nodes off the interface have been eliminated, a system involving both primal variables (nodal variables associated with the cross points) and dual variables (Lagrange multipliers associated with the nodes on the interface that are not cross points) is obtained. Two notable features of the FETI-DP approach are: (i) the evaluation of the operator associated with the dual-primal system no longer involves the solutions of singular problems on floating subdomains, and (ii) a coarse problem is built into the evaluation of the reduced operator associated with the dual variables.

Dirichlet preconditioners for the FETI-DP approach were studied in [30] where polylogarithmic bounds for the condition numbers of the preconditioned systems were obtained for second and fourth order problems on two-dimensional domains. The goal of this paper is to give an alternative derivation of the results in [30] and demonstrate that the bound for second order problems is sharp, using the standard additive Schwarz framework [11, 2, 39, 38, 12, 22, 37, 6]. This paper is therefore a continuation of the earlier work [5, 4, 8]. Note that this approach can also be applied to three-dimensional FETI-DP methods [17, 33, 25] and such an analysis of 3D FETI-DP methods will be carried out in a separate paper.

*Received March 17, 2003. Accepted for publication May 3, 2003. Recommended by O. Widlund.

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The rest of the paper is organized as follows. The description of the FETI-DP algorithm for two-dimensional second and fourth order elliptic boundary value problems is given in Section 2, followed by the definition of FETI-DP preconditioners and some preliminary estimates in Section 3. Condition number estimates for the FETI-DP preconditioners are then given in Section 4. The sharpness of the condition number estimate for second order problems is established in Section 5.

2. Two-Dimensional FETI-DP Methods. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain subdivided into nonoverlapping (open) polygonal subdomains $\Omega_1, \ldots, \Omega_J$ (cf. Figure 2.1), with diameters $H_1, \ldots, H_J$. The maximum of $H_1, \ldots, H_J$ will be denoted by H.

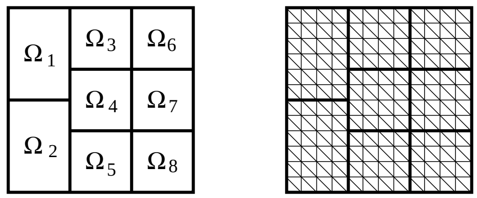

FIG. 2.1. *A nonoverlapping subdivision with an underlying triangulation*

For concreteness we will consider two model problems. We take

$$a_j(v_j, w_j) = \alpha_j \int_{\Omega_j} \nabla v_j \cdot \nabla w_j \, dx \qquad \forall\, v_j, w_j \in H^1(\Omega_j) \tag{2.1}$$

for the second order model problem, and

$$a_j(v_j, w_j) = \alpha_j \int_{\Omega_j} \sum_{1 \le k,l \le 2} \frac{\partial^2 v_j}{\partial x_k \partial x_l} \frac{\partial^2 w_j}{\partial x_k \partial x_l} \, dx \qquad \forall\, v_j, w_j \in H^2(\Omega_j) \tag{2.2}$$

for the fourth order model problem, where the α_j are positive constants.

Let $\mathcal{T}$ be a quasiuniform triangulation of Ω with mesh size h such that each Ω_j is a union of the triangles in $\mathcal{T}$ (cf. Figure 2.1), and $V(\Omega) \subset H_0^m(\Omega)$ be the P_1 finite element space associated with $\mathcal{T}$ when $m = 1$ and the Hsieh-Clough-Tocher macro finite element space (cf. [9]) when $m = 2$. The discrete problem that we want to solve is:

Find $u \in V(\Omega)$ such that

$$a(u, v) = \int_{\Omega} f v \, dx \qquad \forall\, v \in V(\Omega), \tag{2.3}$$

where $f \in L_2(\Omega)$ and

$$a(u, v) = \sum_{j=1}^{J} a_j(u_j, v_j), \tag{2.4}$$

with $u_j = u\big|_{\Omega_j}$ and $v_j = v\big|_{\Omega_j}$.

REMARK 2.1. *The results of this paper also hold for other elements for second and fourth order problems, such as the bilinear finite element and the reduced Hsieh-Clough-Tocher macro element* [10].

The description of the FETI-DP approach requires some terminology from domain decomposition: The set of all the vertices of the polygonal subdomains $\Omega_1, \ldots, \Omega_J$ will be denoted by $\mathcal{V}$. For example, there are 16 such vertices in $\mathcal{V}$ for the domain decomposition depicted in Figure 2.1. The subset of $\mathcal{V}$ consisting of vertices not on $\partial\Omega$ will be denoted by $\mathcal{C}$ (the set of *cross points*). For example, there are 5 cross points on the boundary of the subdomain Ω_4 in Figure 2.1. An open line segment on the boundary of a subdomain between two consecutive vertices in $\mathcal{V}$ will be referred to as an edge of the boundary of the subdomain. For example, there are 5 edges on the boundary of the subdomain Ω_1 in Figure 2.1. The *interface* of the subdomains is the set $\Gamma = \bigcup_{j=1}^{J} \Gamma_j$, where $\Gamma_j = \partial\Omega_j \setminus \partial\Omega$ is the part of $\partial\Omega_j$ disjoint from $\partial\Omega$. The set $\Gamma_j \setminus \mathcal{C}$ will be denoted by Γ'_j and Γ' is the set $\Gamma \setminus \mathcal{C} = \bigcup_{j=1}^{J} \Gamma'_j$.

We will also use $\mathcal{N}_S$ to denote the set of the nodes of the finite element space belonging to the geometric object S and $|G|$ to denote the number of elements in a set G.

Let $V(\Omega_j \cup \Gamma_j)$ be the subdomain finite element space associated with the triangulation on Ω_j induced by $\mathcal{T}$ whose members vanish up to the derivatives of order $m-1$ on $\partial\Omega_j \cap \partial\Omega$. Let V be the subspace of the product space $V(\Omega_1 \cup \Gamma_1) \times \cdots \times V(\Omega_J \cup \Gamma_J)$ defined by

$$V = \{(v_1, \dots, v_J) : v_j \in V(\Omega_j \cup \Gamma_j) \text{ for } 1 \le j \le J \text{ and } v_k = v_\ell \text{ up to the derivatives of order } m-1 \text{ at all the cross points on } \partial\Omega_k \cap \partial\Omega_\ell\}. \tag{2.5}$$

Note that the definition of $a(\cdot,\cdot)$ in (2.4) can be extended to the space V. Furthermore, $a(v,v) = 0$ implies that v_j is a polynomial of degree $\le m-1$ on Ω_j for $1 \le j \le J$. The continuity at the cross points then implies that v is a global polynomial of degree $\le m-1$ on Ω and hence $v = 0$, because of the homogeneous Dirichlet boundary condition(s) on $\partial\Omega$. Therefore, the bilinear form $a(\cdot,\cdot)$ remains symmetric positive definite (SPD) on V.

REMARK 2.2. *Sometimes a cross point is defined to be a point in Ω belonging to the boundaries of at least three subdomains. Let $\tilde{\mathcal{C}}$ be the set of cross points according to this definition. In the case where all the subdomains are convex we have $\tilde{\mathcal{C}} = \mathcal{C}$. But in general $\tilde{\mathcal{C}}$ is a proper subset of $\mathcal{C}$. Note that in the case where a subdomain is surrounded by another, the absence of cross points according to the stricter definition creates a singular problem on the inner subdomain in the FETI-DP approach. This would not happen if the larger set $\mathcal{C}$ is used.*

We can identify the global finite element space $V(\Omega)$ with the subspace of V whose members (or more precisely their components) are continuous up to the derivatives of order $m-1$ along the interface Γ. Moreover, the continuity of the derivatives of the finite element functions on Γ' up to order $m-1$ can be enforced by Lagrange multipliers.

For each $p \in \mathcal{N}_{\Gamma'}$ we introduce the Lagrange multiplier space M_p as follows. Let $p \in \Gamma_k \cap \Gamma_\ell$. In the second order case p is a vertex and M_p is spanned by the linear functional $\mu_{p,k,\ell} \in V'$ defined by

$$\langle \mu_{p,k,\ell}, v\rangle_V = v_\ell(p) - v_k(p) \qquad \forall\, v = (v_1, \cdots, v_J) \in V, \tag{2.6}$$

where $\langle \cdot,\cdot\rangle_V$ represents the canonical bilinear form defined on $V' \times V$. In the fourth order case p is either a vertex or a midpoint. If p is a midpoint, then M_p is spanned by $\mu^n_{p,k,\ell} \in V'$ defined by

$$\langle \mu^n_{p,k,\ell}, v\rangle_V = \frac{\partial v_\ell}{\partial n_\ell}(p) + \frac{\partial v_k}{\partial n_k}(p) \qquad \forall\, v = (v_1, \cdots, v_J) \in V, \tag{2.7}$$

where n_ℓ (respectively n_k) is the outer unit normal of Ω_ℓ (respectively Ω_k). If p is a vertex, then M_p is spanned by $\mu_{p,k,\ell}$, $\mu^n_{p,k,\ell}$, $\mu^t_{p,k,\ell} \in V'$. The linear functionals $\mu_{p,k,\ell}$ and $\mu^n_{p,k,\ell}$ are defined as in (2.6) and (2.7), and $\mu^t_{p,k,\ell}$ is defined by

$$\langle \mu^t_{p,k,\ell}, v\rangle_V = \frac{\partial v_\ell}{\partial t_\ell}(p) + \frac{\partial v_k}{\partial t_k}(p) \qquad \forall\, v = (v_1, \cdots, v_J) \in V, \tag{2.8}$$

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where t_ℓ (respectively t_k) is the unit tangent along $\partial\Omega_k \cap \partial\Omega_\ell$ obtained by rotating the unit normal n_ℓ (respectively n_k) counterclockwise through a right angle.

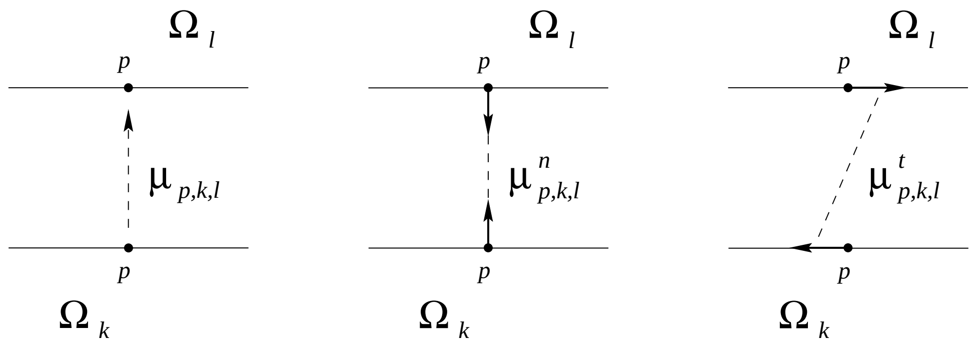

FIG. 2.2. *Lagrange multipliers along the interface of two subdomains*

The three types of Lagrange multipliers are depicted graphically in Figure 2.2. The Lagrange multiplier space M is taken to be

$$M = \bigoplus_{p \in \mathcal{N}_{\Gamma'}} M_p \subset V'.$$

Using M we can characterize the subspace of V corresponding to $V(\Omega)$ as $\{v \in V : \langle \mu, v \rangle_V = 0 \text{ for all } \mu \in M\}$.

The first step in the FETI-DP approach is to replace (2.3) by the following problem:

Find $(\tilde{u}, \phi) \in V \times M$ such that

$$\begin{aligned} \sum_{j=1}^{J} a_j(\tilde{u}_j, v_j) + \langle \phi, v \rangle_V &= \sum_{j=1}^{J} \int_{\Omega_j} f v_j \, dx \qquad \forall\, v \in V, \\ \langle \mu, \tilde{u} \rangle_V &= 0 \qquad\qquad\qquad\qquad \forall\, \mu \in M, \end{aligned} \tag{2.9}$$

where $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_J)$ and $v = (v_1, \ldots, v_J)$. It is easy to check that (2.9) is non-singular and the solution of (2.3) is related to the solution of (2.9) through the relation $\tilde{u} = (u\big|_{\Omega_1}, \ldots, u\big|_{\Omega_J})$.

We can also rewrite (2.9) more concisely as

$$\mathfrak{B}\big((\tilde{u}, \phi), (v, \mu)\big) = \sum_{j=1}^{J} \int_{\Omega_j} f v_j \, dx \qquad \forall\, (v, \mu) \in V \times M, \tag{2.10}$$

where

$$\mathfrak{B}\big((w, \lambda), (v, \mu)\big) = \sum_{j=1}^{J} a_j(w_j, v_j) + \langle \lambda, v \rangle_V + \langle \mu, w \rangle_V \qquad \forall\, (w, \lambda), (v, \mu) \in V \times M.$$

Let $V_{IR} = \{v = (v_1, \ldots, v_J) \in V$: the nodal variables of v_j, $1 \le j \le J$, vanish at all the nodes in $\mathcal{C} \cap \partial\Omega_j\}$ (i.e., v_j vanishes at the cross points on $\partial\Omega_j$ for the P_1 finite element and v_j vanishes up to the first order derivatives at the cross points on $\partial\Omega_j$ for the Hsieh-Clough-Tocher macro element). The space $V \times M$ admits the decomposition

$$V \times M = \mathcal{W} \oplus \mathcal{W}^\circ,$$

where $\mathcal{W} = V_{IR} \times \{0\}$ and $\mathcal{W}^\circ = \{(v,\mu) \in V \times M : \mathfrak{B}\big((v,\mu),(w,0)\big) = 0$ for all $w \in V_{IR}\}$. Indeed, since the bilinear form $\mathfrak{B}(\cdot,\cdot)$ is nonsingular, we have

$$\dim(V \times M) = \dim \mathcal{W} + \dim \mathcal{W}^\circ.$$

Moreover, if $(v,0) \in \mathcal{W} \cap \mathcal{W}^\circ$, then

$$0 = \mathfrak{B}((v,0),(v,0)) = a(v,v),$$

which implies $v = 0$, because $a(\cdot,\cdot)$ is SPD on V.

We can therefore write

$$(\tilde{u},\phi) = (\tilde{u}^{ir},0) + (\tilde{u}^\circ,\phi), \tag{2.11}$$

where $\tilde{u}^{ir} \in V_{IR}$ and $(\tilde{u}^\circ,\phi) \in \mathcal{W}^\circ$.

The second step in the FETI-DP approach is to reduce (2.10) to the following *dual-primal problem*:

Find $(\tilde{u}^\circ,\phi) \in \mathcal{W}^\circ$ such that

$$\mathfrak{B}\big((\tilde{u}^\circ,\phi),(v^\circ,\mu)\big) = \sum_{j=1}^{J} \int_{\Omega_j} f v_j^\circ \, dx \qquad \forall\, (v^\circ,\mu) \in \mathcal{W}^\circ. \tag{2.12}$$

REMARK 2.3. *Since $\tilde{u}^{ir}$ in (2.11) is determined by*

$$\sum_{j=1}^{J} a_j(\tilde{u}_j^{ir}, v_j^{ir}) = \sum_{j=1}^{J} \int_{\Omega_j} f v_j^{ir} \, dx \qquad \forall\, v^{ir} \in V_{IR}$$

and the components of u^{ir} are independent of one another, the reduction in the second step of the FETI-DP approach involves solving SPD problems on the subdomains in parallel.

Let $V_{\mathcal{C}} \subset V$ be the orthogonal complement of V_{IR} with respect to $a(\cdot,\cdot)$. Note that

$$V = V_{IR} \oplus V_{\mathcal{C}}, \tag{2.13}$$

$V_{\mathcal{C}} \times \{0\} \subset \mathcal{W}^\circ$ and $\mathcal{W}^\circ$ has the decomposition

$$\mathcal{W}^\circ = \big(V_{\mathcal{C}} \times \{0\}\big) \oplus \big(V \times \{0\}\big)^\circ,$$

where $\big(V \times \{0\}\big)^\circ = \{(w,\lambda) \in V \times M : \mathfrak{B}\big((w,\lambda),(v,0)\big) = 0$ for all $v \in V\}$. Note also that $(w,\lambda) \in \big(V \times \{0\}\big)^\circ$ is completely determined by λ and therefore can be written as $(T\lambda,\lambda)$, where T is the linear map from M to V defined by

$$\mathfrak{B}\big((T\lambda,\lambda),(v,0)\big) = 0 \qquad \forall\, v \in V. \tag{2.14}$$

Hence, we can write

$$(\tilde{u}^\circ,\phi) = (\tilde{u}^c,0) + (T\phi,\phi), \tag{2.15}$$

where $\tilde{u}^c \in V_{\mathcal{C}}$.

In the final step of the FETI-DP approach the dual-primal problem (2.12) is further reduced to the following problem:

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Find $\phi \in M$ such that

$$\mathfrak{B}\big((T\phi,\phi),(T\mu,\mu)\big) = \sum_{j=1}^{J}\int_{\Omega_j} f(T\mu)_j\,dx \qquad \forall\,\mu\in M. \tag{2.16}$$

The role of a FETI-DP preconditioner is to improve the conditioning of the system (2.16) so that it can be solved efficiently by a preconditioned conjugate gradient method.

REMARK 2.4. *Since $\tilde u^c$ in (2.15) is determined by*

$$\sum_{j=1}^{J} a(\tilde u_j^c, v_j^c) = \sum_{j=1}^{J}\int_{\Omega_j} f v_j^c\,dx \qquad \forall\,v^c\in V_{\mathcal{C}},$$

the reduction in the final step of the FETI-DP approach involves solving a SPD coarse problem whose dimension is

$$\dim V_{\mathcal{C}} = \dim V - \dim V_{IR} = \begin{cases} |\mathcal{C}| & \text{for the } P_1 \text{ finite element,}\\ 3|\mathcal{C}| & \text{for the Hsieh-Clough-Tocher macro element.}\end{cases}$$

The description above of the FETI-DP approach follows the actual solution process given in [16, 17]. It shows that the evaluation of the operator defined by (2.16) on a given $\mu\in M$ involves solving SPD problems on the subdomains in parallel and also solving a SPD coarse problem that provides global communication among the subdomains. But the analysis of FETI-DP preconditioners (cf. [30]) is best carried out through an alternative formulation of (2.16) that is based on a decomposition of V different from (2.13).

Let $V_I = \{v=(v_1,\ldots,v_J)\in V : v_j, 1\le j\le J$, vanishes up to the derivatives of order $(m-1)$ on $\partial\Omega_j\}$. The space V admits the decomposition

$$V = V_I \oplus V_\Gamma,$$

where V_Γ is the orthogonal complement of V_I with respect to $a(\cdot,\cdot)$, i.e.,

$$V_\Gamma = \{v\in V : a(v,w)=0 \quad \forall\,w\in V_I\}. \tag{2.17}$$

Note that $V_\Gamma\times M = \big(V_I\times\{0\}\big)^\circ = \{(w,\lambda)\in V\times M : \mathfrak{B}\big((w,\lambda),(v,0)\big) = 0$ for all $v\in V_I\}$.

REMARK 2.5. *For $v^\Gamma\in V_\Gamma$, the nodal variables of v_j^Γ can take arbitrary values along $\mathcal{N}_{\Gamma'}$ and share common (but arbitrary) values at the cross points. The rest of the nodal variables (at the nodes in $\mathcal{N}_{\Omega_1\cup\cdots\cup\Omega_J}$) are then determined by the $a(\cdot,\cdot)$ orthogonality of V_Γ to V_I. In particular, v_j^Γ is a discrete harmonic function on Ω_j in the second order case and a discrete biharmonic function in the fourth order case.*

REMARK 2.6. *Since $\langle\mu,v\rangle_V = 0$ for all $v\in V_I$ and $\mu\in M$, we will treat M as a subspace of V_Γ'.*

We can write

$$\tilde u = \tilde u^I + \tilde u^\Gamma,$$

where $\tilde u^I\in V_I$ and $\tilde u^\Gamma\in V_\Gamma$, and reduce (2.9) to the following problem:

Find $(\tilde{u}^\Gamma,\phi)\in V_\Gamma\times M$ such that

$$\begin{aligned}(2.18)\qquad \sum_{j=1}^J a_j(\tilde{u}_j^\Gamma,v_j^\Gamma)+\langle\phi,v^\Gamma\rangle_\Gamma &= \sum_{j=1}^J\int_{\Omega_j} f v_j^\Gamma\,dx \qquad \forall\, v^\Gamma\in V_\Gamma,\\ \langle\mu,\tilde{u}^\Gamma\rangle_\Gamma &= 0 \qquad \forall\,\mu\in M,\end{aligned}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ is the canonical bilinear form defined on $V_\Gamma'\times V_\Gamma$.

Let $S: V_\Gamma\longrightarrow V_\Gamma'$ be defined by

$$(2.19)\qquad \langle Sv^\Gamma,w^\Gamma\rangle_\Gamma = a(v^\Gamma,w^\Gamma)=\sum_{j=1}^J a_j(v_j^\Gamma,w_j^\Gamma)$$

for all $v^\Gamma=(v_1^\Gamma,\dots,v_J^\Gamma), w^\Gamma=(w_1^\Gamma,\dots,w_J^\Gamma)\in V_\Gamma$, and $\gamma_{_f}\in V_\Gamma'$ be defined by

$$(2.20)\qquad \langle\gamma_{_f},v^\Gamma\rangle_\Gamma=\sum_{j=1}^J\int_{\Omega_j} f v_j^\Gamma\,dx \qquad \forall\, v^\Gamma=(v_1^\Gamma,\dots,v_J^\Gamma)\in V_\Gamma.$$

We can now rewrite (2.18) as

$$\begin{aligned}(2.21)\qquad \langle S\tilde{u}^\Gamma,v^\Gamma\rangle_\Gamma+\langle\phi,v^\Gamma\rangle_\Gamma &= \langle\gamma_{_f},v^\Gamma\rangle_\Gamma \qquad \forall\, v^\Gamma\in V_\Gamma,\\ \langle\mu,\tilde{u}^\Gamma\rangle_\Gamma &= 0 \qquad \forall\,\mu\in M.\end{aligned}$$

Note that the operator S is SPD, i.e.,

$$\begin{aligned}(2.22)\qquad \langle Sv^\Gamma,w^\Gamma\rangle_\Gamma &= \langle Sw^\Gamma,v^\Gamma\rangle_\Gamma \qquad \forall\, v^\Gamma,w^\Gamma\in V_\Gamma,\\ \langle Sv^\Gamma,v^\Gamma\rangle_\Gamma &> 0 \qquad \forall\, v^\Gamma\in V_\Gamma\setminus\{0\}.\end{aligned}$$

From (2.21) we obtain the following problem for ϕ:

Find $\phi\in M$ such that

$$(2.23)\qquad \langle\mu,S^{-1}\phi\rangle_\Gamma=\langle\mu,S^{-1}\gamma_{_f}\rangle_\Gamma \qquad \forall\,\mu\in M.$$

The two problems (2.16) and (2.23) are identical since they both come from (2.9) by eliminating $\tilde{u}$. Indeed, we have, by (2.14), (2.17), (2.19), (2.20) and (2.22),

$$\begin{aligned} T\mu &= -S^{-1}\mu \qquad &\forall\,\mu\in M,\\ \mathfrak{B}\big((T\phi,\phi),(T\mu,\mu)\big) &= -\langle\mu,S^{-1}\phi\rangle_\Gamma \qquad &\forall\,\mu\in M,\\ \sum_{j=1}^J\int_{\Omega_j} f(T\mu)_j\,dx &= -\langle\mu,S^{-1}\gamma_f\rangle_\Gamma \qquad &\forall\,\mu\in M.\end{aligned}$$

Our analysis of FETI-DP preconditioners will be based on the formulation (2.23).

3. FETI-DP Preconditioners and Preliminary Estimates. Let $\hat{\mathbb{S}}: M\longrightarrow M'$ be defined by

$$(3.1)\qquad \langle\mu_1,\hat{\mathbb{S}}\,\mu_2\rangle_M=\langle\mu_1,S^{-1}\mu_2\rangle_\Gamma \qquad \forall\,\mu_1,\mu_2\in M,$$

where $\langle\cdot,\cdot\rangle_M$ is the canonical bilinear form defined on $(M')'\times M'=M\times M'$. It follows from (2.22) that $\hat{\mathbb{S}}$ is SPD, i.e.,

$$\langle\mu_1,\hat{\mathbb{S}}\,\mu_2\rangle_M=\langle\mu_2,\hat{\mathbb{S}}\,\mu_1\rangle_M \qquad \forall\,\mu_1,\mu_2\in M,$$

$$\langle \mu, \hat{\mathbb{S}}\, \mu \rangle_M > 0 \qquad\qquad \forall\, \mu \in M \setminus \{0\}.$$

We see from (2.23) that $\hat{\mathbb{S}}$ is the operator that needs to be preconditioned in the FETI-DP approach. The preconditioner for $\hat{\mathbb{S}}$ will be constructed using Schur complement operators associated with the subdomains.

Let $V_j \subset V(\Omega_j \cup \Gamma_j)$ be the space of discrete harmonic functions ($m = 1$) or the space of discrete biharmonic functions ($m = 2$) that vanish up to the derivatives of order $m - 1$ at the cross points on $\partial\Omega_j$. In order words, $v_j \in V_j$ is characterized by the following conditions:

(3.2) $\qquad a_j(v_j, w_j) = 0 \quad \forall\, w_j \in V(\Omega_j \cup \Gamma_j) \cap H_0^m(\Omega_j),$

(3.3) $\qquad v_j$ vanishes up to the derivatives of order $m - 1$ at every $p \in \mathcal{C} \cap \mathcal{N}_{\Gamma_j}$.

Note that

$$V_1 \times \cdots \times V_J \subset V_\Gamma. \tag{3.4}$$

The Schur complement operator $S_j : V_j \longrightarrow V_j'$ is defined by

$$\langle S_j v_j, w_j \rangle_j = a_j(v_j, w_j) \qquad \forall\, v_j, w_j \in V_j, \tag{3.5}$$

where $\langle \cdot, \cdot \rangle_j$ is the canonical bilinear form on $V_j' \times V_j$. It is clear that S_j is SPD.

In order to define the FETI-DP preconditioner we need connection maps

$$I_j : V_j' \longrightarrow M.$$

We will treat the second order case and fourth order case separately.

Let $p \in \mathcal{N}_{\Gamma'}$. Then p belongs to the common boundary of two subdomains Ω_k and Ω_ℓ. We define the function $\chi_p : \{k, \ell\} \longrightarrow \{k, \ell\}$ by

$$\chi_p(k) = \ell \quad \text{and} \quad \chi_p(\ell) = k. \tag{3.6}$$

For the second order model problem we define, for arbitrary $\psi_j \in V_j'$,

$$I_j \psi_j = \sum_{p \in \mathcal{N}_{\Gamma_j'}} \frac{\alpha_{\chi_p(j)}}{\alpha_j + \alpha_{\chi_p(j)}} \langle \psi_j, \delta_{p,j} \rangle_j \mu_{p,\chi_p(j),j}, \tag{3.7}$$

where the α_j's are the coefficients appearing in (2.1) and $\delta_{p,j} \in V_j$ satisfies

$$\delta_{p,j}(q) = \begin{cases} 1 & q = p, \\ 0 & q \in \mathcal{N}_{\Gamma_j'} \setminus \{p\}. \end{cases} \tag{3.8}$$

REMARK 3.1. *The scaling $\alpha_{\chi_p(j)}/(\alpha_j + \alpha_{\chi_p(j)})$ in (3.7) and (3.9) below enables us to obtain an estimate for $\lambda_{\max}(\mathbb{B}\hat{\mathbb{S}})$ that is independent of the α_j's. This technique is well-known in the literature (cf. [36, 28, 13, 35, 34, 24, 25, 4, 5]). In fact, the results in this paper remain valid if α_j (respectively $\alpha_{\chi_p(j)}$) in (3.7) and (3.9) is replaced by α_j^t (respectively $\alpha_{\chi_p(j)}^t$) for any $t \geq 1/2$. We choose $t = 1$ in (3.7) and (3.9) for simplicity.*

The definition of I_j for the fourth order model problem follows the same principle. First we introduce the discrete biharmonic functions $\delta_{p,j}$, $\delta_{p,j}^n$, $\delta_{p,j}^t$ and $\delta_{m,j}^n$. Note that a function in V_j is determined by its values and the values of its normal and tangential derivatives at the vertices of $\mathcal{T}$ on Γ', and also the values of its normal derivatives at the midpoints of the edges of $\mathcal{T}$ on Γ'. For a vertex $p \in \mathcal{N}_{\Gamma'}$, we define (i) $\delta_{p,j} \in V_j$ to be the function that

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takes the value 1 at p and takes the value zero for all other nodal variables, (ii) $\delta^n_{p,j} \in V_j$ to be the function whose normal derivative at p is 1 and takes the value zero for all other nodal variables, and (iii) $\delta^t_{p,j} \in V_j$ to be the function whose tangential derivative at p is 1 and takes the value zero for all other nodal variables. For a midpoint $m \in \Gamma'_j$, we define $\delta^n_{m,j} \in V_j$ to be the function whose normal derivative at m is 1 and takes the value zero for all other nodal variables.

We can now define, for arbitrary $\psi_j \in V'_j$,

$$
\begin{aligned}
I_j\psi_j = \sum_{p\in\mathcal{N}_{\Gamma',1}} \frac{\alpha_{\chi_p(j)}}{\alpha_j + \alpha_{\chi_p(j)}} \big[\langle\psi_j, \delta_{p,j}\rangle_j \mu_{p,\chi_p(j),j} + \langle\psi_j, \delta^n_{p,j}\rangle_j \mu^n_{p,\chi_p(j),j} \\
+ \langle\psi_j, \delta^t_{p,j}\rangle_j \mu^t_{p,\chi_p(j),j}\big] \\
+ \sum_{m\in\mathcal{N}_{\Gamma',2}} \frac{\alpha_{\chi_p(j)}}{\alpha_j + \alpha_{\chi_p(j)}} \langle\psi_j, \delta^n_{m,j}\rangle_j \mu^n_{m,\chi_p(j),j},
\end{aligned}
\tag{3.9}
$$

where $\mathcal{N}_{\Gamma'_j,1}$ is the set of the vertices of the triangulation $\mathcal{T}$ on Γ'_j and $\mathcal{N}_{\Gamma'_j,2}$ is the set of the midpoints of the edges of $\mathcal{T}$ on Γ'_j.

Let $E_j : V_j \longrightarrow V_1 \times \cdots \times V_J \subset V_\Gamma$ be the embedding map defined by

$$
(E_j w)_k = \begin{cases} w & k = j \\ 0 & k \neq j \end{cases} \qquad \forall\, w \in V_j, \tag{3.10}
$$

and $R_j : M \longrightarrow V'_j$ be the restriction map defined by

$$
\langle R_j\mu, w\rangle_j = \langle \mu, E_j w\rangle_\Gamma \qquad \forall\, w \in V_j. \tag{3.11}
$$

Then the maps R_j and I_j form a partition of unity on M:

$$
\sum_{j=1}^{J} I_j R_j \mu = \mu \qquad \forall\, \mu \in M, \tag{3.12}
$$

which can be easily verified using (2.6)–(2.8), (3.7) and (3.9).

The FETI-DP preconditioner $\mathbb{B} : M' \longrightarrow M$ is given by the formula

$$
\mathbb{B} = \sum_{j=1}^{J} I_j S_j I_j^t, \tag{3.13}
$$

where $I_j^t : M' \longrightarrow V_j$ is the transpose of I_j with respect to $\langle\cdot,\cdot\rangle_j$ and $\langle\cdot,\cdot\rangle_M$.

Our analysis of the operator $\mathbb{B}\hat{\mathbb{S}} : M \longrightarrow M$ is based on the following well-known characterizations of $\lambda_{\min}(\mathbb{B}\hat{\mathbb{S}})$ and $\lambda_{\max}(\mathbb{B}\hat{\mathbb{S}})$ from the additive Schwarz theory [11, 2, 39, 38, 12, 22, 6]:

$$
\lambda_{\min}(\mathbb{B}\hat{\mathbb{S}}) = \min_{0\neq\mu\in M} \frac{\langle \mu, \hat{\mathbb{S}}\,\mu\rangle_M}{\displaystyle\min_{\substack{\mu=\sum_{j=1}^J I_j\psi_j \\ \psi_j\in V'_j}} \sum_{j=1}^{J} \langle\psi_j, S_j^{-1}\psi_j\rangle_j}, \tag{3.14}
$$

$$
\lambda_{\max}(\mathbb{B}\hat{\mathbb{S}}) = \max_{0\neq\mu\in M} \frac{\langle \mu, \hat{\mathbb{S}}\,\mu\rangle_M}{\displaystyle\min_{\substack{\mu=\sum_{j=1}^J I_j\psi_j \\ \psi_j\in V'_j}} \sum_{j=1}^{J} \langle\psi_j, S_j^{-1}\psi_j\rangle_j}. \tag{3.15}
$$

In the rest of this section we provide characterizations of $\langle\mu,\hat{\mathbb{S}}\,\mu\rangle_M$ and $\langle\psi_j,S_j^{-1}\psi_j\rangle_j$, and derive a lower bound for $\lambda_{\min}(\mathbb{B}\hat{\mathbb{S}})$.

LEMMA 3.2. *Given any $\mu\in M$, we have*

$$-\langle\mu,\hat{\mathbb{S}}\,\mu\rangle_M=\min_{v^\Gamma\in V_\Gamma}\Big[a(v^\Gamma,v^\Gamma)+2\langle\mu,v^\Gamma\rangle_\Gamma\Big]. \tag{3.16}$$

Proof. We can, by (2.19), rewrite the right-hand side of (3.16) as

$$\min_{v\in V_\Gamma}\big[\langle Sv^\Gamma,v^\Gamma\rangle_\Gamma+2\langle\mu,v^\Gamma\rangle_\Gamma\big].$$

Therefore the minimum occurs at $v^\Gamma=-S^{-1}\mu$ and the value of the minimum is, by (3.1),

$$\langle SS^{-1}\mu,S^{-1}\mu\rangle_\Gamma-2\langle\mu,S^{-1}\mu\rangle_\Gamma=-\langle\mu,S^{-1}\mu\rangle_\Gamma=-\langle\mu,\hat{\mathbb{S}}\,\mu\rangle_M.$$

□

The proof of the following lemma is similar.

LEMMA 3.3. *Given any $\psi_j\in V_j'$, we have*

$$-\langle\psi_j,S_j^{-1}\psi_j\rangle_j=\min_{v_j\in V_j}\big[a_j(v_j,v_j)+2\langle\psi_j,v_j\rangle_j\big]. \tag{3.17}$$

A lower bound for $\lambda_{\min}(\mathbb{B}\hat{\mathbb{S}})$ can be derived as a corollary of Lemma 3.2 and Lemma 3.3.

LEMMA 3.4. *For both the second order model problem and the fourth order model problem, it holds that*

$$\lambda_{\min}(\mathbb{B}\hat{\mathbb{S}})\geq 1. \tag{3.18}$$

Proof. Let $\mu\in M$ be arbitrary and $\psi_j=R_j\mu\in V_j'$. It follows from (3.12) that

$$\mu=\sum_{j=1}^J I_j\psi_j. \tag{3.19}$$

Let $v_j\in V_j$ be arbitrary. Then $v^\Gamma=(v_1,\dots,v_J)=\sum_{j=1}^J E_jv_j\in V_\Gamma$ (cf. (3.4) and (3.10)), and in view of (3.11),

$$\langle\mu,v^\Gamma\rangle_\Gamma=\langle\mu,\sum_{j=1}^J E_jv_j\rangle_\Gamma=\sum_{j=1}^J\langle R_j\mu,v_j\rangle_j=\sum_{j=1}^J\langle\psi_j,v_j\rangle_j,$$
$$\sum_{j=1}^J\big[a_j(v_j,v_j)+2\langle\psi_j,v_j\rangle_j\big]=a(v^\Gamma,v^\Gamma)+2\langle\mu,v^\Gamma\rangle_\Gamma.$$

It follows that

$$\sum_{j=1}^J\min_{v_j\in V_j}\big[a_j(v_j,v_j)+2\langle\psi_j,v_j\rangle_j\big]\geq\min_{v^\Gamma\in V_\Gamma}\Big[a(v^\Gamma,v^\Gamma)+2\langle\mu,v^\Gamma\rangle_\Gamma\Big]. \tag{3.20}$$

We deduce from (3.16), (3.17) and (3.20) that

$$-\sum_{j=1}^{J}\langle\psi_j, S_j^{-1}\psi_j\rangle_j \geq -\langle\mu, \hat{\mathbb{S}}\,\mu\rangle_M. \tag{3.21}$$

The estimate (3.18) follows from (3.14), (3.19) and (3.21). □

The estimates for $\lambda_{\max}(\mathbb{B}\hat{\mathbb{S}})$ in the next section requires another characterization of $\langle\mu, \hat{\mathbb{S}}\,\mu\rangle_M^{1/2}$ and $\langle\psi_j, S_j^{-1}\psi_j\rangle_j^{1/2}$ as dual norms.

LEMMA 3.5. *Given any $\mu \in M$, we have*

$$\langle\mu, \hat{\mathbb{S}}\,\mu\rangle_M^{1/2} = \max_{\substack{v^\Gamma\in V_\Gamma\\ v^\Gamma\neq 0}} \frac{\langle\mu, v^\Gamma\rangle_\Gamma}{|v^\Gamma|_a}, \tag{3.22}$$

where

$$|v^\Gamma|_a = a(v^\Gamma, v^\Gamma)^{1/2} = \Big(\sum_{j=1}^{J} a_j(v_j^\Gamma, v_j^\Gamma)\Big)^{1/2}. \tag{3.23}$$

Proof. Let $v_\mu^\Gamma = S^{-1}\mu$. From (2.19), (3.1), (3.23) and a standard duality formula we have

$$\langle\mu, \hat{\mathbb{S}}\,\mu\rangle_M^{1/2} = \langle\mu, S^{-1}\mu\rangle_\Gamma^{1/2} = \langle Sv_\mu^\Gamma, v_\mu^\Gamma\rangle_\Gamma^{1/2} = \max_{\substack{v^\Gamma\in V_\Gamma\\ v^\Gamma\neq 0}} \frac{\langle Sv_\mu^\Gamma, v^\Gamma\rangle_\Gamma}{\langle Sv^\Gamma, v^\Gamma\rangle_\Gamma^{1/2}} = \max_{\substack{v^\Gamma\in V_\Gamma\\ v^\Gamma\neq 0}} \frac{\langle\mu, v^\Gamma\rangle_\Gamma}{|v^\Gamma|_a}.$$

□

The proof of the following lemma is similar.

LEMMA 3.6. *Given any $\psi_j \in V_j'$, we have*

$$\langle\psi_j, S_j^{-1}\psi_j\rangle_j^{1/2} = \max_{\substack{v_j\in V_j\\ v_j\neq 0}} \frac{\langle\psi_j, v_j\rangle_j}{|v_j|_{a_j}}, \tag{3.24}$$

where

$$|v_j|_{a_j} = a_j(v_j, v_j)^{1/2}. \tag{3.25}$$

4. Condition Number Estimates.

We first derive an upper bound for $\lambda_{\max}(\mathbb{B}\hat{\mathbb{S}})$ for the second order model problem.

In order to avoid the proliferation of constants, from here on we use $A \lesssim B$ (or $B \gtrsim A$) to represent the inequality $A \leq \text{constant} \times B$, where the constant is positive and independent of h, H and J. The statement $A \approx B$ is equivalent to $A \lesssim B$ and $A \gtrsim B$.

Let $\mu = \sum_{j=1}^{J}\psi_j$, where $\psi_j \in V_j'$ for $1 \leq j \leq J$. In order to apply (3.15) we need to bound $\langle\mu, \hat{\mathbb{S}}\,\mu\rangle_M$ in terms of $\sum_{j=1}^{J}\langle\psi_j, S_j^{-1}\psi_j\rangle_j$. This will be accomplished by exploiting the characterizations (3.22), (3.24) and well-known estimates for discrete harmonic functions.

Let $v^\Gamma = (v_1^\Gamma, \ldots, v_J^\Gamma) \in V_\Gamma$ be arbitrary and $v^H = (v_1^H, \ldots, v_J^H)$, where v_j^H is the discrete harmonic function on Ω_j with the following properties:

$$v_j^H(p) = v_j^\Gamma(p) \quad \forall\, p \in \partial\Omega_j \cap \mathcal{V} \quad \text{and} \quad v^H \text{ is piecewise linear on } \partial\Omega_j,$$

i.e., v_j^H agrees with v_j^Γ at all the vertices in $\partial\Omega_j \cap \mathcal{V}$ and is linear on the edge between any two consecutive vertices.

Our first observation is that

$$v_j = v_j^\Gamma - v_j^H \in V_j, \tag{4.1}$$

and

$$\langle \mu, v^\Gamma \rangle_\Gamma = \langle \mu, v^\Gamma - v^H \rangle_\Gamma = \langle \mu, \sum_{k=1}^{J} E_k v_k \rangle_\Gamma = \sum_{j,k=1}^{J} \langle I_j \psi_j, E_k v_k \rangle_\Gamma, \tag{4.2}$$

where we have used the fact that the components of v^H are continuous across the interface of the subdomains and therefore $\langle \mu, v^H \rangle_\Gamma = 0$.

REMARK 4.1. *The relation $\langle \mu, v^\Gamma \rangle_\Gamma = \langle \mu, v^\Gamma - v^H \rangle_\Gamma$ allows us to use well-known estimates for the BPS preconditioner* [3] *in the study of the FETI-DP preconditioner. In this sense the FETI-DP algorithm is dual to the iterative substructuring algorithm while the original FETI algorithm is dual to the balancing domain decomposition algorithm* [27].

Our second observation is that

$$\langle I_j \psi_j, E_k v_k \rangle_\Gamma = 0 \quad \text{unless } \Gamma_j \text{ and } \Gamma_k \text{ share a common edge.} \tag{4.3}$$

Let e be an edge on Γ_k. We will denote by $v_{k,e}$ the discrete harmonic function on Ω_k that agrees with v_k on e and vanishes on $\partial\Omega_k \setminus e$, i.e.,

$$v_{k,e} = \sum_{p \in \mathcal{N}_e} v_k(p) \delta_{p,k}. \tag{4.4}$$

Clearly, we have

$$v_k = \sum_{e \in \mathcal{E}_k} v_{k,e}, \tag{4.5}$$

where $\mathcal{E}_k$ is the set of the edges on Γ_k.

Let $e \in \mathcal{E}(\Gamma)$, the set of edges on Γ. We denote by σ_e the set of the indices of the two subdomains sharing e. From (4.2)–(4.5), we have

$$\langle \mu, v^\Gamma \rangle_\Gamma = \sum_{e \in \mathcal{E}(\Gamma)} \sum_{j,k \in \sigma_e} \langle I_j \psi_j, E_k v_{k,e} \rangle_\Gamma. \tag{4.6}$$

Note that the inner sum on the right-hand side of (4.6) is given by

$$\langle I_j \psi_j, E_j v_{j,e} + E_k v_{k,e} \rangle_\Gamma + \langle I_k \psi_k, E_k v_{k,e} + E_j v_{j,e} \rangle_\Gamma,$$

where $\{j, k\} = \sigma_e$. Below we will focus on the term $\langle I_j \psi_j, E_j v_{j,e} + E_k v_{k,e} \rangle_\Gamma$, where e is a common edge of Γ_j and Γ_k (cf. Figure 4.1), since a similar result also holds for the term $\langle I_k \psi_k, E_k v_{k,e} + E_j v_{j,e} \rangle_\Gamma$.

We have, by (2.6), (3.7), (3.10) and (4.4),

$$\begin{aligned}
\langle I_j \psi_j, E_j v_{j,e} \rangle_\Gamma &= \frac{\alpha_k}{\alpha_j + \alpha_k} \sum_{p \in \mathcal{N}_e} \langle \psi_j, \delta_{p,j} \rangle_j \langle \mu_{p,k,j}, E_j v_{j,e} \rangle_V \\
&= \frac{\alpha_k}{\alpha_j + \alpha_k} \langle \psi_j, \sum_{p \in \mathcal{N}_e} v_{j,e}(p) \delta_{p,j} \rangle_j = \frac{\alpha_k}{\alpha_j + \alpha_k} \langle \psi_j, v_{j,e} \rangle_j,
\end{aligned}$$

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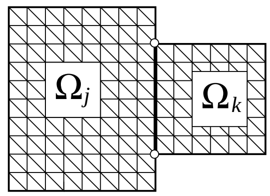

FIG. 4.1. *A common edge of* Γ_j *and* Γ_k

and hence it follows from (3.24) that

$$\langle I_j\psi_j, E_jv_{j,e}\rangle_\Gamma \le \langle \psi_j, S_j^{-1}\psi_j\rangle_j^{1/2}|v_{j,e}|_{a_j}. \tag{4.7}$$

Let β_j be the mean of v_j^Γ over Ω_j, i.e.,

$$\beta_j = \frac{1}{|\Omega_j|}\int_{\Omega_j} v_j^\Gamma\,dx. \tag{4.8}$$

We have by scaling

$$|v_j^H - \beta_j|_{H^{1/2}(e)} \lesssim \|v_j^H - \beta_j\|_{L_\infty(e)}, \tag{4.9}$$

and it follows easily from the definition of v^H that

$$\|v_j^H - \beta_j\|_{L_\infty(e)} \le \|v_j^\Gamma - \beta_j\|_{L_\infty(\Omega_j)}. \tag{4.10}$$

We can then apply the discrete Sobolev inequality, the Poincaré-Friedrichs inequality, (4.1), (4.8)–(4.10), and well-known estimates from the theory of nonoverlapping domain decomposition (cf. [3, 6]) to obtain

$$\begin{aligned}
|v_{j,e}|_{a_j} &= \alpha_j^{1/2}|v_{j,e}|_{H^1(\Omega_j)}\\
&\approx \alpha_j^{1/2}|v_{j,e}|_{H^{1/2}_{00}(e)}\\
&\lesssim \alpha_j^{1/2}\Big[|v_{j,e}|_{H^{1/2}(e)} + \big(1+\ln(H_j/h)\big)^{1/2}\|v_{j,e}\|_{L_\infty(e)}\Big]\\
&\lesssim \alpha_j^{1/2}\Big[|(v_j^\Gamma - \beta_j) - (v_j^H - \beta_j)|_{H^{1/2}(e)}\\
&\qquad + \big(1+\ln(H_j/h)\big)^{1/2}\|(v_j^\Gamma - \beta_j) - (v_j^H - \beta_j)\|_{L_\infty(e)}\Big]\\
&\lesssim \alpha_j^{1/2}\Big[|v_j^\Gamma|_{H^1(\Omega_j)} + \big(1+\ln(H_j/h)\big)^{1/2}\|v_j^\Gamma - \beta_j\|_{L_\infty(\Omega_j)}\Big]\\
&\lesssim \alpha_j^{1/2}\Big[|v_j^\Gamma|_{H^1(\Omega_j)} + \big(1+\ln(H_j/h)\big)\big(H_j^{-1}\|v_j^\Gamma - \beta_j\|_{L_2(\Omega_j)} + |v_j^\Gamma|_{H^1(\Omega_j)}\big)\Big]\\
&\lesssim \big(1+\ln(H_j/h)\big)\alpha_j^{1/2}|v_j^\Gamma|_{H^1(\Omega_j)} = \big(1+\ln(H_j/h)\big)|v_j^\Gamma|_{a_j}.
\end{aligned} \tag{4.11}$$

Let $\tilde v_{j,e}$ be the discrete harmonic function on Ω_j that agrees with $v_{k,e}$ at the nodes in $\mathcal{N}_e$ and vanishes at the other nodes of $\partial\Omega_j$. Then we have, by (3.7) and (3.10),

$$\langle I_j\psi_j, E_kv_{k,e}\rangle_\Gamma = \frac{\alpha_k}{\alpha_j+\alpha_k}\sum_{p\in\mathcal{N}_e}\langle\psi_j,\delta_{p,j}\rangle_j\langle\mu_{p,k,j}, E_kv_{k,e}\rangle_V$$

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$$= -\frac{\alpha_k}{\alpha_j+\alpha_k}\langle \psi_j, \sum_{p\in\mathcal{N}_e} v_{k,e}(p)\delta_{p,j}\rangle_j = -\frac{\alpha_k}{\alpha_j+\alpha_k}\langle \psi_j, \tilde v_{j,e}\rangle_j,$$

and hence it follows from (3.24) that

$$\begin{aligned} \langle I_j\psi_j, E_k v_{k,e}\rangle_\Gamma &\leq \frac{\alpha_k}{\alpha_j+\alpha_k}\langle \psi_j, S_j^{-1}\psi_j\rangle_j^{1/2}|\tilde v_{j,e}|_{a_j} \\ &\leq \frac{1}{2}\langle \psi_j, S_j^{-1}\psi_j\rangle_j^{1/2}(\alpha_k\alpha_j^{-1})^{1/2}|\tilde v_{j,e}|_{a_j}. \end{aligned} \tag{4.12}$$

Moreover, the analog of (4.11) for $v_{k,e}$ implies that

$$\begin{aligned} (\alpha_k\alpha_j^{-1})^{1/2}|\tilde v_{j,e}|_{a_j} &\lesssim \alpha_k^{1/2}|\tilde v_{j,e}|_{H_{00}^{1/2}(e)} \\ &= \alpha_k^{1/2}|v_{k,e}|_{H_{00}^{1/2}(e)} \lesssim \big(1+\ln(H_k/h)\big)|v_k^\Gamma|_{a_k}. \end{aligned} \tag{4.13}$$

We then obtain from (4.6), (4.7) and (4.11)–(4.13) the estimate

$$\begin{aligned} \langle \mu, v^\Gamma\rangle_\Gamma &\lesssim \big(1+\ln(H/h)\big)\sum_{e\in\mathcal{E}(\Gamma)}\Big(\sum_{j\in\sigma_e}\langle\psi_j, S_j^{-1}\psi_j\rangle_j\Big)^{1/2}\Big(\sum_{j\in\sigma_e}|v_j^\Gamma|_{a_j}^2\Big)^{1/2} \\ &\lesssim \big(1+\ln(H/h)\big)\Big(\sum_{j=1}^J\langle\psi_j, S_j^{-1}\psi_j\rangle_j\Big)^{1/2}\Big(\sum_{j=1}^J|v_j^\Gamma|_{a_j}^2\Big)^{1/2} \\ &= \big(1+\ln(H/h)\big)\Big(\sum_{j=1}^J\langle\psi_j, S_j^{-1}\psi_j\rangle_j\Big)^{1/2}|v^\Gamma|_a \qquad \forall\, v^\Gamma\in V_\Gamma. \end{aligned} \tag{4.14}$$

Combining (3.22) and (4.14) we find

$$\langle \mu, \hat{\mathbb{S}}\,\mu\rangle_M^{1/2} \lesssim \big(1+\ln(H/h)\big)\Big(\sum_{j=1}^J\langle\psi_j, S_j^{-1}\psi_j\rangle_j\Big)^{1/2}, \tag{4.15}$$

whenever $\mu = \sum_{j=1}^J I_j\psi_j$ and $\psi_j \in V_j'$ for $1\leq j\leq J$. The following lemma is then a simple consequence of (3.15) and (4.15).

LEMMA 4.2. *For the second order model problem, it holds that*

$$\lambda_{\max}(\mathbb{B}\hat{\mathbb{S}}) \leq C\big(1+\ln(H/h)\big)^2, \tag{4.16}$$

where the positive constant C is independent of h, H and J.

The derivation of an upper bound for $\lambda_{\max}(\mathbb{B}\hat{\mathbb{S}})$ for the fourth order model problem follows a similar line, with the necessary modifications of some of the definitions.

Let $\mu = \sum_{j=1}^J I_j\psi_j$, where $\psi_j\in V_j'$. For an arbitrary $v^\Gamma = (v_1^\Gamma,\ldots,v_J^\Gamma)\in V_\Gamma$, we define $v_j^H = (v_1^H,\ldots,v_J^H)$ to be the discrete biharmonic function on Ω_j with the following properties:

v^H agrees with v^Γ up to the first order derivatives at $\partial\Omega_j\cap\mathcal{V}$,
v^H is piecewise cubic on $\partial\Omega_j$ and $\partial v^H/\partial n$ is piecewise linear on $\partial\Omega_j$.

Note that the components of v^H are continuous up to the first order derivatives along the interface of the subdomains and therefore (4.1)–(4.6) remain valid, provided we take $v_{j,e}$ to be the discrete biharmonic function on Ω_j that is identical with v_j up to the first order derivatives on e and vanish up to the first order derivatives on $\partial\Omega_j\setminus e$.

Let e be an common edge of Γ_j and Γ_k. Again (4.7) holds because of (3.9) and (3.24). Let $w_j = b_j x_1 + c_j x_2$ be defined by the property

$$\int_{\Omega_j} \frac{\partial}{\partial x_1}(v_j^\Gamma - w_j)\,dx = \int_{\Omega_j} \frac{\partial}{\partial x_2}(v_j^\Gamma - w_j)\,dx = 0. \tag{4.17}$$

We can then apply the discrete Sobolev inequality, the Poincaré-Friedrichs inequality, (4.17) and well-known estimates from the nonoverlapping domain decomposition theory for fourth order problems (cf. [39, 26, 7]) to obtain the following analog of (4.11):

$$\begin{aligned}
|v_{j,e}|_{a_j} &= \alpha_j^{1/2} |v_{j,e}|_{H^2(\Omega_j)} \\
&\approx \alpha_j^{1/2} |\nabla v_{j,e}|_{H_{00}^{1/2}(e)} \\
&\lesssim \alpha_j^{1/2} \Big[|v_j^\Gamma|_{H^2(\Omega_j)} + \big(1+\ln(H_j/h)\big)^{1/2} \|\nabla (v_j^\Gamma - w_j)\|_{L_\infty(\Omega_j)} \Big] \\
&\lesssim \alpha_j^{1/2} \Big[|v_j^\Gamma|_{H^2(\Omega_j)} + \big(1+\ln(H_j/h)\big)\big(H_j^{-1} \|\nabla (v_j^\Gamma - w_j)\|_{L_2(\Omega_j)} + |v_j^\Gamma|_{H^2(\Omega_j)}\big) \Big] \\
&\lesssim \big(1+\ln(H_j/h)\big) \alpha_j^{1/2} |v_j^\Gamma|_{H^2(\Omega_j)} = \big(1+\ln(H_j/h)\big) |v_j^\Gamma|_{a_j}.
\end{aligned} \tag{4.18}$$

If we define $\tilde{v}_{j,e}$ to be the discrete biharmonic function on Ω_j such that $\tilde{v}_{j,e}$ agrees with $v_{k,e}$ up to the first order derivatives on e and $\tilde{v}_{j,e}$ vanishes up to the first order derivatives on $\partial\Omega_j \setminus e$, then the estimates (4.12) and (4.13) remain valid. Finally (4.6), (4.7), (4.12), (4.13) and (4.18) imply (4.14) and (4.15), and we have proved the following analog of Lemma 4.2.

LEMMA 4.3. *For the fourth order model problem, it holds that*

$$\lambda_{\max}(\mathbb{B}\hat{\mathbb{S}}) \le C\big(1+\ln(H/h)\big)^2, \tag{4.19}$$

where the positive constant C is independent of h, H and J.

Combining Lemma 3.4, Lemma 4.2 and Lemma 4.3, we have the following theorem for two-dimensional FETI-DP preconditioners.

THEOREM 4.4. *For both the second order model problem and the fourth order model problem, it holds that*

$$\kappa(\mathbb{B}\hat{\mathbb{S}}) = \frac{\lambda_{\max}(\mathbb{B}\hat{\mathbb{S}})}{\lambda_{\min}(\mathbb{B}\hat{\mathbb{S}})} \le C\big(1+\ln(H/h)\big)^2, \tag{4.20}$$

where the positive constant C is independent of h, H and J.

REMARK 4.5. *A closer inspection of (4.16) and (4.19) reveals that the constant C in (4.20) depends on the quasi-uniformity of the triangulation, the shape regularity of the subdomains, and the numbers and positions of the cross points on the boundaries of the subdomains.*

5. Sharpness of the Condition Number Estimate for Second Order Problems. In this section we will show that the bound in Theorem 4.4 is sharp for second order problems. We will take Ω to be the unit square and the subdomains to be nonoverlapping squares with side H obtained by a uniform subdivision (cf. Figure 5.1). The triangulation $\mathcal{T}$ is then obtained by a uniform subdivision of the subdomains into triangles. Furthermore, we take the coefficients α_j to be 1 identically. We need to show that

$$\lambda_{\min}(\mathbb{B}\hat{\mathbb{S}}) \lesssim 1 \quad \text{and} \quad \lambda_{\max}(\mathbb{B}\hat{\mathbb{S}}) \gtrsim \big(1+\ln(H/h)\big)^2.$$

Without loss of generality, we may assume $(H/h) \gg 1$.

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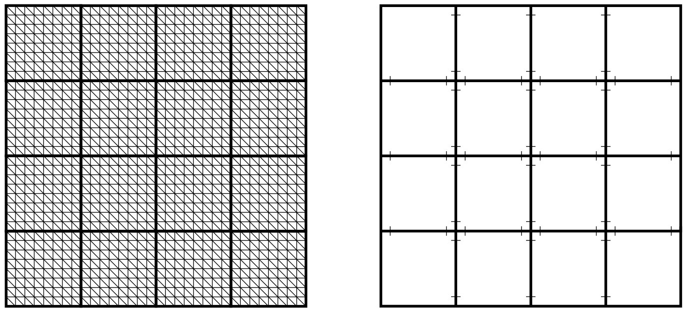

FIG. 5.1. *A uniformly subdivided square*

Let $V_{\Gamma_*} = \{v^\Gamma = (v_1^\Gamma, \ldots, v_J^\Gamma) \in V_\Gamma : v_j^\Gamma$, for $1 \le j \le J$, vanishes at every node on Γ_j whose distance from the nearest cross point is less than or equal to $H/8\}$ (cf. Figure 5.1, where we have marked every node whose distance to the nearest cross point is $H/8$). It is clear that $\dim V_{\Gamma_*} \approx 3/4 \dim V_\Gamma$ and $\dim M = 1/2 \dim V_\Gamma$. Let $V_{\Gamma_\circ} \subset V_\Gamma$ be the orthogonal complement of V_{Γ_*} with respect to $a(\cdot,\cdot)$. Since the dimension of the subspace $V_{\Gamma_\circ}^\perp$ of V_Γ' defined by $V_{\Gamma_\circ}^\perp = \{\eta \in V_\Gamma' : \langle \eta, v^\Gamma \rangle_\Gamma = 0$ for all $v^\Gamma \in V_{\Gamma_\circ}\}$ equals the dimension of V_{Γ_*}, the intersection of M and $V_{\Gamma_\circ}^\perp$ is nontrivial. Let $\mu_* \in M \cap V_{\Gamma_\circ}^\perp$ be a nontrivial Lagrange multiplier. Then we have, by (3.22),

$$\langle \mu_*, \hat{\mathbb{S}}\, \mu_* \rangle_M^{1/2} = \frac{\langle \mu_*, v_*^\Gamma \rangle_\Gamma}{|v_*^\Gamma|_a}, \tag{5.1}$$

for some $v_*^\Gamma = (v_{*1}^\Gamma, \ldots, v_{*J}^\Gamma) \in V_{\Gamma_*} \subset V_1 \times \cdots \times V_J$.

Given any decomposition

$$\mu_* = \sum_{j=1}^J I_j \psi_j, \tag{5.2}$$

where $\psi_j \in V_j'$, we have, in view of (4.3),

$$\langle \mu_*, v_*^\Gamma \rangle_\Gamma = \langle \mu, \sum_{k=1}^J E_k v_{*k}^\Gamma \rangle = \sum_{e \in \mathcal{E}(\Gamma)} \sum_{j,k \in \sigma_e} \langle I_j \psi_j, E_k v_{k,e} \rangle_\Gamma, \tag{5.3}$$

where $v_{k,e}$ is the discrete harmonic function on Ω_k that agrees with v_{*k}^Γ on e and vanishes on $\partial\Omega_k \setminus e$. Note that

$$\sum_{j,k \in \sigma_e} \langle I_j \psi_j, E_k v_{k,e} \rangle_\Gamma = \langle I_j \psi_j, E_j v_{j,e} + E_k v_{k,e} \rangle_\Gamma + \langle I_k \psi_j, E_k v_{k,e} + E_j v_{j,e} \rangle_\Gamma, \tag{5.4}$$

where $\{j,k\} = \sigma_e$, and it suffices to analyze the first term on the right-hand side of (5.4).

Let $\tilde{v}_{j,e}$ be the discrete harmonic function on Ω_j that agrees with $v_{k,e}$ (or equivalently v_{*k}^Γ) on e and vanishes on $\partial\Omega_j \setminus e$. We can then write, by (2.6), (3.7) and (3.10),

$$\langle I_j \psi_j, E_j v_{j,e} + E_k v_{k,e} \rangle_\Gamma = \frac{1}{2} \langle \psi_j, v_{j,e} - \tilde{v}_{j,e} \rangle_j. \tag{5.5}$$

Observe that since v_{*j}^Γ (respectively v_{*k}^Γ) vanishes at all the nodes within a distance of $H/8$ from the corners of Ω_j (respectively Ω_k), we have

$$|v_{j,e}|_{a_j} = |v_{j,e}|_{H^1(\Omega_j)} \lesssim |v_{j,e}|_{H_{00}^{1/2}(e)} \lesssim |v_{*j}^\Gamma|_{H^{1/2}(e)} \lesssim |v_{*j}^\Gamma|_{H^1(\Omega_j)} = |v_{*j}^\Gamma|_{a_j}, \tag{5.6}$$

and similarly

$$|\tilde{v}_{j,e}|_{a_j} \lesssim |v_{k,e}|_{H_{00}^{1/2}(e)} \lesssim |v^{\Gamma}_{*k}|_{a_k}. \tag{5.7}$$

It follows from (3.24) and (5.5)–(5.7) that

$$\begin{aligned} \langle I_j\psi_j, E_j v_{j,e} + E_k v_{k,e}\rangle_\Gamma &\lesssim \langle \psi_j, S_j^{-1}\psi_j\rangle_j^{1/2}\big(|v_{j,e}|_{a_j} + |\tilde{v}_{j,e}|_{a_j}\big) \\ &\lesssim \langle \psi_j, S_j^{-1}\psi_j\rangle_j^{1/2}\big(|v^{\Gamma}_{*j}|_{a_j} + |v^{\Gamma}_{*k}|_{a_k}\big). \end{aligned} \tag{5.8}$$

Combining (5.3), (5.4) and (5.8) we find

$$\langle \mu_*, v_*^{\Gamma}\rangle_\Gamma \lesssim \Big(\sum_{j=1}^{J}\langle \psi_j, S_j^{-1}\psi_j\rangle_j\Big)^{1/2}|v_*^{\Gamma}|_a,$$

which together with (5.1) implies

$$\langle \mu_*, \hat{\mathbb{S}}\,\mu_*\rangle_M^{1/2} \lesssim \Big(\sum_{j=1}^{J}\langle \psi_j, S_j^{-1}\psi_j\rangle_j\Big)^{1/2}, \tag{5.9}$$

whenever (5.2) holds. We conclude from (3.14) and (5.9) that

$$\lambda_{\min}(\mathbb{B}\hat{\mathbb{S}}) \lesssim 1. \tag{5.10}$$

Now we turn to the derivation of a lower bound for $\lambda_{\max}(\mathbb{B}\hat{\mathbb{S}})$, which involves the special piecewise linear functions constructed in [8] for proving the sharpness of the condition number estimates for the BPS and the Neumann-Neumann preconditioners.

By Corollary 3.6 and Corollary 3.9 in [8], there exists a continuous piecewise linear function g_h on the real line with respect to the uniform subdivision with nodes jh ($j \in \mathbb{Z}$) such that

$$g_h(x) = 0 \qquad \text{for} \quad |x| \geq H/4, \tag{5.11}$$

$$g_h(x) = g_h(-x) \qquad \forall\, x \in \mathbb{R}, \tag{5.12}$$

$$g_h(0) = \|g_h\|_{L_\infty(\mathbb{R})} \approx 1 + \ln(H/h), \tag{5.13}$$

$$|g_h|^2_{H^{1/2}(\mathbb{R})} \approx 1 + \ln(H/h), \tag{5.14}$$

$$|g_h - \Pi_{(H/4)}g_h|^2_{H_{00}^{1/2}(-H/4,0)} = |g_h - \Pi_{(H/4)}g_h|^2_{H_{00}^{1/2}(0,H/4)} \approx \big(1 + \ln(H/h)\big)^3, \tag{5.15}$$

where $\Pi_{(H/4)}$ is the piecewise linear nodal interpolation operator with respect to the nodes $j(H/4)$ for $j \in \mathbb{Z}$.

LEMMA 5.1. *Let $\ell_h(x)$ be the continuous piecewise linear function on $[0,H]$ defined by*

$$\ell_h(x) = \begin{cases} g_h(x) - g_h(0) & 0 \leq x \leq H/4, \\ -g_h(0) & H/4 \leq x \leq 3H/4, \\ g_h(x-H) - g_h(0) & 3H/4 \leq x \leq H. \end{cases} \tag{5.16}$$

Then we have

$$|\ell_h|^2_{H_{00}^{1/2}(0,H)} \approx \big(1 + \ln(H/h)\big)^3. \tag{5.17}$$

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Proof. We can write $\ell_h = (\ell_h - \Pi_{(H/4)}\ell_h) + \Pi_{(H/4)}\ell_h$. Note that $\Pi_{(H/4)}\ell_h$ is linear between the four nodes 0, $H/4$, $3H/4$ and H, equal to 0 at 0 and H, and equal to $-g_h(0)$ at $H/4$ and $3H/4$. Therefore, in view of (5.13) and a scaling argument, we have

$$|\Pi_{(H/4)}\ell_h|^2_{H^{1/2}_{00}(0,H)} \approx [g_h(0)]^2 \approx \big(1+\ln(H/h)\big)^2. \tag{5.18}$$

On the other hand, $\ell_h - \Pi_{(H/4)}\ell_h$ agrees with $g_h - \Pi_{(H/4)}g_h$ on $[0, H/4]$, vanishes on $[H/4, 3H/4]$ and agrees with $\big(g_h - \Pi_{(H/4)}g_h\big)(\cdot - H)$ on $[3H/4, H]$. Therefore, it follows from (5.15) that

$$\begin{aligned} &|\ell_h - \Pi_{(H/4)}\ell_h|^2_{H^{1/2}_{00}(0,H)} \\ &\qquad \approx |g_h - \Pi_{(H/4)}g_h|^2_{H^{1/2}_{00}(-H/4,0)} + |g_h - \Pi_{(H/4)}g_h|^2_{H^{1/2}_{00}(0,H/4)} \\ &\qquad \approx \big(1+\ln(H/h)\big)^3. \end{aligned} \tag{5.19}$$

The estimate (5.17) follows from (5.18) and (5.19). ☐

Our derivation of a lower bound for $\lambda_{\max}(\mathbb{B}\hat{\mathbb{S}})$ involves the 9 subdomains depicted in Figure 5.2. First we define $v_\dagger$ to be the discrete harmonic function on Ω_1 that is identical with the function ℓ_h defined by (5.16) on the common side of Ω_1 and Ω_9 and vanishes on the other three sides of Ω_1. From (5.17) we have

$$|v_\dagger|^2_{a_1} = |v_\dagger|^2_{H^1(\Omega_1)} \approx |v_\dagger|^2_{H^{1/2}(\partial\Omega_1)} \approx |v_\dagger|^2_{H^{1/2}_{00}(e)} \approx \big(1+\ln(H/h)\big)^3. \tag{5.20}$$

Moreover, there exists, by (3.24) and a simple dimension argument (or the Hahn-Banach theorem), $\psi_\dagger \in V_1'$ such that

$$\langle \psi_\dagger, S_1^{-1}\psi_\dagger\rangle_1^{1/2} = \frac{\langle \psi_\dagger, v_\dagger\rangle_1}{|v_\dagger|_{a_1}}, \tag{5.21}$$

and we define

$$\mu_\dagger = I_1\psi_\dagger. \tag{5.22}$$

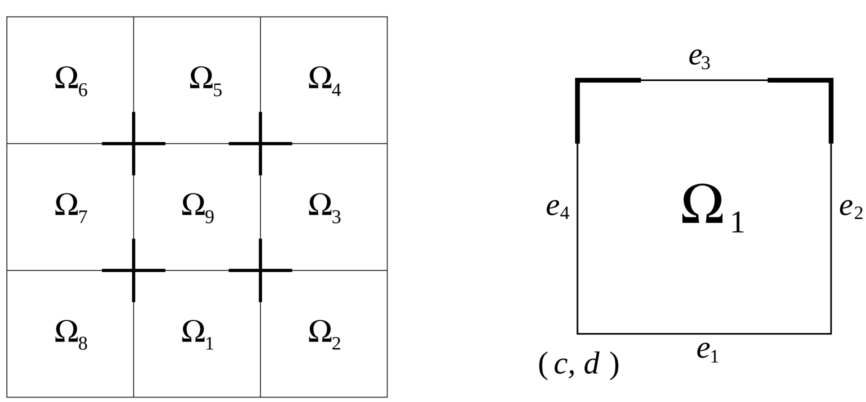

FIG. 5.2. *The definition of* $v_\diamond^\Gamma$

Next we define a special $v_\diamond^\Gamma \in V_\Gamma$ as follows: (i) $v_{\diamond j}^\Gamma = 0$ except for $1 \le j \le 9$; (ii) $v_{\diamond j}^\Gamma$, for $1 \le j \le 8$, is the discrete harmonic function on Ω_j whose trace on $\partial\Omega_j$ is identical with g_h around the corners indicated in Figure 5.2 and zero at all other nodes (so each of $v_{\diamond 1}^\Gamma$, $v_{\diamond 3}^\Gamma$, $v_{\diamond 5}^\Gamma$ and $v_{\diamond 7}^\Gamma$ are nontrivial around two corners, and each of $v_{\diamond 2}^\Gamma$, $v_{\diamond 4}^\Gamma$, $v_{\diamond 6}^\Gamma$ and $v_{\diamond 8}^\Gamma$ are nontrivial around only one corner); (iii) $v_{\diamond 9}^\Gamma$ is the constant function $g_h(0)$.

A more precise definition of, say $v_{\diamond 1}^\Gamma$, is as follows. Let (c, d) be the lower left corner of Ω_1 and the four edges of $\partial\Omega_1$ be $e_1, \ldots, e_4$ (cf. Figure 5.2). Then $v_{\diamond 1}^\Gamma$ vanishes on e_1. On the edges e_2 and e_4, we have

$$v_{\diamond 1}^\Gamma(c+H, d+x_2) = v_{\diamond 1}^\Gamma(c, d+x_2) = \begin{cases} 0 & 0 \le x_2 \le 3H/4, \\ g_h(x_2 - H) & 3H/4 \le x_2 \le H, \end{cases}$$

and $v_{\diamond 1}^\Gamma$ on e_3 is defined by

$$v_{\diamond 1}^\Gamma(c+x_1, d+H) = \begin{cases} g_h(x_1) & 0 \le x_1 \le H/4, \\ 0 & H/4 \le x_1 \le 3H/4, \\ g_h(x_1 - H) & 3H/4 \le x_1 \le H. \end{cases}$$

Now we make two crucial observations. The first is that, by (3.7), (5.16), (5.22) and the definitions of $v_\dagger$ and $v_\diamond^\Gamma$, we have

$$\langle \mu_\dagger, v_\diamond^\Gamma \rangle_\Gamma = \frac{1}{2} \langle \psi_\dagger, v_\dagger \rangle_1. \tag{5.23}$$

The second observation is that (5.12), (5.14) and the definition of $v_\diamond^\Gamma$ imply

$$|v_\diamond^\Gamma|_a^2 = \sum_{j=1}^{9} |v_{\diamond j}|_{H^1(\Omega_j)}^2 \approx 1 + \ln(H/h). \tag{5.24}$$

It follows from (3.24), (5.20), (5.21), (5.23) and (5.24) that

$$\begin{aligned} \langle \mu_\dagger, v_\diamond^\Gamma \rangle_\Gamma^2 &\approx \langle \psi_\dagger, S_1^{-1} \psi_\dagger \rangle_1 |v_\dagger|_{a_1}^2 \\ &\approx \big(1 + \ln(H/h)\big)^3 \langle \psi_\dagger, S_1^{-1} \psi_\dagger \rangle_1 \approx \big(1 + \ln(H/h)\big)^2 |v_\diamond^\Gamma|_a^2 \langle \psi_\dagger, S_1^{-1} \psi_\dagger \rangle_1. \end{aligned} \tag{5.25}$$

Combining (3.22) and (5.25) we find

$$\langle \mu_\dagger, \hat{\mathbb{S}}\, \mu_\dagger \rangle_M \gtrsim \big(1 + \ln(H/h)\big)^2 \langle \psi_\dagger, S_1^{-1} \psi_\dagger \rangle_1. \tag{5.26}$$

Finally (3.15), (5.22) and (5.26) yield

$$\lambda_{\max}(\mathbb{B}\hat{\mathbb{S}}) \gtrsim \big(1 + \ln(H/h)\big)^2. \tag{5.27}$$

Therefore, for the second order model problem we have, by (5.10) and (5.27),

$$\kappa(\mathbb{B}\hat{\mathbb{S}}) = \frac{\lambda_{\max}(\mathbb{B}\hat{\mathbb{S}})}{\lambda_{\min}(\mathbb{B}\hat{\mathbb{S}})} \ge C\big(1 + \ln(H/h)\big)^2, \tag{5.28}$$

where C is independent of h, H and J.

In summary we have established the following theorem.

THEOREM 5.2. *The condition number estimate* (4.20) *is sharp for the second order model problem.*

REMARK 5.3. *The sharpness of (4.20) for the fourth order model problem can also be established within the additive Schwarz framework. Indeed (5.10) can be obtained in a similar fashion with only minor modifications. However, the derivation of (5.27) requires the construction of special piecewise quadratic polynomial functions whose symmetry properties are different from the piecewise linear functions constructed in* [8]. *Such piecewise quadratic polynomial functions will also be useful for showing that the condition number estimates in* [39] *for iterative substructuring algorithms are sharp. The investigations in this direction will be pursued elsewhere.*

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