Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Refinements of the Heinz inequalities for matrices

Yanxiong Yan[1,2], Yuming Feng[3] and Guiyun Chen[1*]

*Correspondence:
gychen1963@163.com
[1]School of Mathematics and Statistics, Southwest University, Chongqing, 400715, P.R. China
Full list of author information is available at the end of the article

Abstract

This article aims to discuss Heinz inequalities involving unitarily invariant norms. We use a similar method to (Feng in J. Inequal. Appl. 2012:18, 2012; Wang in J. Inequal. Appl. 2013:424, 2013) and we get different refinements of the Heinz inequalities for matrices. Our results are better than some given in (Kittaneh in Integral Equ. Oper. Theory 68:519-527, 2010) and they are different from (Feng in J. Inequal. Appl. 2012:18, 2012; Wang in J. Inequal. Appl. 2013:424, 2013).

Keywords: convex function; Heinz inequality; Hermite-Hadamard inequality; unitarily invariant norm

1 Introduction

If A, B, X are operators on a complex separable Hilbert space such that A and B are positive, then for every unitarily invariant norm $|||\cdot|||$, the function $f(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||$ is convex on the interval $[0,1]$, attains its minimum at $\nu = \frac{1}{2}$, and attains its maximum at $\nu = 0$ and $\nu = 1$. Moreover, $f(\nu) = f(1-\nu)$ for $0 \leq \nu \leq 1$. From [1] we know that for every unitarily invariant norm, we have the Heinz inequalities

$$2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| \leq \left|\left|\left|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\right|\right|\right| \leq |||AX + XB|||. \tag{1}$$

In [2], Feng used the following inequalities to get refinements of (1):

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{4}\left(f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right) \leq \frac{f(a)+f(b)}{2},$$

where f is a real-valued function which is convex on the interval $[a,b]$. With a similar method, Wang [3] got some new refinements of (1).

In this paper, we use a similar method to [2, 3] and we get different refinements of (1).

When we consider $|||T|||$, we are implicitly assuming that the operator T belongs to the norm ideal associated with $|||\cdot|||$. Our results are better than those in [4] and different from [2, 3].

2 Main results

From page 122 of [5], we know the following Hermite-Hadamard integral inequality for convex functions.

Lemma 1 (Hermite-Hadamard integral inequality) *Let f be a real-valued function which is convex on the interval $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}.$$

We will use the following lemma.

Lemma 2 *Let f be a real-valued function which is convex on the interval $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{8}\left(3f(a)+2f\left(\frac{a+b}{2}\right)+3f(b)\right) \le \frac{f(a)+f(b)}{2}.$$

Proof Using the previous lemma, we can easily verify the inequality

$$\frac{1}{8}\left(3f(a)+2f\left(\frac{a+b}{2}\right)+3f(b)\right) \le \frac{f(a)+f(b)}{2}.$$

Next, we will prove the following inequality:

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{8}\left(3f(a)+2f\left(\frac{a+b}{2}\right)+3f(b)\right).$$

From the previous lemma, we have

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(t)\,dt &= \frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}} f(t)\,dt + \int_{\frac{a+b}{2}}^b f(t)\,dt\right) \\
&\le \frac{1}{b-a}\left(\frac{f(a)+f(\frac{a+b}{2})}{2}\cdot\frac{b-a}{2} + \frac{f(\frac{a+b}{2})+f(b)}{2}\cdot\frac{b-a}{2}\right) \\
&= \frac{1}{4}\left(f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right) \\
&= \frac{1}{8}\left(2f(a)+4f\left(\frac{a+b}{2}\right)+2f(b)\right) \\
&\le \frac{1}{8}\left(2f(a)+2f\left(\frac{a+b}{2}\right)+\bigl(f(a)+f(b)\bigr)+2f(b)\right) \\
&= \frac{1}{8}\left(3f(a)+2f\left(\frac{a+b}{2}\right)+3f(b)\right).
\end{aligned}$$

□

Applying the previous lemma to the function $f(\nu) = |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu|||$ on the interval $[\mu, 1-\mu]$ when $0 \le \mu \le \frac{1}{2}$, and on the interval $[1-\mu, \mu]$ when $\frac{1}{2} \le \mu \le 1$, we obtain a refinement of the first inequality in (1).

Theorem 1 *Let A, B, X be operators such that A, B are positive. Then for $0 \le \mu \le 1$ and for every unitarily invariant norm, we have*

$$\begin{aligned}
2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\le \frac{1}{|1-2\mu|}\left|\int_\mu^{1-\mu} |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu|||\,d\nu\right| \\
&\le \frac{1}{4}\bigl(3|||A^\mu XB^{1-\mu} + A^{1-\mu}XB^\mu||| + 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\bigr) \\
&\le |||A^\mu XB^{1-\mu} + A^{1-\mu}XB^\mu|||.
\end{aligned} \tag{2}$$

Proof First assume that $0 \leq \mu \leq \frac{1}{2}$. Then it follows by the previous lemma that

$$\begin{aligned} f\left(\frac{1-\mu+\mu}{2}\right) &\leq \frac{1}{1-2\mu}\int_{\mu}^{1-\mu} f(t)\,dt \\ &\leq \frac{1}{8}\left(3f(\mu)+2f\left(\frac{1-\mu+\mu}{2}\right)+3f(1-\mu)\right) \\ &\leq \frac{f(\mu)+f(1-\mu)}{2}, \end{aligned}$$

and so

$$\begin{aligned} f\left(\frac{1}{2}\right) &\leq \frac{1}{1-2\mu}\int_{\mu}^{1-\mu} f(t)\,dt \\ &\leq \frac{1}{4}\left(3f(\mu)+f\left(\frac{1}{2}\right)\right) \\ &\leq f(\mu). \end{aligned}$$

Thus,

$$\begin{aligned} 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| &\leq \frac{1}{1-2\mu}\int_{\mu}^{1-\mu} \left|\left|\left|A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}\right|\right|\right|\,d\nu \\ &\leq \frac{1}{4}\left(3\left|\left|\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\right|\right|+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\ &\leq \left|\left|\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\right|\right|. \end{aligned} \tag{3}$$

Now, assume that $\frac{1}{2} \leq \mu \leq 1$. Then by applying (3) to $1-\mu$, it follows that

$$\begin{aligned} 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| &\leq \frac{1}{2\mu-1}\int_{1-\mu}^{\mu} \left|\left|\left|A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}\right|\right|\right|\,d\nu \\ &\leq \frac{1}{4}\left(3\left|\left|\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\right|\right|+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\ &\leq \left|\left|\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\right|\right|. \end{aligned} \tag{4}$$

Since

$$\begin{aligned} &\lim_{\mu\to\frac{1}{2}} \frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu} \left|\left|\left|A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}\right|\right|\right|\,d\nu\right| \\ &\quad= \lim_{\mu\to\frac{1}{2}} \frac{1}{4}\left(3\left|\left|\left|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right|\right|\right|+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\ &\quad= 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|, \end{aligned}$$

the inequalities in (2) follow by combining (3) and (4). □

Applying the previous lemma to the function $f(\nu) = |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||$ on the interval $[\mu, \frac{1}{2}]$ when $0 \leq \mu \leq \frac{1}{2}$, and on the interval $[\frac{1}{2}, \mu]$ when $\frac{1}{2} \leq \mu \leq 1$, we obtain the following.

Theorem 2 *Let A, B, X be operators such that A, B are positive. Then for $0 \le \mu \le 1$ and for every unitarily invariant norm, we have*

$$\begin{aligned}
&\big|\big|\big|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}} + A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\big|\big|\big| \\
&\quad\le \frac{2}{|1-2\mu|}\left|\int_{\mu}^{\frac{1}{2}} \big|\big|\big|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\big|\big|\big|\,d\nu\right| \\
&\quad\le \frac{1}{8}\big(3\big|\big|\big|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\big|\big|\big| + 2\big|\big|\big|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}} + A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\big|\big|\big| + 6\big|\big|\big|A^{\frac{1}{2}}XB^{\frac{1}{2}}\big|\big|\big|\big) \\
&\quad\le \frac{1}{2}\big(\big|\big|\big|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\big|\big|\big| + 2\big|\big|\big|A^{\frac{1}{2}}XB^{\frac{1}{2}}\big|\big|\big|\big).
\end{aligned} \tag{5}$$

The inequality (5) and the first inequality in (1) yield the following refinement of the first inequality in (1).

Corollary 1 *Let A, B, X be operators such that A, B are positive. Then for $0 \le \mu \le 1$ and for every unitarily invariant norm, we have*

$$\begin{aligned}
&2\big|\big|\big|A^{\frac{1}{2}}XB^{\frac{1}{2}}\big|\big|\big| \\
&\quad\le \big|\big|\big|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}} + A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\big|\big|\big| \\
&\quad\le \frac{2}{|1-2\mu|}\left|\int_{\mu}^{\frac{1}{2}} \big|\big|\big|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\big|\big|\big|\,d\nu\right| \\
&\quad\le \frac{1}{8}\big(3\big|\big|\big|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\big|\big|\big| + 2\big|\big|\big|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}} + A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\big|\big|\big| + 6\big|\big|\big|A^{\frac{1}{2}}XB^{\frac{1}{2}}\big|\big|\big|\big) \\
&\quad\le \frac{1}{2}\big(\big|\big|\big|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\big|\big|\big| + 2\big|\big|\big|A^{\frac{1}{2}}XB^{\frac{1}{2}}\big|\big|\big|\big) \\
&\quad\le \big|\big|\big|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\big|\big|\big|.
\end{aligned} \tag{6}$$

Applying the previous lemma to the function $f(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||$ on the interval $[0, \mu]$ when $0 \le \mu \le \frac{1}{2}$, and on the interval $[\mu, 1]$ when $\frac{1}{2} \le \mu \le 1$, we obtain the following theorem.

Theorem 3 *Let A, B, X be operators such that A, B are positive. Then*

(1) *for $0 \le \mu \le \frac{1}{2}$ and for every unitarily norm,*

$$\begin{aligned}
&\big|\big|\big|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\big|\big|\big| \\
&\quad\le \frac{1}{\mu}\int_{0}^{\mu} \big|\big|\big|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\big|\big|\big|\,d\nu \\
&\quad\le \frac{1}{8}\big(3|||AX + XB||| + 2\big|\big|\big|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\big|\big|\big| + 3\big|\big|\big|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\big|\big|\big|\big) \\
&\quad\le \frac{1}{2}\big(|||AX + XB||| + \big|\big|\big|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\big|\big|\big|\big);
\end{aligned} \tag{7}$$

(2) *for $\frac{1}{2} \le \mu \le 1$ and for every unitarily norm,*

$$\begin{aligned}
&\big|\big|\big|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}} + A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\big|\big|\big| \\
&\quad\le \frac{1}{1-\mu}\int_{\mu}^{1} \big|\big|\big|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\big|\big|\big|\,d\nu
\end{aligned}$$

$$\le \frac{1}{8}\bigl(3|||AX+XB|||+2\big|\big|\big|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}+A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\big|\big|\big|+3\big|\big|\big|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\big|\big|\big|\bigr)$$
$$\le \frac{1}{2}\bigl(|||AX+XB|||+\big|\big|\big|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\big|\big|\big|\bigr). \tag{8}$$

Since the function $f(\nu) = |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||$ is decreasing on the interval $[0,\frac{1}{2}]$ and increasing on the interval $[\frac{1}{2},1]$, and using the inequalities (7) and (8), we obtain the refinement of the second inequality in (1).

Corollary 2 *Let A, B, X be operators such that A, B are positive. Then for $0\le\mu\le1$ and for every unitarily invariant norm, we have the following.*

(1) *For $0\le\mu\le\frac{1}{2}$ and for every unitarily norm,*

$$\begin{aligned}
&\big|\big|\big|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\big|\big|\big|\\
&\quad\le \big|\big|\big|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}}+A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\big|\big|\big|\\
&\quad\le \frac{1}{\mu}\int_0^{\mu}\big|\big|\big|A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}\big|\big|\big|\,d\nu\\
&\quad\le \frac{1}{8}\bigl(3|||AX+XB|||+2\big|\big|\big|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}}+A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\big|\big|\big|+3\big|\big|\big|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\big|\big|\big|\bigr)\\
&\quad\le \frac{1}{2}\bigl(|||AX+XB|||+\big|\big|\big|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\big|\big|\big|\bigr)\\
&\quad\le |||AX+XB|||.
\end{aligned}\tag{9}$$

(2) *For $\frac{1}{2}\le\mu\le1$ and for every unitarily norm,*

$$\begin{aligned}
&\big|\big|\big|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\big|\big|\big|\\
&\quad\le \big|\big|\big|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}+A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\big|\big|\big|\\
&\quad\le \frac{1}{1-\mu}\int_{\mu}^{1}\big|\big|\big|A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}\big|\big|\big|\,d\nu\\
&\quad\le \frac{1}{8}\bigl(3|||AX+XB|||+2\big|\big|\big|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}+A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\big|\big|\big|+3\big|\big|\big|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\big|\big|\big|\bigr)\\
&\quad\le \frac{1}{2}\bigl(|||AX+XB|||+\big|\big|\big|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\big|\big|\big|\bigr)\\
&\quad\le |||AX+XB|||.
\end{aligned}\tag{10}$$

It should be noticed that in the inequalities (7) to (10), we have

$$\begin{aligned}
&\lim_{\mu\to0}\frac{1}{\mu}\int_0^{\mu}\big|\big|\big|A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}\big|\big|\big|\,d\nu\\
&\quad\le \lim_{\mu\to1}\frac{1}{1-\mu}\int_{\mu}^{1}\big|\big|\big|A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}\big|\big|\big|\,d\nu\\
&\quad= |||AX+XB|||.
\end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

YY carried out convex function. YF carried out unitarily invariant norm. GC carried out the calculation. All authors read and approved the final manuscript.

Author details

[1]School of Mathematics and Statistics, Southwest University, Chongqing, 400715, P.R. China. [2]Department of Mathematics and Information Engineering, Chongqing University of Education, Chongqing, 400067, P.R. China. [3]School of Electronic Information Engineering, Southwest University, Chongqing, 400715, P.R. China.

Acknowledgements

This work is supported by NSF of China (Grant Nos. 11171364 and 11271301).

Received: 10 November 2013 Accepted: 8 January 2014 Published: 31 Jan 2014

References

1. Bhatia, R, Davis, C: More matrix forms of the arithmetic-geometric mean inequality. SIAM J. Matrix Anal. Appl. **14**, 132-136 (1993)
2. Feng, Y: Refinements of the Heinz inequalities. J. Inequal. Appl. **2012**, 18 (2012). doi:10.1186/1029-242X-2012-18
3. Wang, S: Some new refinements of Heinz inequalities of matrices. J. Inequal. Appl. **2013**, 424 (2013). doi:10.1186/1029-242X-2013-424
4. Kittaneh, F: On the convexity of the Heinz means. Integral Equ. Oper. Theory **68**, 519-527 (2010)
5. Bullen, PS: A Dictionary of Inequalities. Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 97. Longman, Harlow (1998)

10.1186/1029-242X-2014-50
Cite this article as: Yan et al.: Refinements of the Heinz inequalities for matrices. *Journal of Inequalities and Applications* 2014, 2014:50