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Some generalized Gronwall-Bellman type nonlinear delay integral inequalities for discontinuous functions

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Abstract

In this work, some new generalized Gronwall-Bellman type nonlinear delay integral inequalities for discontinuous functions are established, which can be used as a handy tool in the quantitative and qualitative analysis of solutions of certain integral equations. The established results generalize the main results of Gao and Meng in (Math. Pract. Theory 39(22):198-203, 2009).

MSC: 26D10; 26D15

Keywords: delay integral inequality; discontinuous function; integral equation; bounded

1 Introduction

In recent years, the research of various mathematical inequalities has been paid much attention to by many authors, and many integral inequalities have been established, which provide handy tools for investigating the quantitative and qualitative properties of solutions of integral and differential equations. Among these inequalities, Gronwall-Bellman type integral inequalities are of particular importance as such inequalities provide explicit bounds for unknown functions, and a lot of generalizations of Gronwall-Bellman type integral inequalities have been established (for example, see [1–16] and the references therein). But to our knowledge, most of the known integral inequalities are concerned with continuous functions, while very few authors undertake research in integral inequalities for discontinuous functions [14–16]. Furthermore, delay integral inequalities containing integration on infinite intervals for discontinuous functions have not been reported in the literature so far.

In [1], Gao and Meng established a generalized Gronwall-Bellman type integral inequality, named Mate-Nevai type nonlinear integral inequality for continuous functions, which is one case of inequalities containing integration on infinite intervals for continuous functions. The related theorem reads as follows.

Theorem A *Suppose that $u \in C(\mathbb{R}_+, \mathbb{R}_+)$, $H, F \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, and $H(t,s)$, $F(t,s)$ are decreasing in t for every fixed s. $\phi \in C(\mathbb{R}_+, \mathbb{R}_+)$ is strictly increasing, and $\phi(\infty) = \infty$. $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ is increasing. $\psi(\phi^{-1}) \in C(\mathbb{R}_+, \mathbb{R}_+)$ is strictly increasing, and $\psi(\phi^{-1})$ is sub-*

multiplicative. If for $C \geq 0$ *and* $t \in \mathbb{R}_+$, $u(t)$ *satisfies*

$$\phi\bigl(u(t)\bigr) \leq C + \int_t^\infty H(t,s)\phi\bigl(u(s)\bigr)\,ds + \int_t^\infty F(t,s)\psi\bigl(u(s)\bigr)\,ds,$$

then for $0 \leq T \leq t \leq \infty$, *we have*

$$u(t) \leq \phi^{-1}\biggl\{\exp\biggl[\int_t^\infty H(t,s)\,ds\biggr]G^{-1}\biggl[G(C) + \int_t^\infty F(t,s)\psi\biggl(\phi^{-1}\biggl(\exp\biggl(\int_t^\infty H(t,s)\,d\xi\biggr)\biggr)\biggr)\,ds\biggr]\biggr\},$$

where $G(z) = \int_{z_0}^z \frac{1}{\psi(\phi^{-1}(s))}\,ds$, $z \geq z_0 > 0$, G^{-1} *is the inverse of* G, *and* $T \in \mathbb{R}_+$ *is chosen so that*

$$G(C) + \int_t^\infty F(t,s)\psi\biggl(\phi^{-1}\biggl(\exp\biggl(\int_t^\infty H(t,s)\,d\xi\biggr)\biggr)\biggr)\,ds \in \operatorname{Dom}\bigl(G^{-1}\bigr), \quad T \leq t < \infty.$$

The inequality above appears to be useful in deriving bounds of solutions of certain integral equations, but it is inadequate to deal with boundedness for solutions of both integral equations for discontinuous functions and delay integral equations.

In the present paper, motivated by the work above, we establish some new generalized Mate-Nevai type nonlinear delay integral inequalities for discontinuous functions, which extend the main results in [1]. The establishment for the desired inequalities will be discussed in 1-D and 2-D cases respectively.

2 Main results

In the rest of the paper, we denote the set of real numbers as $\mathbb{R}$, and $\mathbb{R}_+ = [0,\infty)$.

Theorem 2.1 *Suppose that* $u(t)$ *is a nonnegative continuous function defined on* $\mathbb{R}_+$ *with the first kind of discontinuities at the points* t_i, $i = 1,2,\ldots,n$, *and* $0 \leq t_1 < t_2 < \cdots < t_n < t_{n+1} = \infty$. $f,g \in C(\mathbb{R}_+ \times \mathbb{R}_+,\mathbb{R}_+)$, $f_2', g_2' \in C(\mathbb{R}_+ \times \mathbb{R}_+,\mathbb{R})$, *and* $f(s,t)$, $g(s,t)$ *are decreasing in* t *for every fixed* s. $p > 0$ *is a constant.* $\tau \in C(\mathbb{R}_+,\mathbb{R}_+)$ *with* $\tau(t) \leq t$, *and* $\tau(t) \geq t_i$ *for* $\forall t \in [t_i, t_{i+1})$, $i = 0,1,\ldots,n$. $\omega \in C(\mathbb{R}_+,\mathbb{R}_+)$, *and* ω *is nondecreasing with* $\omega(u) > 0$ *for* $u > 0$. *Furthermore,* ω *is submultiplicative, that is,* $\omega(\alpha\gamma) \leq \omega(\alpha)\omega(\gamma)$ *for* $\forall \alpha,\gamma \in \mathbb{R}_+$. C, β_i *are constants, and* $C \geq 0$, $\beta_i > 0$, $i = 1,2,\ldots,n$. *If for* $t \in \mathbb{R}_+$, $u(t)$ *satisfies the following inequality*:

$$u(t) \leq C + \int_t^\infty f(s,t)u\bigl(\tau(s)\bigr)\,ds + \int_t^\infty g(s,t)\omega\bigl(u\bigl(\tau(s)\bigr)\bigr)\,ds + \sum_{t<t_j<\infty} \beta_j u(t_j - 0), \tag{2.1}$$

then

$$u(t) \leq G_{i-1}^{-1}\biggl[-\int_t^{t_i} \widetilde{g}_{i-1}(s)\exp\bigl(-\widetilde{f}_{i-1}(s)\bigr)\omega\bigl(\exp\bigl(\widetilde{f}_{i-1}(s)\bigr)\bigr)\,ds\biggr]\exp\bigl(\widetilde{f}_{i-1}(t)\bigr),$$
$$t \in [t_{i-1}, t_i), i = 1,2,\ldots,n+1, \tag{2.2}$$

where

$$\begin{cases} G_{i-1}(z) = \int_{b_{i-1}}^{z} \frac{1}{\omega(s)}\,ds, \quad i = 1,2,\ldots,n+1, \\ b_n = C, \qquad b_{i-1} = a_i + \beta_i a_i, \quad i = 1,2,\ldots n, \\ a_i = G_i^{-1}[-\int_{t_i}^{t_{i+1}} \widetilde{g}_i(s)\exp(-\widetilde{f}_i(s))\omega(\exp(\widetilde{f}_i(s)))\,ds]\exp(\widetilde{f}_i(t_i)), \quad i = 1,2,\ldots,n, \\ \widetilde{f}_{i-1}(t) = \int_t^{t_i} f(s,t)\,ds, \quad i = 1,2,\ldots,n+1, \\ \widetilde{g}_{i-1}(t) = \frac{d(\int_t^{t_i} g(s,t)\,ds)}{dt}, \quad i = 1,2,\ldots,n+1. \end{cases} \tag{2.3}$$

Proof Denote the right-hand side of (2.1) by $v(t)$. Then

$$u(t) \le v(t). \tag{2.4}$$

Define

$$\widetilde{v}_i(t) = b_i + \int_t^{\infty} f(s,t)u\big(\tau(s)\big)\,ds + \int_t^{\infty} g(s,t)\omega\big(u\big(\tau(s)\big)\big)\,ds, \quad i = 0,1,\ldots,n. \tag{2.5}$$

Case 1: If $t \in [t_n,\infty)$ (in fact, $t_{n+1} = \infty$), then $v(t) = \widetilde{v}_n(t)$. From the definition of τ, we have $\tau(t) \ge t_n$ for $t \in [t_n,\infty)$. So, $\tau(t) \in [t_n,\infty)$, and from (2.4) we have

$$u\big(\tau(t)\big) \le \widetilde{v}_n\big(\tau(t)\big) \le \widetilde{v}_n(t). \tag{2.6}$$

Then

$$\widetilde{v}_n(t) \le C + \int_t^{\infty} f(s,t)\widetilde{v}_n(s)\,ds + \int_t^{\infty} g(s,t)\omega\big(\widetilde{v}_n(s)\big)\,ds \tag{2.7}$$

and

$$\begin{aligned} \widetilde{v}_n'(t) &= -f(t,t)u\big(\tau(t)\big) + \int_t^{\infty} \frac{\partial f(s,t)}{\partial t} u\big(\tau(s)\big)\,ds - g(t,t)\omega\big(u\big(\tau(t)\big)\big) \\ &\quad + \int_t^{\infty} \frac{\partial g(s,t)}{\partial t}\omega\big(u\big(\tau(s)\big)\big)\,ds \\ &\ge -f(t,t)\widetilde{v}_n(t) + \int_t^{\infty} \frac{\partial f(s,t)}{\partial t}\widetilde{v}_n(s)\,ds - g(t,t)\omega\big(\widetilde{v}_n(t)\big) + \int_t^{\infty} \frac{\partial g(s,t)}{\partial t}\omega\big(\widetilde{v}_n(s)\big)\,ds \\ &\ge \left[-f(t,t) + \int_t^{\infty} \frac{\partial f(s,t)}{\partial t}\,ds\right]\widetilde{v}_n(t) + \left[-g(t,t) + \int_t^{\infty} \frac{\partial g(s,t)}{\partial t}\,ds\right]\omega\big(\widetilde{v}_n(t)\big) \\ &= \frac{d(\int_t^{\infty} f(s,t)\,ds)}{dt}\widetilde{v}_n(t) + \frac{d(\int_t^{\infty} g(s,t)\,ds)}{dt}\omega\big(\widetilde{v}_n(t)\big) \\ &= \frac{d\widetilde{f}_n(t)}{dt}\widetilde{v}_n(t) + \widetilde{g}_n(t)\omega\big(\widetilde{v}_n(s)\big), \end{aligned}$$

where $\widetilde{f}_n$, $\widetilde{g}_n$ are defined in (2.3), and obviously $\widetilde{g}_n(t) \le 0$. Furthermore, we have

$$\widetilde{v}_n'(t) - \frac{d\widetilde{f}_n(t)}{dt}\widetilde{v}_n(t) \ge \widetilde{g}_n(t)\omega\big(\widetilde{v}_n(t)\big). \tag{2.8}$$

Multiplying $\exp(\widetilde{f}_n(t))$ on both sides of (2.8) yields

$$\bigl[\widetilde{v}_n(t)\exp\bigl(-\widetilde{f}_n(t)\bigr)\bigr]' \ge \widetilde{g}_n(t)\exp\bigl(-\widetilde{f}_n(t)\bigr)\omega\bigl(\widetilde{v}_n(t)\bigr). \tag{2.9}$$

Setting $t = s$ in (2.9), by $\widetilde{v}_n(\infty) = C = b_n$, an integration for (2.9) with respect to s from t to ∞ yields

$$b_n - \widetilde{v}_n(t)\exp\bigl(-\widetilde{f}_n(t)\bigr) \ge \int_t^\infty \widetilde{g}_n(s)\exp\bigl(-\widetilde{f}_n(s)\bigr)\omega\bigl(\widetilde{v}_n(s)\bigr)\,ds, \tag{2.10}$$

which implies

$$\widetilde{v}_n(t) \le \left\{b_n - \int_t^\infty \widetilde{g}_n(s)\exp\bigl(-\widetilde{f}_n(s)\bigr)\omega\bigl(\widetilde{v}_n(s)\bigr)\,ds\right\}\exp\bigl(\widetilde{f}_n(t)\bigr). \tag{2.11}$$

Define $c(t) = b_n - \int_t^\infty \widetilde{g}_n(s)\exp(-\widetilde{f}_n(s))\omega(\widetilde{v}_n(s))\,ds$. Then

$$\widetilde{v}_n(t) \le c(t)\exp\bigl(\widetilde{f}_n(t)\bigr). \tag{2.12}$$

Since $\widetilde{g}_n(t) \le 0$, and ω is submultiplicative, we have

$$\begin{aligned}
c'(t) &= \widetilde{g}_n(t)\exp\bigl(-\widetilde{f}_n(t)\bigr)\omega\bigl(\widetilde{v}_n(t)\bigr) \\
&\ge \widetilde{g}_n(t)\exp\bigl(-\widetilde{f}_n(t)\bigr)\omega\bigl(c(t)\exp\bigl(\widetilde{f}_n(t)\bigr)\bigr) \\
&\ge \widetilde{g}_n(t)\exp\bigl(-\widetilde{f}_n(t)\bigr)\omega\bigl(c(t)\bigr)\omega\bigl(\exp\bigl(\widetilde{f}_n(t)\bigr)\bigr),
\end{aligned}$$

that is,

$$\frac{c'(t)}{\omega(c(t))} \ge \widetilde{g}_n(t)\exp\bigl(-\widetilde{f}_n(t)\bigr)\omega\bigl(\exp\bigl(\widetilde{f}_n(t)\bigr)\bigr). \tag{2.13}$$

Setting $t = s$ in (2.13), an integration for (2.13) with respect to s from t to ∞ yields

$$G_n\bigl(c(\infty)\bigr) - G_n\bigl(c(t)\bigr) \ge \int_t^\infty \widetilde{g}_n(s)\exp\bigl(-\widetilde{f}_n(s)\bigr)\omega\bigl(\exp\bigl(\widetilde{f}_n(s)\bigr)\bigr)\,ds.$$

By $c(\infty) = b_n$, $G_n(b_n) = 0$, and G_n is increasing, we obtain

$$c(t) \le G_n^{-1}\left[-\int_t^\infty \widetilde{g}_n(s)\exp\bigl(-\widetilde{f}_n(s)\bigr)\omega\bigl(\exp\bigl(\widetilde{f}_n(s)\bigr)\bigr)\,ds\right]. \tag{2.14}$$

Combining (2.12) and (2.14), we obtain

$$u(t) \le \widetilde{v}_n(t) \le G_n^{-1}\left[-\int_t^\infty \widetilde{g}_n(s)\exp\bigl(-\widetilde{f}_n(s)\bigr)\omega\bigl(\exp\bigl(\widetilde{f}_n(s)\bigr)\bigr)\,ds\right]\exp\bigl(\widetilde{f}_n(t)\bigr). \tag{2.15}$$

Especially,

$$u(t_n) \le \widetilde{v}_n(t_n) \le G_n^{-1}\left[-\int_{t_n}^\infty \widetilde{g}_n(s)\exp\bigl(-\widetilde{f}_n(s)\bigr)\omega\bigl(\exp\bigl(\widetilde{f}_n(s)\bigr)\bigr)\,ds\right]\exp\bigl(\widetilde{f}_n(t_n)\bigr) = a_n, \tag{2.16}$$

where a_n is defined in (2.3). At the same time, we have

$$u(t_n - 0) \le \widetilde{v}_n(t_n - 0) \le a_n. \tag{2.17}$$

Case 2: If $t \in [t_{n-1}, t_n)$, then from (2.1), (2.16) and (2.17), we have

$$\begin{aligned}
u(t) &\le C + \int_t^{\infty} f(s,t)u\big(\tau(s)\big)\,ds + \int_t^{\infty} g(s,t)\omega\big(u\big(\tau(s)\big)\big)\,ds + \beta_n u(t_n - 0) \\
&= b_n + \int_{t_n}^{\infty} f(s,t)u\big(\tau(s)\big)\,ds + \int_{t_n}^{\infty} g(s,t)\omega\big(u\big(\tau(s)\big)\big)\,ds \\
&\quad + \int_t^{t_n} f(s,t)u\big(\tau(s)\big)\,ds + \int_t^{t_n} g(s,t)\omega\big(u\big(\tau(s)\big)\big)\,ds + \beta_n u(t_n - 0) \\
&\le a_n + \beta_n a_n + \int_t^{t_n} f(s,t)u\big(\tau(s)\big)\,ds + \int_t^{t_n} g(s,t)\omega\big(u\big(\tau(s)\big)\big)\,ds \\
&= b_{n-1} + \int_t^{t_n} f(s,t)u\big(\tau(s)\big)\,ds + \int_t^{t_n} g(s,t)\omega\big(u\big(\tau(s)\big)\big)\,ds = \widetilde{v}_{n-1}(t).
\end{aligned} \tag{2.18}$$

Then, following in a similar manner as in Case 1, we obtain

$$\begin{aligned}
u(t) &\le \widetilde{v}_{n-1}(t) \\
&\le G_{n-1}^{-1}\left[-\int_t^{t_n} \widetilde{g}_{n-1}(s)\exp\big(-\widetilde{f}_{n-1}(s)\big)\omega\big(\exp\big(\widetilde{f}_{n-1}(s)\big)\big)\,ds\right]\exp\big(\widetilde{f}_{n-1}(t)\big).
\end{aligned} \tag{2.19}$$

Especially,

$$\begin{cases}
u(t_{n-1}) \le \widetilde{v}_{n-1}(t_{n-1}) \\
\qquad \le G_{n-1}^{-1}[-\int_{t_{n-1}}^{t_n} \widetilde{g}_{n-1}(s)\exp(-\widetilde{f}_{n-1}(s))\omega(\exp(\widetilde{f}_{n-1}(s)))\,ds]\exp(\widetilde{f}_{n-1}(t_{n-1})) = a_{n-1}, \\
u(t_{n-1} - 0) \le \widetilde{v}_{n-1}(t_{n_1} - 0) \le a_{n-1},
\end{cases}$$

where a_{n-1} is defined in (2.3).

Case 3: If for $t \in [t_j, t_{j+1})$, $j = i, i+1, \dots, n$, the following inequalities hold:

$$\begin{cases}
u(t) \le \widetilde{v}_j(t) \le G_j^{-1}[-\int_t^{t_{j+1}} \widetilde{g}_j(s)\exp(-\widetilde{f}_j(s))\omega(\exp(\widetilde{f}_j(s)))\,ds]\exp(\widetilde{f}_j(t)), \\
u(t_j) \le \widetilde{v}_j(t_j) \le G_j^{-1}[-\int_{t_j}^{t_{j+1}} \widetilde{g}_j(s)\exp(-\widetilde{f}_j(s))\omega(\exp(\widetilde{f}_j(s)))\,ds]\exp(\widetilde{f}_j(t_j)) = a_j, \\
u(t_j - 0) \le \widetilde{v}_j(t_j - 0) \le a_j,
\end{cases} \tag{2.20}$$

where a_j is defined in (2.3). Then for $t \in [t_{i-1}, t_i)$, from (2.1) we obtain

$$\begin{aligned}
u(t) &\le C + \int_t^{\infty} f(s,t)u\big(\tau(s)\big)\,ds + \int_t^{\infty} g(s,t)\omega\big(u\big(\tau(s)\big)\big)\,ds + \sum_{t<t_j<\infty} \beta_j u(t_j - 0) \\
&= C + \int_{t_i}^{\infty} f(s,t)u\big(\tau(s)\big)\,ds + \int_{t_i}^{\infty} g(s,t)\omega\big(u\big(\tau(s)\big)\big)\,ds \\
&\quad + \int_t^{t_i} f(s,t)u\big(\tau(s)\big)\,ds + \int_t^{t_i} g(s,t)\omega\big(u\big(\tau(s)\big)\big)\,ds + \sum_{t<t_j<\infty} \beta_j u(t_j - 0)
\end{aligned}$$

$$\begin{aligned}&\leq a_i + \beta_i a_i + \int_t^{t_i} f(s,t)u\big(\tau(s)\big)\,ds + \int_t^{t_i} g(s,t)\omega\big(u\big(\tau(s)\big)\big)\,ds\\&= b_{i-1} + \int_t^{t_i} f(s,t)u\big(\tau(s)\big)\,ds + \int_t^{t_i} g(s,t)\omega\big(u\big(\tau(s)\big)\big)\,ds = \widetilde{v}_{i-1}(t).\end{aligned}\tag{2.21}$$

Then, following in a similar manner as in Case 1, we obtain

$$u(t) \leq \widetilde{v}_{i-1}(t) \leq G_{i-1}^{-1}\left[-\int_t^{t_i} \widetilde{g}_{i-1}(s)\exp\big(-\widetilde{f}_{i-1}(s)\big)\omega\big(\exp\big(\widetilde{f}_{i-1}(s)\big)\big)\,ds\right]\exp\big(\widetilde{f}_{i-1}(t)\big),\tag{2.22}$$

and the proof is complete. □

Remark 2.1 If we take $u(t) = \phi(\widetilde{u}(t))$, $\omega = \psi\phi^{-1}$, $\tau(t) = t$, and furthermore $u(t)$ is continuous on $\mathbb{R}_+$, then Theorem 2.1 reduces to Theorem A (*i.e.*, [1, Theorem 1]).

Remark 2.2 Under the assumptions of Remark 2.1, furthermore, if we take $\phi(\widetilde{u}(t)) = \widetilde{u}^p(t)$, then Theorem 2.1 reduces to [1, Corollary 1].

Remark 2.3 If we take $\omega(u(t)) = u^q(t)$ in Theorem 2.1, Remark 2.1 and Remark 2.2, then we can obtain three corollaries, which are omitted here.

Based on Theorem 2.1, we will establish another Mate-Nevai type inequality for discontinuous functions in the 2-D case.

Theorem 2.2 *Suppose $u(x,y)$ is a nonnegative continuous function on $\Omega = \bigcup_{i,j\geq 1}\Omega_{i,j}$, $\Omega_{i,j} = \{(x,y)|x_{i-1} \leq x < x_i, y_{j-1} \leq y < y_j\}$ with the exception at the points (x_i,y_i), $i = 1,2,\ldots,n$, where there are finite jumps, and $0 \leq x_0 < x_1 < \cdots < x_n < x_{n+1} = \infty$, $0 \leq y_0 < y_1 < \cdots < y_n < y_{n+1} = \infty$. $f,g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$. $\tau_1(x) \in C(\mathbb{R}_+,\mathbb{R}_+)$ with $\tau_1(x) \leq x$, and $\tau_1(x) \geq x_i$ for $\forall x \in [x_i,x_{i+1})$, $i = 0,1,\ldots,n$. $\tau_2(y) \in C(\mathbb{R}_+,\mathbb{R}_+)$ with $\tau_2(y) \leq y$, and $\tau(y) \geq y_i$ for $\forall y \in [y_i,y_{i+1})$, $i = 0,1,\ldots,n$. ω is defined the same as in Theorem 2.1. Furthermore, $f(x,y) = 0$, $g(x,y) = 0$ for $(x,y) \in \Omega_{i,j}$, $i \neq j$. $C > 0$ is a constant. If for $(x,y) \in \Omega$, $u(x,y)$ satisfies the following inequality*:

$$\begin{aligned}u(x,y) \leq{}& C + \int_x^\infty\int_y^\infty f(s,t)u\big(\tau_1(s),\tau_2(t)\big)\,dt\,ds + \int_x^\infty\int_y^\infty g(s,t)\omega\big(u\big(\tau_1(s),\tau_2(t)\big)\big)\,dt\,ds\\&+ \sum_{x<x_j<\infty,y<y_j<\infty}\beta_j u(x_j-0,y_j-0),\end{aligned}\tag{2.23}$$

then

$$\begin{aligned}u(x,y) \leq{}& \widetilde{G}_{i-1}^{-1}\left[-\int_x^{x_i}\widetilde{g}_{i-1}(s,y)\exp\big(-\widetilde{f}_{i-1}(s,y)\big)\omega\big(\exp\big(\widetilde{f}_{i-1}(s,y)\big)\big)\,ds\right]\exp\big(\widetilde{f}_{i-1}(x,y)\big),\\&(x,y)\in\Omega_{i,i}, i = 1,2,\ldots,n+1,\end{aligned}\tag{2.24}$$

where

$$\begin{cases} \widetilde{G}_{i-1}(z) = \int_{\widetilde{b}_{i-1}}^{z} \frac{1}{\omega(s)}\,ds, \quad i = 1,2,\ldots,n+1, \\ \widetilde{b}_n = C, \qquad \widetilde{b}_{i-1} = \widetilde{a}_i + \beta_i \widetilde{a}_i, \quad i = 1,2,\ldots,n, \\ \widetilde{a}_i = \widetilde{G}_i^{-1}[-\int_{x_i}^{x_{i+1}} \widetilde{g}_i(s,y_i)\exp(-\widetilde{f}_i(s,y_i))\omega(\exp(\widetilde{f}_i(s,y_i)))\,ds]\exp(\widetilde{f}_i(x_i,y_i)), \\ \quad i = 1,2,\ldots,n, \\ \widetilde{f}_{i-1}(x,y) = \int_x^{x_i}\int_y^{y_i} f(s,t)\,dt\,ds, \quad i = 1,2,\ldots,n+1, \\ \widetilde{g}_{i-1}(x,y) = -\int_y^{y_i} g(x,t)\,dt, \quad i = 1,2,\ldots,n+1. \end{cases} \tag{2.25}$$

Proof Denote the right-hand side of (2.23) by $v(x,y)$. Then

$$u(x,y) \le v(x,y). \tag{2.26}$$

Define

$$\widetilde{v}_i(x,y) = b_i + \int_x^\infty \int_y^\infty f(s,t)u\big(\tau_1(s),\tau_2(t)\big)\,dt\,ds + \int_t^\infty g(s,t)\omega\big(u\big(\tau_1(s),\tau_2(t)\big)\big)\,dt\,ds,$$
$$i = 0,1,\ldots,n. \tag{2.27}$$

Case 1: If $(x,y) \in \Omega_{n+1,n+1}$, then $v(x,y) = \widetilde{v}_n(x,y)$. As $\tau_1(x) \ge x_n$ for $x \in [x_n,\infty)$ and $\tau_2(y) \ge y_n$ for $y \in [y_n,\infty)$, so $(\tau_1(x),\tau_2(y)) \in \Omega_{n+1,n+1}$, and from (2.26) we have

$$u\big(\tau_1(x),\tau_2(y)\big) \le \widetilde{v}_n\big(\tau_1(x),\tau_2(y)\big) \le \widetilde{v}_n(x,y). \tag{2.28}$$

Then

$$\widetilde{v}_n(x,y) \le C + \int_x^\infty \int_y^\infty f(s,t)\widetilde{v}_n(s,t)\,dt\,ds + \int_x^\infty \int_y^\infty g(s,t)\omega\big(\widetilde{v}_n(s,t)\big)\,dt\,ds \tag{2.29}$$

and

$$\begin{aligned} \big[\widetilde{v}_n(x,y)\big]_x &\ge -\left[\int_y^\infty f(x,t)\widetilde{v}_n(x,t)\,dt + \int_y^\infty g(x,t)\omega\big(\widetilde{v}_n(x,t)\big)\,dt\right] \\ &\ge -\left[\int_y^\infty f(x,t)\,dt\right]\widetilde{v}_n(x,y) - \left[\int_y^\infty g(x,t)\,dt\right]\omega\big(\widetilde{v}_n(x,y)\big) \\ &= \big[\widetilde{f}_n(x,y)\big]_x \widetilde{v}_n(x,y) + \widetilde{g}_n(x,y)\omega\big(\widetilde{v}_n(x,y)\big), \end{aligned}$$

where $\widetilde{f}_n$, $\widetilde{g}_n$ are defined in (2.25), and obviously $\widetilde{g}_n(x,y) \le 0$.

Furthermore, we have

$$\big[\widetilde{v}_n(x,y)\big]_x - \big[\widetilde{f}_n(x,y)\big]_x \widetilde{v}_n(x,y) \ge \widetilde{g}_n(x,y)\omega\big(\widetilde{v}_n(x,y)\big). \tag{2.30}$$

Multiplying $\exp(\widetilde{f}_n(x,y))$ on both sides of (2.30) yields

$$\big[\widetilde{v}_n(x,y)\exp\big(-\widetilde{f}_n(x,y)\big)\big]_x \ge \widetilde{g}_n(x,y)\exp\big(-\widetilde{f}_n(x,y)\big)\omega\big(\widetilde{v}_n(x,y)\big). \tag{2.31}$$

Setting $x = s$ in (2.31), by $\widetilde{v}_n(\infty, y) = C = \widetilde{b}_n$, an integration for (2.31) with respect to s from x to ∞ yields

$$\widetilde{b}_n - \widetilde{v}_n(x,y)\exp\bigl(-\widetilde{f}_n(x,y)\bigr) \geq \int_x^\infty \widetilde{g}_n(s,y)\exp\bigl(-\widetilde{f}_n(s,y)\bigr)\omega\bigl(\widetilde{v}_n(s,y)\bigr)\,ds, \tag{2.32}$$

which implies

$$\widetilde{v}_n(x,y) \leq \left\{\widetilde{b}_n - \int_x^\infty \widetilde{g}_n(s,y)\exp\bigl(-\widetilde{f}_n(s,y)\bigr)\omega\bigl(\widetilde{v}_n(s,y)\bigr)\,ds\right\}\exp\bigl(\widetilde{f}_n(x,y)\bigr). \tag{2.33}$$

Denote $c(x,y)$ by $\widetilde{b}_n - \int_x^\infty \widetilde{g}_n(s,y)\exp(-\widetilde{f}_n(s,y))\omega(\widetilde{v}_n(s,y))\,ds$. Then

$$\widetilde{v}_n(x,y) \leq c(x,y)\exp\bigl(\widetilde{f}_n(x,y)\bigr). \tag{2.34}$$

By $\widetilde{g}_n(x,y) \leq 0$, we have

$$\begin{aligned}\bigl[c(x,y)\bigr]_x &= \widetilde{g}_n(x,y)\exp\bigl(-\widetilde{f}_n(x,y)\bigr)\omega\bigl(\widetilde{v}_n(x,y)\bigr)\\ &\geq \widetilde{g}_n(x,y)\exp\bigl(-\widetilde{f}_n(x,y)\bigr)\omega\bigl(c(x,y)\exp\bigl(\widetilde{f}_n(x,y)\bigr)\bigr)\\ &\geq \widetilde{g}_n(x,y)\exp\bigl(-\widetilde{f}_n(x,y)\bigr)\omega\bigl(c(x,y)\bigr)\omega\bigl(\exp\bigl(\widetilde{f}_n(x,y)\bigr)\bigr),\end{aligned}$$

that is,

$$\frac{[c(x,y)]_x}{\omega(c(x,y))} \geq \widetilde{g}_n(x,y)\exp\bigl(-\widetilde{f}_n(x,y)\bigr)\omega\bigl(\exp\bigl(\widetilde{f}_n(x,y)\bigr)\bigr). \tag{2.35}$$

Setting $x = s$ in (2.35), and an integration for (2.35) with respect to s from x to ∞ yields

$$\widetilde{G}_n\bigl(c(\infty,y)\bigr) - \widetilde{G}_n\bigl(c(x,y)\bigr) \geq \int_x^\infty \widetilde{g}_n(s,y)\exp\bigl(-\widetilde{f}_n(s,y)\bigr)\omega\bigl(\exp\bigl(\widetilde{f}_n(s,y)\bigr)\bigr)\,ds.$$

Since $c(\infty,y) = \widetilde{b}_n$, $\widetilde{G}_n(\widetilde{b}_n) = 0$, and $\widetilde{G}_n$ is increasing, it follows that

$$c(x,y) \leq \widetilde{G}_n^{-1}\left[-\int_x^\infty \widetilde{g}_n(s,y)\exp\bigl(-\widetilde{f}_n(s,y)\bigr)\omega\bigl(\exp\bigl(\widetilde{f}_n(s,y)\bigr)\bigr)\,ds\right]. \tag{2.36}$$

Combining (2.34) and (2.36), we obtain

$$\begin{aligned}u(x,y) &\leq \widetilde{v}_n(x,y)\\ &\leq G_n^{-1}\left[-\int_x^\infty \widetilde{g}_n(s,y)\exp\bigl(-\widetilde{f}_n(s,y)\bigr)\omega\bigl(\exp\bigl(\widetilde{f}_n(s,y)\bigr)\bigr)\,ds\right]\exp\bigl(\widetilde{f}_n(x,y)\bigr).\end{aligned} \tag{2.37}$$

Especially,

$$\begin{aligned}u(x_n,y_n) \leq \widetilde{v}_n(x_n,y_n) \leq G_n^{-1}&\left[-\int_{x_n}^\infty \widetilde{g}_n(s,y_n)\exp\bigl(-\widetilde{f}_n(s,y_n)\bigr)\omega\bigl(\exp\bigl(\widetilde{f}_n(s,y_n)\bigr)\bigr)\,ds\right]\\ &\times\exp\bigl(\widetilde{f}_n(x_n,y_n)\bigr) = \widetilde{a}_n,\end{aligned} \tag{2.38}$$

where $\widetilde{a}_n$ is defined in (2.25). At the same time, we have

$$u(x_n - 0, y_n - 0) \le \widetilde{v}_n(x_n - 0, y_n - 0) \le \widetilde{a}_n. \tag{2.39}$$

Case 2: If $(x,y) \in \Omega_{n,n}$, then from (2.23), (2.38) and (2.39), under the given conditions $f(x,y) = 0$, $g(x,y) = 0$ for $(x,y) \in \Omega_{i,j}$, $i \neq j$, we have

$$\begin{aligned}
u(x,y) &\le C + \int_x^\infty \int_y^\infty f(s,t) u\big(\tau_1(s), \tau_2(t)\big)\,dt\,ds + \int_x^\infty \int_y^\infty g(s,t)\omega\big(u\big(\tau_1(s), \tau_2(t)\big)\big)\,dt\,ds \\
&\quad + \beta_n u(x_n - 0, y_n - 0), \\
&= \widetilde{b}_n + \int_{x_n}^\infty \int_{y_n}^\infty f(s,t) u\big(\tau_1(s), \tau_2(t)\big)\,ds + \int_{x_n}^\infty \int_{y_n}^\infty g(s,t)\omega\big(u\big(\tau_1(s), \tau_2(t)\big)\big)\,ds \\
&\quad + \int_x^{x_n} \int_y^{y_n} f(s,t) u\big(\tau_1(s), \tau_2(t)\big)\,ds + \int_x^{x_n} \int_y^{y_n} g(s,t)\omega\big(u\big(\tau_1(s), \tau_2(t)\big)\big)\,ds \\
&\quad + \beta_n u(x_n - 0, y_n - 0) \\
&\le \widetilde{a}_n + \beta_n \widetilde{a}_n + \int_x^{x_n} \int_y^{y_n} f(s,t) u\big(\tau_1(s), \tau_2(t)\big)\,ds \\
&\quad + \int_x^{x_n} \int_y^{y_n} g(s,t)\omega\big(u\big(\tau_1(s), \tau_2(t)\big)\big)\,ds \\
&= \widetilde{b}_{n-1} + \int_x^{x_n} \int_y^{y_n} f(s,t) u\big(\tau_1(s), \tau_2(t)\big)\,ds + \int_x^{x_n} \int_y^{y_n} g(s,t)\omega\big(u\big(\tau_1(s), \tau_2(t)\big)\big)\,ds \\
&= \widetilde{v}_{n-1}(x,y).
\end{aligned} \tag{2.40}$$

Then, following in a similar manner as in Case 1, we obtain

$$\begin{aligned}
u(x,y) \le \widetilde{v}_{n-1}(x,y) &\le \widetilde{G}_{n-1}^{-1}\left[- \int_x^{x_n} \widetilde{g}_{n-1}(s,y) \exp\big(-\widetilde{f}_{n-1}(s,y)\big)\omega\big(\exp\big(\widetilde{f}_{n-1}(s,y)\big)\big)\,ds \right] \\
&\quad \times \exp\big(\widetilde{f}_{n-1}(x,y)\big).
\end{aligned} \tag{2.41}$$

Especially,

$$\begin{cases}
u(x_{n-1}, y_{n-1}) \le \widetilde{v}_{n-1}(x_{n-1}, y_{n-1}) \\
\quad \le \widetilde{G}_{n-1}^{-1}[-\int_{x_{n-1}}^{x_n} \widetilde{g}_{n-1}(s,y)\exp(-\widetilde{f}_{n-1}(s,y))\omega(\exp(\widetilde{f}_{n-1}(s,y)))\,ds] \\
\qquad \times \exp(\widetilde{f}_{n-1}(x_{n-1}, y_{n-1})) = \widetilde{a}_{n-1}, \\
u(x_{n-1} - 0, y_{n-1} - 0) \le \widetilde{v}_{n-1}(x_{n-1} - 0, y_{n-1} - 0) \le \widetilde{a}_{n-1},
\end{cases}$$

where $\widetilde{a}_{n-1}$ is defined in (2.25).

Case 3: If for $(x,y) \in \Omega_{j,j}$, $j = i+1, \dots, n$, the following inequalities hold:

$$\begin{cases}
u(x,y) \le \widetilde{v}_j(x,y) \\
\qquad \le G_j^{-1}[-\int_x^{t_{j+1}} \widetilde{g}_j(s,y)\exp(-\widetilde{f}_j(s,y))\omega(\exp(\widetilde{f}_j(s,y)))\,ds] \exp(\widetilde{f}_j(x,y)), \\
u(x_j, y_j) \le \widetilde{v}_j(x_j, y_j) \le G_j^{-1}[-\int_{x_j}^{x_{j+1}} \widetilde{g}_j(s,y)\exp(-\widetilde{f}_j(s,y))\omega(\exp(\widetilde{f}_j(s,y)))\,ds] \\
\qquad \times \exp(\widetilde{f}_j(x_j, y_j)) = \widetilde{a}_j, \\
u(x_j - 0, y_j - 0) \le \widetilde{v}_j(x_j - 0, y_j - 0) \le \widetilde{a}_j,
\end{cases} \tag{2.42}$$

where $\widetilde{a}_j$ is defined in (2.25), then for $(x,y) \in \Omega_{i,i}$, by (2.23), we obtain

$$\begin{aligned}
u(x,y) &\le C + \int_x^\infty \int_y^\infty f(s,t)u\big(\tau_1(s),\tau_2(t)\big)\,dt\,ds + \int_x^\infty \int_y^\infty g(s,t)\omega\big(u\big(\tau_1(s),\tau_2(t)\big)\big)\,dt\,ds \\
&\quad + \sum_{x<x_j<\infty, y<y_j<\infty} \beta_j u(x_j-0,y_j-0) \\
&= C + \int_{x_i}^\infty \int_{y_i}^\infty f(s,t)u\big(\tau_1(s),\tau_2(t)\big)\,dt\,ds + \int_{x_i}^\infty \int_{y_i}^\infty g(s,t)\omega\big(u\big(\tau_1(s),\tau_2(t)\big)\big)\,dt\,ds \\
&\quad + \int_x^{x_i} \int_y^{y_i} f(s,t)u\big(\tau_1(s),\tau_2(t)\big)\,dt\,ds + \int_x^{x_i} \int_y^{y_i} g(s,t)\omega\big(u\big(\tau_1(s),\tau_2(t)\big)\big)\,dt\,ds \\
&\quad + \sum_{x<x_j<\infty, y<y_j<\infty} \beta_j u(x_j-0,y_j-0) \\
&\le \widetilde{a}_i + \beta_i\widetilde{a}_i + \int_x^{x_i} \int_y^{y_i} f(s,t)u\big(\tau_1(s),\tau_2(t)\big)\,dt\,ds \\
&\quad + \int_x^{x_i} \int_y^{y_i} g(s,t)\omega\big(u\big(\tau_1(s),\tau_2(t)\big)\big)\,dt\,ds \\
&= \widetilde{b}_{i-1} + \int_x^{x_i} \int_y^{y_i} f(s,t)u\big(\tau_1(s),\tau_2(t)\big)\,dt\,ds + \int_x^{x_i} \int_y^{y_i} g(s,t)\omega\big(u\big(\tau_1(s),\tau_2(t)\big)\big)\,dt\,ds \\
&= \widetilde{v}_{i-1}(x,y).
\end{aligned} \tag{2.43}$$

Then, following in a similar manner as in Case 1, we obtain

$$\begin{aligned}
u(x,y) &\le \widetilde{v}_{i-1}(x,y) \le \widetilde{G}_{i-1}^{-1}\biggl[-\int_x^{t_i} \widetilde{g}_{i-1}(s,y)\exp\bigl(-\widetilde{f}_{i-1}(s,y)\bigr)\omega\bigl(\exp\bigl(\widetilde{f}_{i-1}(s,y)\bigr)\bigr)\,ds\biggr] \\
&\quad \times \exp\bigl(\widetilde{f}_{i-1}(x,y)\bigr),
\end{aligned} \tag{2.44}$$

and the proof is complete. □

Remark 2.4 If we take $u(x,y) = \phi(\widetilde{u}(x,y))$ and, furthermore, $\phi(u(x,y)) = u^q(x,y)$, or take $\omega(u) = u^q$ in Theorem 2.2, then we can obtain corresponding corollaries, which are omitted here.

3 Application

In this section, we present one application to illustrate the validity of our results in deriving explicit bounds for the discontinuous solutions of certain integral equations.

Consider an integral equation of the form

$$u(t) = u(\infty) + \int_t^\infty M\big(s,t,u\big(\tau(s)\big)\big)\,ds + \sum_{t<t_j<\infty} \beta_j u(t_j-0), \quad \forall t \in \mathbb{R}_+, \tag{3.1}$$

where u is a continuous function defined on $\mathbb{R}_+$ with the first kind of discontinuities at the points t_i, $i = 1,2,\ldots,n$, and $0 \le t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = \infty$. $\tau \in C(\mathbb{R}_+,\mathbb{R}_+)$ with $\tau(t) \le t$, and $\tau(t) \ge t_i$ for $\forall t \in [t_i,t_{i+1})$, $i = 0,1,\ldots,n$. $\beta_j \ge 0$, $j = 1,2,\ldots,n$, and $M \in C(\mathbb{R}_+^2 \times \mathbb{R},\mathbb{R})$.

Theorem 3.1 *Assume that $u(t)$ is a solution of Eq.* (3.1), *and*

$$\begin{cases} |u(\infty)| \le C, \\ |M(s,t,u)| \le f(s,t)|u| + g(s,t)|u|^q, \\ \omega(z) = z^q, \quad z \in \mathbb{R}_+, \end{cases} \tag{3.2}$$

where q, C are constants with $0 < q < 1$, $C \ge 0$, $f, g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $f_2', g_2' \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$, and $f(s,t)$, $g(s,t)$ are decreasing in t for every fixed s. Then we have

$$\begin{aligned} \big|u(t)\big| &\le G_{i-1}^{-1}\left[-\int_t^{t_i} \widetilde{g}_{i-1}(s)\exp\big(-\widetilde{f}_{i-1}(s)\big)\omega\big(\exp\big(\widetilde{f}_{i-1}(s)\big)\big)\,ds\right]\exp\big(\widetilde{f}_{i-1}(t)\big), \\ &\qquad t \in [t_{i-1}, t_i), i = 1, 2, \ldots, n+1, \end{aligned} \tag{3.3}$$

where G_i, b_i, a_i, $\widetilde{f}_i$, $\widetilde{g}_i$ are defined the same as in (2.3), *and $t_{n+1} = \infty$.*

Proof From (3.1) we have

$$\begin{aligned} \big|u(t)\big| &\le \big|u(\infty)\big| + \int_t^\infty \big|M\big(s,t,u\big(\tau(s)\big)\big)\big|\,ds + \sum_{t<t_j<\infty} \beta_j\big|u(t_j - 0)\big| \\ &\le C + \int_t^\infty \big[f(s,t)\big|u\big(\tau(s)\big)\big| + g(s,t)\big|u\big(\tau(s)\big)\big|^q\big]\,ds + \sum_{t<t_j<\infty} \beta_j\big|u(t_j - 0)\big| \\ &= C + \int_t^\infty \big[f(s,t)\big|u\big(\tau(s)\big)\big| + g(s,t)\omega\big(\big|u\big(\tau(s)\big)\big|\big)\big]\,ds + \sum_{t<t_j<\infty} \beta_j\big|u(t_j - 0)\big|. \end{aligned}$$

Then a suitable application of Theorem 2.1 yields the desired result. □

Theorem 3.2 *Under the conditions of Theorem* 3.1, *furthermore, we have*

$$\begin{aligned} \big|u(t)\big| &\le \left\{(1-q)\left[-\int_t^{t_i} \widetilde{g}_{i-1}(s)\exp\big(-\widetilde{f}_{i-1}(s)\big)\exp\big(q\widetilde{f}_{i-1}(s)\big)\,ds\right] + b_{i-1}^{1-q}\right\}^{\frac{1}{1-q}} \\ &\qquad \times \exp\big(\widetilde{f}_{i-1}(t)\big). \end{aligned} \tag{3.4}$$

Proof As long as we notice $\omega(z) = z^q$, and

$$G_{i-1}(z) = \int_{b_{i-1}}^z \frac{1}{\omega(s)}\,ds = \int_{b_{i-1}}^z \frac{1}{s^q}\,ds = \frac{1}{1-q}\big(z^{1-q} - b_{i-1}^{1-q}\big), \quad i = 1, 2, \ldots, n+1, \tag{3.5}$$

combining (3.5) and Theorem 3.1, we can easily deduce the desired result. □

Competing interests
The author declares that they have no competing interests.

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