Review

Global chaos synchronization for four-scroll attractor by nonlinear control

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This paper investigates the global synchronization method of two identical systems and two different chaotic systems. The proposed method is effective and convenient to synchronize two identical systems and two different chaotic systems by using nonlinear control functions. The method has been applied successfully to make two identical four-scroll attractor and also two different chaotic systems; four-scroll attractor and Lorenz system globally synchronized. Numerical simulations are given to validate the proposed synchronization methods.

Key words: Chaotic system, global synchronization, four-scroll attractor, Lorenz system, Lyapunov function.

INTRODUCTION

Chaos has been developed and thoroughly studied over the past two decades. A chaotic system is a nonlinear deterministic system that displays complex and unpredictable behavior. The sensitive dependence on the initial conditions and on the system's parameter variation is a prominent characteristic of chaotic behavior. Research efforts have investigated chaos control and chaos synchronization problems in many physical chaotic systems.

Chaos control and synchronization have attracted a great deal of attention from various fields since Huber published the first paper on chaos control in 1989 (Hubler, 1989). Over the last decades, many methods and techniques have been developed, such as OGY method (Ott et al., 1990), PC method (Pecora and Carroll, 1990; Carroll and Pecora, 1991), feedback approach, adaptive method, nonlinear control, active control, and backstepping design technique (Chen and Dong, 1989; Wang et al., 2001; Huang et al., 2004; Elabbasy et al., 2004; Liao and Tsai, 2000; Jiang et al., 2002) etc.

In 1963, Lorenz found the first classical chaotic attractor. In 1999, Chen found another similar but not topological equivalent chaotic attractor the Chen attractor (Chen et al., 1999). In 2002, Lü and Chen found a new

critical chaotic system (Lü and Chen 2002), bearing the name of Lü system. It is noticed that these systems can be classified into three different types by the definition of Vaněček and Čelikovsky (Vaněček and Čelikovsky, 1996): the Lorenz system (Sparrow, 1982; Lorenze, 1963) satisfies the condition $a_{12} a_{21} > 0$, the Chen system satisfies $a_{12} a_{21} < 0$ and the Lü system (Lü and Chen, 2002) satisfies $a_{12} a_{21} = 0$, where a_{12} and a_{21} are the corresponding elements in the linear part matrix $A = \left[a_{ij} \right]_{3 \times 3}$ of the system.

In this paper we study the synchronization of the system (Liu and Chen, 2004) which is described by the following equation:

$$\dot{x} = ax - yz$$
$$\dot{y} = -by + xz \qquad (1)$$
$$\dot{z} = -cz + xy$$

Where a, b and c are positive control parameters. This system exhibits a chaotic attractor at the parameter values a = 0.4, b = 12 and c = 5. This system bridges the gap between the Lorenz and Chen attractors (Lü et al., 2002).

Differing from other known similar systems, system (1) has five equilibria, and does not have Hopf and pitch bifurcations (Lü et al., 2004). Of most interesting is the

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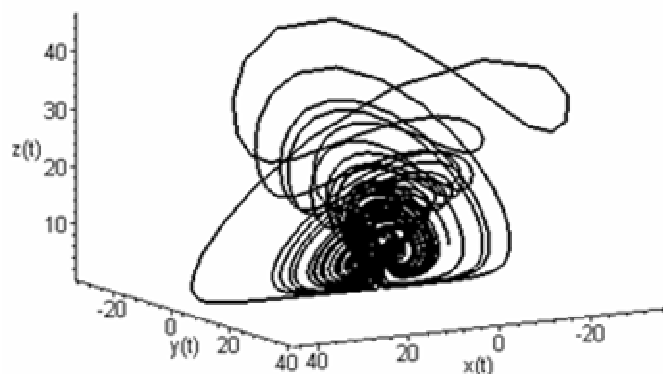*Corresponding Author's E-Mail: emelabbasy@mans.edu.eg.

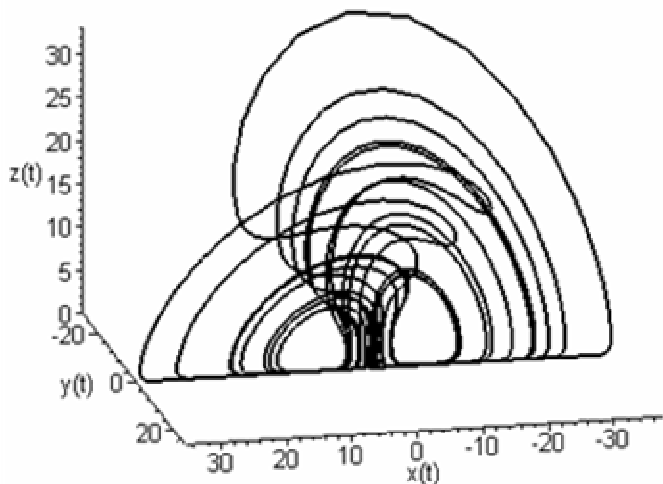**Figure 1.** Shows the chaotic attractor of two-scroll chaotic attractor at a = 4.5, b = 12 and c = 5.

Figure 2. Shows the chaotic attractor of four-scroll chaotic attractor at a = 0.4, b = 12 and c = 5.

observation that this chaotic system not only can display a two-scroll chaotic attractor when a = 4.5, b = 12 and c = 5 (see Figure 1), but also can display a four-scroll chaotic attractor when a = 0.4, b = 12 and c = 5 (see Figure 2). Although, system (1) exhibits by computer simulation four-scroll chaotic attractor for certain values of the parameters no complete answer for the following challenging question. Is it true that a three-dimensional smooth quadratic autonomous system can generate a truly single four-scroll Attractor? On the other hand, however, from the engineering applications point of view, even a numerical four-scroll chaotic attractor can be quite useful due to its strong randomness and complex topological properties with a wider power spectrum. It implies that one can take advantage of these phenomena and use this kind of numerical chaotic signals for better and wider use in digital or electronic devices for some good engineering applications such as random signal generation and secure communication.

The paper is organized as follows. In Section 2, a modified control scheme based on Lyapunov stability theory is proposed. In Section 3, the control method is applied to synchronize two identical systems of four-scroll attractor and numerical simulations are presented to show the effectiveness of the proposed method. In Section 4, the proposed scheme is applied to synchronize two different chaotic systems (four-scroll attractor and Lorenz system). Also numerical simulations are presented in order to validate the proposed synchronization approach. Finally, in Section 5 the conclusion of the paper is given.

Modification based on Lyapunov stability theory to design a controller

Consider the following chaotic system described by

$$\dot{X} = AX + f(X) \qquad \text{(D)}$$

Where $X(t) \in R^n$ is a n-dimensional state vector of the system, $A \in R^{n \times n}$ is the matrix of the system parameter, and $f : R^n \rightarrow R^n$ is the nonlinear part of the system. Equation (1) is considered as a drive system.

The controller $U \in R^n$ is added into the system (D) to get the new system.

$$\dot{Y} = BY + g(Y) + U \qquad \text{(R)}$$

Where $Y(t) \in R^n$ denotes the state vector of the response system, $B \in R^{n \times n}$ is the matrix of the response system parameter, and $g : R^n \rightarrow R^n$ is the nonlinear part of the response system. The system (R) is known as the response system. If $A = B$ and $g(Y) = f(X)$, then X and Y are the states of two identical chaotic systems. If $A \neq B$ and $g(Y) \neq f(X)$, then X and Y are the states of two different chaotic systems.

The synchronization problem is to design a controller U, which synchronizes the states of both the drive and response systems. The dynamics of synchronization errors can be expressed

$$\dot{e} = BY + g(Y) - AX - f(X) + U \qquad \text{(E)}$$

Where $e = Y - X$. The aim of synchronization is to make $\lim_{t \to \infty} \|e(t)\| = 0$.

The problem of synchronization between the drive and response systems can be translated into a problem of

how to realize the asymptotic stabilization of the system (E). So the objective is to design a controller U for stabilizing the error dynamical system (E) at origin.

If we take the Lyapunov function to be $V(e) = e^T Pe$, and the matrix P is a positive definite matrix, then $V(e)$ is a positive definite function. Assuming that the parameters of the drive and response systems are known and the states of both systems are measurable. One may achieve the synchronization by selecting a nonlinear controller U to make $\dfrac{dV}{dt} = -e^T Qe$ be a negative definite function, i.e., the matrix Q is also a positive definite matrix. Then the states of the response system and drive system are globally asymptotically synchronized.

Synchronization of tow identical four-scroll chaotic attractor

In this section, we apply the technique detailed in the previous section to four scroll attractor (Liu et al., 2004)

$$\dot{x} = ax - yz$$
$$\dot{y} = -by + xz \qquad (2)$$
$$\dot{z} = -cz + xy$$

Where a > 0, b > 0, c > 0 and b + c > a.

In order to observe the synchronization behavior in the four-scroll system, we have two four-scroll systems where the drive system with three state variables denoted by the subscript 1 drives the response system having identical equations denoted by the subscript 2. However, the initial condition on the drive system is different from that of the response system. The two four-scroll systems are described, respectively, by the following equations:

$$\dot{x}_1 = ax_1 - y_1 z_1$$
$$\dot{y}_1 = -by_1 + x_1 z_1 \qquad (3)$$
$$\dot{z}_1 = -cz_1 + x_1 y_1$$

and

$$\dot{x}_2 = ax_2 - y_2 z_2 + u_1(t)$$
$$\dot{y}_2 = -by_2 + x_2 z_2 + u_2(t) \qquad (4)$$
$$\dot{z}_2 = -cz_2 + x_2 y_2 + u_3(t)$$

Where $U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ is the nonlinear controller to be designed. The aim of this section is to determine the controller U for the global synchronization of two identical four-scroll chaotic attractors. For this purpose, the error dynamical system between the drive system (3) and response system (4) can be expressed by

$$\dot{e}_x = ae_x - e_y e_z - z_1 e_y - y_1 e_z + u_1(t)$$
$$\dot{e}_y = -be_y + e_x e_z + z_1 e_x + x_1 e_z + u_2(t) \qquad (5)$$
$$\dot{e}_z = -ce_z + e_x e_y + y_1 e_x + x_1 e_y + u_3(t)$$

Where

$$e_x = x_2 - x_1, e_y = y_2 - y_1, \ e_z = z_2 - z_1 \ and \ e = \begin{bmatrix} e_x & e_y & e_z \end{bmatrix}^T$$
.

We choose a controller $U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ as follows:

$$u_1(t) = z_1 e_y + y_1 e_z - (\frac{1}{2} + a)e_x$$
$$u_2(t) = (b-1)e_y - z_1 e_x - x_1 e_z \qquad (6)$$
$$u_3(t) = (c-1)e_z - y_1 e_x - x_1 e_y$$

Then the error system can be rewritten as:

$$\dot{e}_x = -\frac{1}{2}e_x - e_y e_z$$
$$\dot{e}_y = -e_y + e_x e_z \qquad (7)$$
$$\dot{e}_z = -e_z + e_x e_y$$

Let us consider the Lyapunov function $V(e)$ which is defined by

$$V(e) = e_x^2 + \frac{1}{2}(e_y^2 + e_z^2) \qquad (8)$$
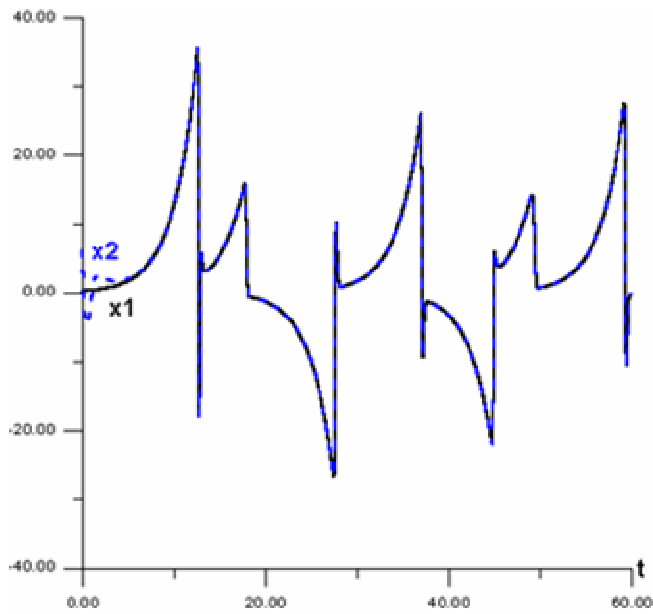
Then we can rewrite (8) as the following

$$V(e) = e^T Pe \ \ and \ \ P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{2} & 0 \\ 0 & 0 & \dfrac{1}{2} \end{bmatrix}$$

It is clear that the Lyapunov function $V(e)$ is a positive definite function. Now, taking the time derivative of equation (8), then we get
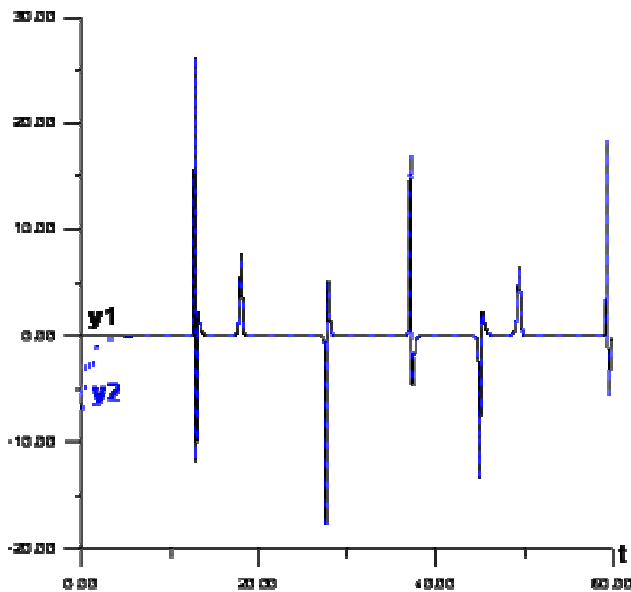
$$\frac{dV}{dt} = -(e_x^2 + e_y^2 + e_z^2)$$

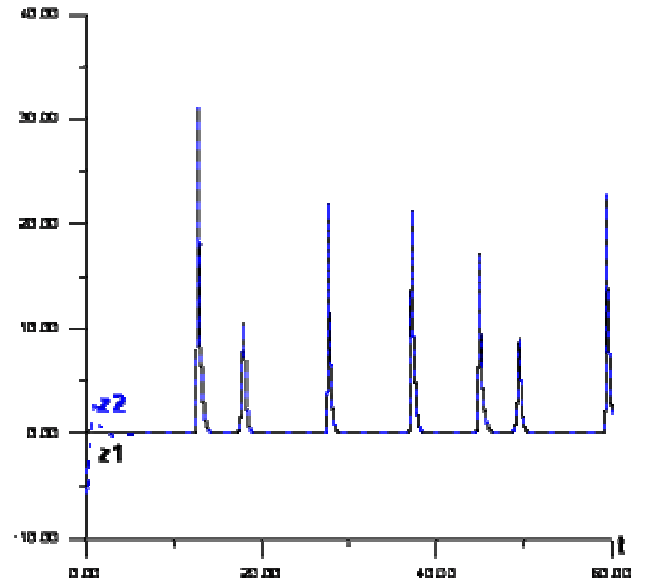$$= -e^t\ Q\ e \quad where \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (9)$$

Based on Lyapunov stability theory, this translates to $\lim_{t\to\infty}\|e(t)\| = 0$. Thus the response system and drive systems are globally asymptotically synchronized.

(a)

(b)

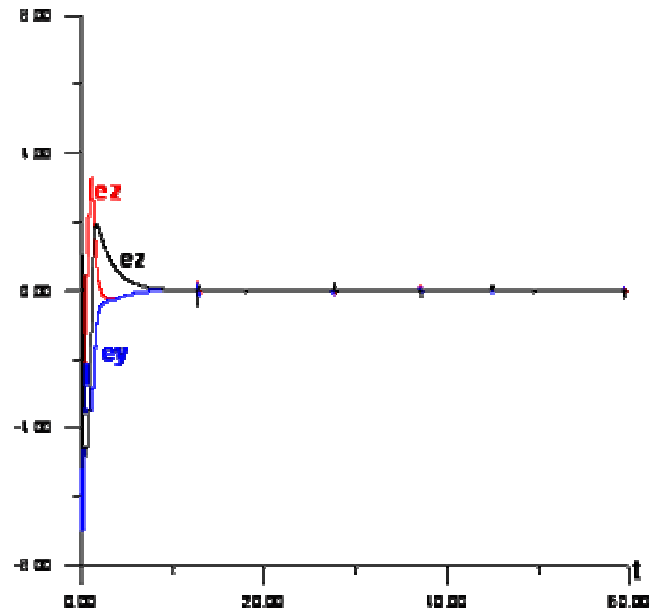(c)

(d)

Figure 3. Shows that the time response of states for drive system (x_1, y_1, z_1) and response system (x_2, y_2, z_2) with active control law (6) is activated, (a) Signals $x_1(t)\ and\ x_2(t)$: (b) Signals $y_1(t)\ and\ y_2(t)$: (c) Signals $z_1(t)\ and\ z_2(t)$ and (d) show that the error system (5) tends to zero (x_1, y_1 and z_1 _____: x_2, y_2 and z_2 ----------).

Numerical results

Fourth-order Runge-Kutta method is used to solve the systems of differential equations (3), (4), and (5). In addition, a time step of size $h = 10^{-4}$ is employed. The

parameters are chosen as a = 0.4, b = 12 and c = 5 in all simulations so that the four-scroll system exhibits a chaotic behavior when no control is applied (see Figure 2).

The initial states of the drive system are $x_1(0) = 0.23$, $y_1(0) = 0.1$ and $z_1(0) = 0.32$ and initial states of the response system are $x_2(0) = 6.23$, $y_2(0) = -5.1$ and $z_2(0) = -3$ hence the error system has the initial values $e_x(0) = 6$, $e_y(0) = -5.2$ and $e_z(0) = -3.32$.

The results of the two identical four-scroll systems with active control are shown in Figure 3: (a) displays the trajectories $x_1(t)$ and $x_2(t)$, (b) displays the traject-ories $y_1(t)$ and $y_2(t)$, (c) displays the trajectoryies $z_1(t)$ and $z_2(t)$ and (d) shows that the trajectories of $e_x(t)$, $e_y(t)$ and $e_z(t)$ of the error system tended to zero.
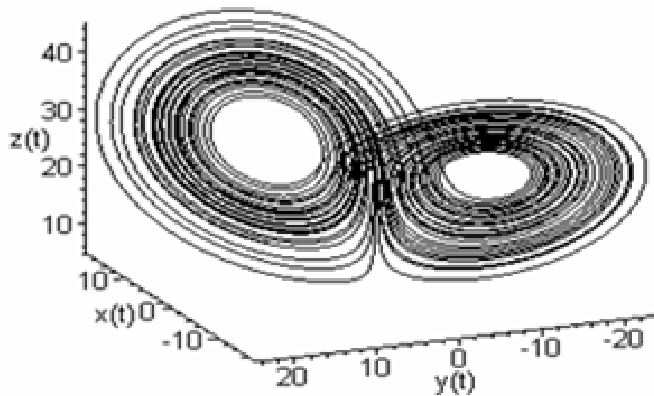

Figure 4. Shows the chaotic attractor of the Lorenze system

Synchronization of two different chaotic systems

In this section, the application of the above method is used to synchronize two different chaotic systems. One is the familiar Lorenz system (Sparrow, 1982; Lorenz, 1963) considered as the drive system. The Lorenz system is a third-order autonomous system with only two quadratic multiplication terms but it can display very complex dynamical behaviors. The other is the four-scroll chaotic attractor considered as the response system. Our aim is to design a controller and make the response system trace the drive system and become ultimately the same.

In this section we take the Lorenz system, as a drive system, which is described by the following equation:

$$\dot{x}_1 = \sigma(y_1 - x_1)$$
$$\dot{y}_1 = \gamma x_1 - y_1 - x_1 z_1 \qquad (10)$$
$$\dot{z}_1 = x_1 y_1 - \beta z_1$$

Where x_1, y_1 and z_1 are the state variables and σ, β and γ are parameters of the system. The Lorenz system has a chaotic attractor for some typical parameter values: $\sigma = 10$, $\beta = \dfrac{8}{3}$ and $\gamma = 28$, as shown by Figure 4. Form equations (3) and (10), the following error system equation can be obtained:

at $\sigma = 10$, $\beta = \dfrac{8}{3}$ and $\gamma = 28$.

$$\dot{e}_x = ae_x - e_y e_z - z_1 e_y - y_1 z_2 - \sigma y_1 + (\sigma + a)x_1 + u_1(t)$$
$$\dot{e}_y = -be_y + e_x e_z + z_1 e_x + x_1 e_z - \gamma x_1 + 2x_1 z_1 + (1-b)y_1 + u_2(t)$$
$$\dot{e}_z = -ce_z + e_x e_y + y_1 e_x + x_1 e_y + (\beta - c)z_1 + u_3(t)$$
$$(11)$$

Where

$e_x = x_2 - x_1, e_y = y_2 - y_1, e_z = z_2 - z_1$ and $e = \begin{bmatrix} e_x & e_y & e_z \end{bmatrix}^T$. We choose a controller $U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ as follows:

$$u_1(t) = z_1 e_y + (z_2 + \sigma)y_1 - (\sigma + a)x_1 - (\frac{1}{2} + a)e_x$$
$$u_2(t) = -z_1 e_x - x_1 e_z + \gamma x_1 - 2x_1 z_1 - (1-b)y_1 + (b-1)e_y$$
$$u_3(t) = (c-1)e_z - y_1 e_x - x_1 e_y - (\beta - c)z_1$$
$$(12)$$

Then the error system can be rewritten as:

$$\dot{e}_x = -\frac{1}{2}e_x - e_y e_z$$
$$\dot{e}_y = -e_y + e_x e_z$$
$$\dot{e}_z = -e_z + e_x e_y$$
$$(13)$$

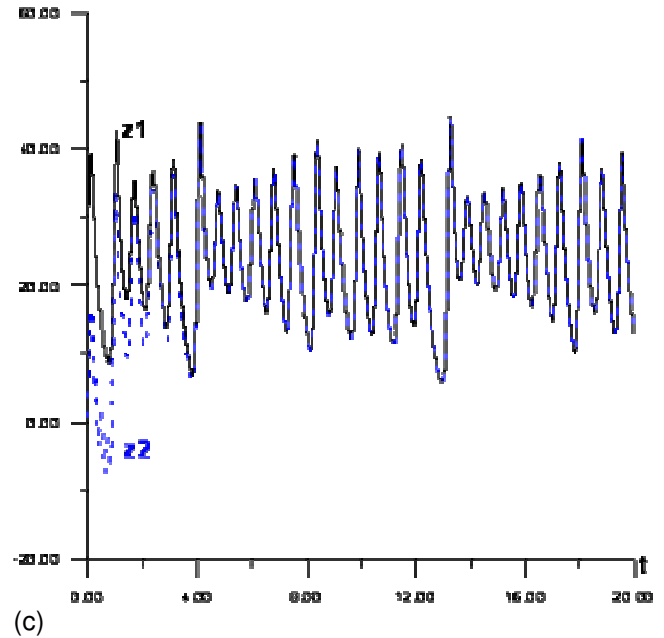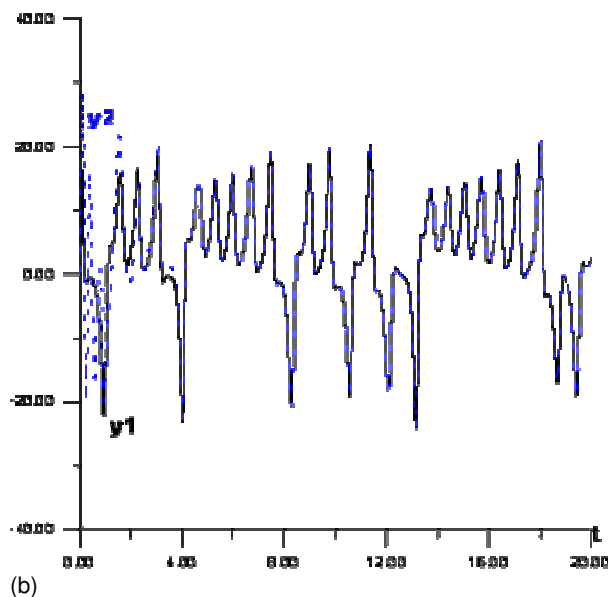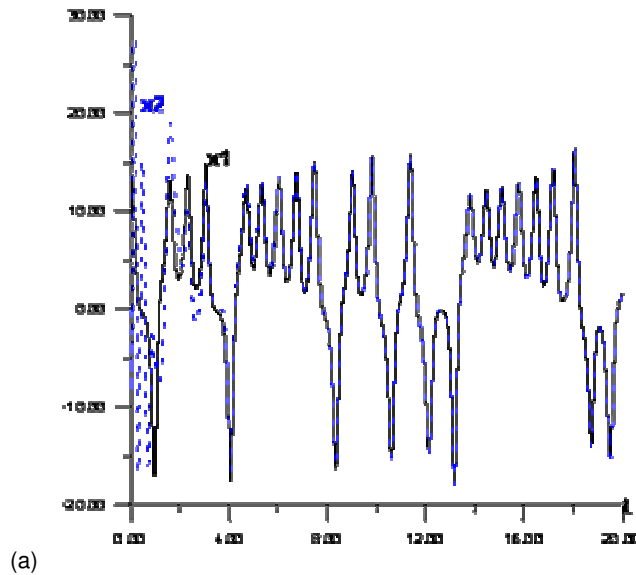Let us consider the Lyapunov function $V(e)$ which is defined by

$$V(e) = e_x^2 + \frac{1}{2}(e_y^2 + e_z^2) \qquad (14)$$

It is clear that the Lyapunov function $V(e)$ is a positive

definite function. Now, taking the time derivative of $V(e)$ defined in equation (14), we get

$$\frac{dV}{dt} = -(e_x^2 + e_y^2 + e_z^2) \qquad (15)$$

It is found that $V(e)$ and $\dfrac{dV}{dt}$ is positive and negative

definite functions. Also, $V(e) \to \infty$ as $\|e\| \to \infty$. Hence, by Lyapunov stability theory, the error dynamics is globally asymptotically stable. Therefore, this choice will lead the error states $e_x(t)$, $e_y(t)$ and $e_z(t)$ to converge to zero as time t tends to infinity and hence the global synchronization of two different chaotic systems is achieved.

(a)

(b)

(c)

Numerical results

In this subsection, numerical simulations are also given to verify the proposed method. The parameters are selected as follows: $\sigma = 10$, $\beta = \dfrac{8}{3}$ and $\gamma = 28$ with initial values

$x_1(0) = 1$, $y_1(0) = 1$, $z_1(0) = 1$, $x_2(0) = 0.23$, $y_2(0) = 0.1$

and $z_2(0) = 0.32$. The simulation results are illustrated in Figure 5 (a) - (d). It can be seen that the synchronization error will converge to zero and two different chaotic systems are indeed achieving chaos synchronization.

CONCLUSIONS

In this paper, modification based on Lyapunov stability theory to design a nonlinear controller is proposed to synchronize two identical chaotic systems and two different chaotic systems. Numerical simulations are also given to validate the proposed synchronization approach. The simulation results show that the states of two identical four-scroll attractor are globally asymptotically synchronized. For two different chaotic systems, the four-scroll attractor is forced to trace the Lorenz system and the states of two systems become ultimately the same. Since the Lyapunov exponents are not required for the calculation, this method is effective and convenient to synchronize two identical systems and two different chaotic systems.

ACKNOWLEDGMENT

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