



Spectral analysis of quantum Dirac systems

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Abstract

In this study, we establish the quantum calculus analogue of the classical Dirac system. Moreover, we investigate the Jost solution, eigenvalues, spectral singularities and some quantitative properties of the spectrum of this new system. ©2017 All rights reserved.

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1. Introduction

Spectral theory of operators is a branch of functional analysis which has a number of significant applications from quantum physics to engineering. Spectral analysis of differential operators can be traced back to Naimark [16]. Basic problems in the spectral theory of Sturm-Liouville and one dimensional Dirac type operators were discussed in the book of Levitan and Sargsjan [14]. Also, differential operators with spectral singularities have been main topic of various studies [12, 15].

Besides differential operators that we have mentioned up to here, difference operators have also become a rising research area for its possible applications to real life problems [4, 11]. Krall et al. studied some spectral properties of the non-selfadjoint discrete Schrödinger operator by taking into account the principal functions and spectral singularities [13]. Adivar and Bairamov investigated the spectrum, the spectral singularities, and the properties of the principal vectors corresponding to the spectral singularities of discrete Sturm-Liouville and discrete Dirac operators [1]. In [6], discrete Dirac operator was considered in terms of spectrum and spectral expansion.

Succeeding the developments in ordinary and discrete calculus, some new topics of quantum calculus have been introduced by Kac and Cheung [10]. This new point of view to concepts of ordinary calculus like q -derivative, q -integration etc. required the redefinition of classical equations. In particular, some problems of q -difference equations have been studied by Berg et al. [7, 8]. Adivar and Bohner introduced the quantum analogue of the classical Sturm-Liouville equation and studied the spectral properties of this

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equation [2]. In their study, they proved that the Jost solution of this equation has analytic continuation to the lower half-plane and the finiteness of the eigenvalues and spectral singularities of the quantum difference equation is achieved as a result of this analytic continuation. Related to the study [2], principal functions of the second order quantum difference equation was treated in [3]. As a continuation of the papers [2] and [3], the presence of the spectral parameter not only in the quantum difference equation but also in the boundary condition has been considered by Aygar and Bohner [5].

The present paper was motivated by the above mentioned studies. In this paper, we let $q > 1$ and use the notation

$$q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\},$$

where \mathbb{N}_0 denotes the set of nonnegative integers. We define the Hilbert space $l_2(q^{\mathbb{N}}, \mathbb{C}^2)$ consisting of all

sequences $y = \{y(t)\} = \left\{ \begin{array}{c} y^{(1)}(t) \\ y^{(2)}(t) \end{array} \right\}$ with the inner product,

$$\langle y, f \rangle_{(q^{\mathbb{N}}, \mathbb{C}^2)} := \sum_{t \in q^{\mathbb{N}}} (y^{(1)}(t) \overline{f^{(1)}(t)} + y^{(2)}(t) \overline{f^{(2)}(t)}), \quad y, f : q^{\mathbb{N}} \rightarrow \mathbb{C},$$

and the norm

$$\|y(t)\|_{(q^{\mathbb{N}}, \mathbb{C}^2)} = \left(\sum_{t \in q^{\mathbb{N}}} (|y^{(1)}(t)|^2 + |y^{(2)}(t)|^2) \right)^{\frac{1}{2}} \quad \text{for } y : q^{\mathbb{N}} \rightarrow \mathbb{C}.$$

By using the definition of quantum derivative [2, 8], we construct the system of equations

$$\begin{cases} (y^{(2)}(t))^{\Delta} + p(t)y^{(1)}(t) = \frac{\lambda y^{(1)}(t)}{\mu(t)}, \\ -(y^{(1)}(\frac{t}{q}))^{\Delta} + r(t)y^{(2)}(t) = \frac{\lambda y^{(2)}(t)}{\mu(t)}, \end{cases} \quad t \in q^{\mathbb{N}}, \quad (1.1)$$

which is quantum analogue of the well-known Dirac system

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & q(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

introduced by Levitan and Sargsjan [14]. Therefore the system (1.1) is called a quantum Dirac system.

The purpose of this paper is investigation of the spectral properties of the non-selfadjoint boundary value problem (BVP)

$$\begin{cases} y^{(2)}(qt) - y^{(2)}(t) + p(t)\mu(t)y^{(1)}(t) = \lambda y^{(1)}(t), \\ y^{(1)}(\frac{t}{q}) - y^{(1)}(t) + r(t)\mu(t)y^{(2)}(t) = \lambda y^{(2)}(t), \end{cases} \quad t \in q^{\mathbb{N}}, \quad (1.2)$$

$$y^{(1)}(1) = 0, \quad (1.3)$$

where $\{p(t)\}_{t \in q^{\mathbb{N}}}$ and $\{r(t)\}_{t \in q^{\mathbb{N}}}$ are complex sequences, $\mu(t)$ is graininess function and λ is an eigenparameter. In essence, our ideas come from [1, 2]. In the next section, we find the Jost solution and Jost function of (1.2)-(1.3). Finally, we discuss the eigenvalues and the spectral singularities of the BVP (1.2)-(1.3) using the classical definitions of spectral analysis and get some quantitative properties of them under the Pavlov's condition

$$\sup_{t \in q^{\mathbb{N}}} \left\{ \exp \left[\varepsilon \left(\frac{\ln t}{\ln q} \right)^{\delta} \right] (|p(t)\mu(t)| + |r(t)\mu(t)|) \right\} < \infty, \quad (1.4)$$

where $\varepsilon > 0$, $\frac{1}{2} \leq \delta \leq 1$.

2. Jost solution and Jost function

Suppose that the condition

$$\sum_{t \in q^{\mathbb{N}}} \frac{\ln t}{\ln q} (|p(t)\mu(t)| + |r(t)\mu(t)|) < \infty, \quad (2.1)$$

is satisfied.

Theorem 2.1. *Under the condition (2.1), the BVP (1.2)-(1.3) has unique solution*

$$f^{(1)}(1, z) = e^{i\frac{z}{2}} \left[1 + \sum_{r \in q^{\mathbb{N}_0}} A_{1r}^{11} e^{i\frac{\ln r}{\ln q} z} \right] - i \sum_{r \in q^{\mathbb{N}_0}} A_{1r}^{12} e^{i\frac{\ln r}{\ln q} z}, \quad (2.2)$$

$$f(t, z) = \begin{pmatrix} f^{(1)}(t, z) \\ f^{(2)}(t, z) \end{pmatrix} = \left\{ \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{r \in q^{\mathbb{N}_0}} A_{tr} e^{i\frac{\ln r}{\ln q} z} \right] \begin{pmatrix} e^{i\frac{z}{2}} \\ -i \end{pmatrix} e^{i\frac{\ln t}{\ln q} z} \right\}, \quad (2.3)$$

for $\lambda = 2 \sin \frac{z}{2}$, where $z \in \overline{\mathbb{C}}_+ := \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$ and

$$A_{tr} = \begin{pmatrix} A_{tr}^{11} & A_{tr}^{12} \\ A_{tr}^{21} & A_{tr}^{22} \end{pmatrix}.$$

Proof. Substituting the function (2.3) in (1.2), we get

$$\begin{aligned} A_{tq}^{12} &= - \sum_{s \in [qt, \infty) \cap q^{\mathbb{N}}} p(s)\mu(s) + r(s)\mu(s), \\ A_{tq}^{11} &= \sum_{s \in [qt, \infty) \cap q^{\mathbb{N}}} p(s)\mu(s) A_{sq}^{12}, \\ A_{tq}^{22} &= \sum_{s \in [t, \infty) \cap q^{\mathbb{N}}} p(s)\mu(s) A_{sq}^{12}, \\ A_{tq^2}^{12} &= -r(qt)\mu(qt)A_{tq}^{11} - \sum_{s \in [qt, \infty) \cap q^{\mathbb{N}}} [p(s)\mu(s) + r(qs)\mu(qs)] A_{sq}^{11}, \\ A_{tq}^{21} &= p(t)\mu(t)A_{tq}^{11} + A_{tq}^{12} - A_{tq^2}^{11}, \\ A_{tq^2}^{11} &= -A_{qt, q}^{22} - \sum_{s \in [qt, \infty) \cap q^{\mathbb{N}}} r(s)\mu(s) - p(s)\mu(s) A_{s, q^2}^{12}, \\ A_{t, q^2}^{22} &= A_{tq}^{22} - A_{tq}^{11} + A_{\frac{t}{q}, q^2}^{22} + r(t)\mu(t)A_{tq}^{21}, \end{aligned}$$

for $r \geq q^3$

$$\begin{aligned} A_{tr}^{12} &= A_{tr}^{21} - \sum_{s \in [t, \infty) \cap q^{\mathbb{N}}} p(s)\mu(s) A_{sr}^{11} - r(s)\mu(s) A_{sr}^{22}, \\ A_{tr}^{11} &= -A_{tq, \frac{t}{q}}^{22} - \sum_{s \in [qt, \infty) \cap q^{\mathbb{N}}} p(s)\mu(s) A_{sr}^{11} - r(s)\mu(s) A_{s, \frac{t}{q}}^{22}, \\ A_{tr}^{22} &= A_{qt, \frac{t}{q}}^{22} + p(t)\mu(t)A_{tr}^{12} - A_{t, \frac{t}{q}}^{11} + A_{tr}^{11}, \\ A_{tr}^{21} &= p(t)\mu(t)A_{tr}^{11} + A_{tr}^{12} + A_{qt, \frac{t}{q}}^{21} - A_{t, qr}^{12}. \end{aligned}$$

Due to the condition (2.1), the series in the definition of A_{tr}^{ij} ($i, j = 1, 2$) are absolutely convergent. Therefore, A_{tr}^{ij} ($i, j = 1, 2$) can be uniquely determined by $p(t)$, $r(t)$ and $\mu(t)$ ($t \in q^{\mathbb{N}}$), i.e., the BVP (1.2)-(1.3) for $\lambda = 2 \sin \frac{z}{2}$ has the solution given by (2.2) and (2.3). \square

The solution f is called Jost solution of the BVP (1.2)-(1.3). Using the equalities for A_{tr}^{ij} ($i, j = 1, 2$) given in Theorem 2.1, one can obtain the inequality

$$|A_{tr}^{ij}| \leq C \sum_{s \in [tq^{\lfloor \frac{\ln r}{2 \ln q} \rfloor}, \infty) \cap q^{\mathbb{N}}} (|p(t)\mu(t)| + |r(t)\mu(t)|), \quad i, j = 1, 2, \quad (2.4)$$

by induction, where $\lfloor \frac{\ln r}{2 \ln q} \rfloor$ is the integer part of $\frac{\ln r}{2 \ln q}$, $C > 0$ is a constant and $t, r \in q^{\mathbb{N}}$. Therefore the function f is analytic in C_+ , continuous in \overline{C}_+ and $f(z) = f(z + 4\pi)$.

3. Eigenvalues and spectral singularities

Let $\varphi(z) = \tilde{\varphi}(\lambda) = \{\tilde{\varphi}(t, \lambda)\}$, $t \in q^{\mathbb{N}}$ be the solution of the BVP (1.2)-(1.3) subject to the initial conditions

$$\varphi^{(1)}(1, z) = 0, \quad \varphi^{(2)}(q, z) = 1.$$

Then φ is entire function and

$$\varphi(z) = \varphi(z + 4\pi).$$

Definition 3.1. The Wronskian of two solutions $y = \{y(t, z)\}_{t \in q^{\mathbb{N}}}$ and $u = \{u(t, z)\}_{t \in q^{\mathbb{N}}}$ of the BVP (1.2)-(1.3) is, for $t \in q^{\mathbb{N}}$, defined by

$$W[y, u](t) := \{y(t, z)u(qt, z) - y(qt, z)u(t, z)\}.$$

Let us define the semi-strips $P_0 := \{z : z \in \mathbb{C}, z = \xi + i\tau, 0 \leq \xi \leq 4\pi, \tau > 0\}$ and $P := P_0 \cup [0, 4\pi]$. Then, from Definition 3.1 for $z \in \overline{C}_+$, we get

$$\begin{aligned} W[f(z), \varphi(z)] &= [f^{(1)}(t, z)\varphi^{(2)}(qt, z) - f^{(2)}(qt, z)\varphi^{(1)}(t, z)] \\ &= f^{(1)}(1, z). \end{aligned}$$

We will denote respectively the set of all eigenvalues and spectral singularities of the BVP (1.2)-(1.3) by σ_d and σ_{ss} . It is obvious that

$$\sigma_d = \left\{ \lambda : \lambda = 2 \sin \frac{z}{2}, z \in P_0, f^{(1)}(1, z) = 0 \right\}, \quad (3.1)$$

$$\sigma_{ss} = \left\{ \lambda : \lambda = 2 \sin \frac{z}{2}, z \in [0, 4\pi], f^{(1)}(1, z) = 0 \right\} \setminus \{0\}. \quad (3.2)$$

Let

$$F(z) := f^{(1)}(1, z)e^{-i\frac{z}{2}}, \quad (3.3)$$

then the function F is analytic in C_+ , continuous in \overline{C}_+ . Moreover, the functions $f^{(1)}$ and F have the same zeros in the semi-strip P . Clearly we can write

$$F(z) = 1 + \sum_{r \in q^{\mathbb{N}_0}} A_{1r}^{11} e^{i\frac{\ln r}{\ln q} z} - i \sum_{r \in q^{\mathbb{N}_0}} A_{1r}^{12} e^{i(\frac{\ln r}{\ln q} - \frac{1}{2})z}, \quad (3.4)$$

and hence,

$$F(z + 4\pi) = F(z).$$

Using (3.1), (3.2), (3.3), we find the eigenvalues and spectral singularities of the BVP (1.2)-(1.3) as

$$\sigma_d = \left\{ \lambda : \lambda = 2 \sin \frac{z}{2}, z \in P_0, F(z) = 0 \right\}, \quad (3.5)$$

$$\sigma_{ss} = \left\{ \lambda : \lambda = 2 \sin \frac{z}{2}, z \in [0, 4\pi], F(z) = 0 \right\} \setminus \{0\}. \quad (3.6)$$

Definition 3.2. The multiplicity of a zero of F in P is called the multiplicity of the corresponding eigenvalue or spectral singularity of the BVP (1.2)-(1.3).

From (3.5) and (3.6) we conclude that, in order to investigate the quantitative properties of the sets σ_d and σ_{ss} , we require to discuss the quantitative properties of the zeros of F in P .

Define

$$\begin{aligned} Q_1 &:= \{z : z \in P_0, F(z) = 0\}, \\ Q_2 &:= \{z : z \in [0, 4\pi], F(z) = 0\}. \end{aligned} \quad (3.7)$$

We also denote the set of all limit points of Q_1 by Q_3 and the set of all zeros of F with infinite multiplicity by Q_4 . So from (3.5), (3.6) and (3.7), we conclude the following:

$$\begin{aligned} \sigma_d &= \left\{ \lambda : \lambda = 2 \sin \frac{z}{2}, z \in Q_1 \right\}, \\ \sigma_{ss} &= \left\{ \lambda : \lambda = 2 \sin \frac{z}{2}, z \in Q_2 \right\} \setminus \{0\}. \end{aligned} \quad (3.8)$$

Theorem 3.3. If (2.1) holds, then

- (i) The set Q_1 is bounded and countable.
- (ii) $Q_1 \cap Q_3 = \emptyset$, $Q_1 \cap Q_4 = \emptyset$.
- (iii) The set Q_2 is compact and $\mu(Q_2) = 0$, where μ is Lebesgue measure in the real axis.
- (iv) $Q_3 \subset Q_2$, $Q_4 \subset Q_2$, $\mu(Q_3) = \mu(Q_4) = 0$.
- (v) $Q_3 \subset Q_4$.

Proof. From (2.4) and (3.4), we have

$$F(z) = 1 + o(1), \quad z \in P, \quad |z| \rightarrow \infty. \quad (3.9)$$

Equation (3.9) shows that the set Q_1 is bounded. Since $F(z)$ is analytic in \mathbb{C}_+ and is a 4π periodic function, we find that Q_1 has at most a countable number of elements. This proves (i). From the boundary uniqueness theorems of analytic functions, we obtain (ii)-(iv) [9]. Using the continuity of all derivatives of F on $[0, 4\pi]$, we get (v). \square

From Theorem 3.3 and (3.8), we have the following.

Theorem 3.4. Under the condition (2.1),

- (i) the set of eigenvalues of the BVP (1.2)-(1.3) is bounded, has at most a countable number of elements, and its limit points can lie only in $[-2, 2]$.
- (ii) $\sigma_{ss} \subset [-2, 2]$ and $\mu(\sigma_{ss}) = 0$.

In preparation for the next result, we assume that the complex sequences $\{p(t)\}_{t \in \mathbb{q}^N}$ and $\{r(t)\}_{t \in \mathbb{q}^N}$ satisfy

$$\sum_{t \in \mathbb{q}^N} \exp \left(\varepsilon \frac{\ln t}{\ln q} \right) (|p(t)\mu(t)| + |r(t)\mu(t)|) < \infty \quad (3.10)$$

for some $\varepsilon > 0$.

Note that for $\delta = 1$, the condition (1.4) reduces to (3.10).

Theorem 3.5. Under condition (3.10), the BVP (1.2)-(1.3) has a finite number of eigenvalues and spectral singularities with a finite multiplicity.

Proof. Using (2.4) and (3.10), we obtain that

$$|A(t, r)| \leq C \exp\left[-\frac{\varepsilon}{2} \frac{\ln r}{\ln q}\right], \quad t = 1, \quad r \in q^{\mathbb{N}_0}, \quad (3.11)$$

where $C > 0$ is a constant. Using (3.4) and (3.11), we observe that the function F has an analytic continuation to the half-plane $\operatorname{Im} z > -\frac{\varepsilon}{2}$. Since F is a 4π periodic function, the limit points of its zeros in P cannot lie in $[0, 4\pi]$. Also, we obtain that the bounded sets Q_1 and Q_2 have a finite number of elements from Theorem 3.3. Using the analyticity of F in the region $\operatorname{Im} z > -\frac{\varepsilon}{2}$, we find that all zeros of F in P have a finite multiplicity. Therefore, we get the finiteness of the eigenvalues and the spectral singularities of the BVP (1.2)-(1.3). \square

Now let us suppose that

$$\sum_{t \in q^{\mathbb{N}}} \exp\left[\varepsilon \left(\frac{\ln t}{\ln q}\right)^\delta\right] (|p(t)\mu(t)| + |r(t)\mu(t)|) < \infty, \quad \varepsilon > 0, \quad \frac{1}{2} \leq \delta < 1, \quad (3.12)$$

which is weaker than (3.10). It is seen that the condition (3.10) undertakes the analytic continuation of F from the real axis to the lower half-plane. So the finiteness of the eigenvalues and the spectral singularities of the BVP (1.2)-(1.3) are achieved as a result of this analytic continuation. It is evident that, under the condition (3.12), the function F is analytic in \mathbb{C}_+ and infinitely differentiable on the real axis. However F does not have an analytic continuation from the real axis to the lower half-plane. Therefore, under the condition (3.12), the finiteness of the eigenvalues and the spectral singularities of the BVP (1.2)-(1.3) cannot be proved in a way similar to Theorem 3.5.

Under the condition (3.12), to show that the eigenvalues and the spectral singularities of the BVP (1.2)-(1.3) are of finite number, we will use the following:

Theorem 3.6 ([6]). *Let us assume that the 4π periodic function g is analytic in \mathbb{C}_+ , all of its derivatives are continuous in $\overline{\mathbb{C}}_+$, and*

$$\sup_{z \in P} |g^{(k)}(z)| \leq \eta_k, \quad k \in \mathbb{N} \cup \{0\}.$$

If the set $G \subset [0, 4\pi]$ with Lebesgue measure zero is the set of all zeros of the function g with infinite multiplicity in P , and if

$$\int_0^\omega \ln T(s) d\mu(G_s) = -\infty,$$

where $T(s) = \inf_k \frac{\eta_k s^k}{k!}$ and $\mu(G_s)$ is the Lebesgue measure of s -neighborhood of G and $\omega \in (0, 4\pi)$ is an arbitrary constant, then $g \equiv 0$ in $\overline{\mathbb{C}}_+$.

It follows from (2.4) and (3.4) that

$$|F^{(k)}(z)| \leq \eta_k, \quad k \in \mathbb{N} \cup \{0\},$$

where

$$\eta_k = B \sum_{r \in q^{\mathbb{N}_0}} \left(\frac{\ln r}{\ln q}\right)^k \exp\left(-\frac{\varepsilon}{2} \left(\frac{\ln r}{\ln q}\right)^\delta\right), \quad k \in \mathbb{N}_0,$$

and $B > 0$ is a constant.

We can get the estimate for η_k

$$\eta_k \leq B \int_0^\infty x^k \exp\left(-\frac{\varepsilon}{2} x^\delta\right) dx \leq D d^k k! k^{\frac{1-\delta}{\delta}}, \quad (3.13)$$

where D and d are constants depending on B , ε and δ .

Theorem 3.7. If (3.12) holds, then $Q_4 = \emptyset$.

Proof. Using Theorem 3.6, we obtain that the function F satisfies the condition

$$\int_0^\omega \ln T(s) d\mu(Q_{4,s}) > -\infty, \quad (3.14)$$

where $T(s) = \inf_k \frac{\eta_k s^k}{k!}$, $k \in \mathbb{N}_0$, $\mu(Q_{4,s})$ is the Lebesgue measure of the s -neighborhood of Q_4 , and η_k is defined by (3.13). Now we get

$$T(s) \leq D \exp \left\{ -\frac{1-\delta}{\delta} e^{-1} d^{-\frac{\delta}{1-\delta}} s^{-\frac{\delta}{1-\delta}} \right\}, \quad (3.15)$$

by (3.13). It follows from (3.14) and (3.15) that

$$\int_0^\omega s^{-\frac{\delta}{1-\delta}} d\mu(Q_{4,s}) < \infty. \quad (3.16)$$

Since $\frac{\delta}{1-\delta} \geq 1$, from (3.16), we obtain that, for arbitrary s , $\mu(Q_{4,s}) = 0$ or $Q_4 = \emptyset$. \square

Theorem 3.8. Assume (3.12) holds. Then, the BVP (1.2)-(1.3) has a finite number of eigenvalues and spectral singularities with a finite multiplicity.

Proof. In order to prove the theorem, we need to show that the function F has a finite number of zeros with finite multiplicities in the region P .

Using Theorem 3.3 and Theorem 3.7, we conclude that the set $Q_3 = \emptyset$. Thus, the bounded sets Q_1 and Q_2 do not have accumulation points, that is to say, the function F has only a finite number of zeros in P . Since $Q_4 = \emptyset$, these zeros are of finite multiplicity. \square

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