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Approximation properties of bivariate complex q-Balàzs-Szabados operators of tensor product kind

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Abstract

In this study, we consider the bivariate complex q-Balàzs-Szabados operators of the tensor product kind. Approximation properties of these operators attached to analytic functions on compact polydisks are investigated by using the results in the univariate case obtained for q-Balàzs-Szabados operators in (İspir and Yıldız Özkan in J. Inequal. Appl. 2013:361, 2013). In this sense, the upper estimate, the Voronovskaja-type theorem, and the lower estimate are obtained. The exact degree of its approximation is also given.

MSC: 30E10; 41A25

Keywords: complex approximation; q-Balàzs-Szabados operators; order of convergence; Voronovskaja-type theorem; exact degree of approximation

1 Introduction

The approximation properties of the q-analogue operators in compact disks have recently been an active area of the research in the field of the approximation theory [1–8]. Details of the q-calculus can be found in [9–11].

Balázs [12] defined the Bernstein-type rational functions. She gave an estimate for the order of its convergence and proved an asymptotic approximation theorem and a convergence theorem concerning the derivative of these operators. In [13], Balázs and Szabados obtained the best possible estimate under more restrictive conditions, in which both the weight and the order of convergence would be better than [12]. They applied their results to the approximation of certain improper integrals by quadrature sums of positive coefficients based on a finite number of equidistant nodes. The q-form of these operators was given by Doğru. He investigated Korovkin-type statistical approximation properties of these operators for the functions of one and two variables [14]. Atakut and Ispir [15] defined the bivariate real Bernstein-type rational functions of the Bernstein-type rational functions given by Balázs in [12] and proved the approximation theorems for these functions. Ispir and Gupta [16] studied the Bézier variant of generalized Kantrovich-type Balazs operators.

Approximation properties of the rational Balázs-Szabados operators on compact disks in the complex plane were investigated by Gal [17]. He proved the upper estimate in an approximation of these operators. Also, he obtained the exact degree of its approximation by using a Voronovskaja-type result. In [18], the approximation properties given by Gal in the complex plane was extended to the bivariate case.

The complex q-Balázs-Szabados operators was defined in [19] as follows:

$$R_n^q(f;z) = \frac{1}{\prod_{s=0}^{n-1}(1+q^s a_n z)} \sum_{j=0}^{n} q^{j(j-1)/2} f\left(\frac{[j]_q}{b_n}\right) \begin{bmatrix} n \\ j \end{bmatrix}_q (a_n z)^j,$$

where $f : D_R \cup [R,\infty) \to \mathbb{C}$ is uniformly continuous and bounded on $[0,\infty)$ with $D_R = \{z \in \mathbb{C} : |z| < R\}$ for $R > 0$, $a_n = [n]_q^{\beta-1}$, $b_n = [n]_q^{\beta}$, $q \in (0,1]$, $0 < \beta \leq \frac{2}{3}$, $n \in \mathbb{N}$, $z \in \mathbb{C}$, and $z \neq -\frac{1}{q^s a_n}$ for $s = 0,1,2,\ldots$.

We consider the following complex bivariate q-Balázs-Szabados operators of the tensor product kind:

$$R_{n,m}^{q_1,q_2}(f)(z_1,z_2) = \sum_{k=0}^{n}\sum_{j=0}^{m} f\left(\frac{[k]_{q_1}}{b_n}, \frac{[j]_{q_2}}{b_m}\right) p_{n,k}(z_1) p_{m,j}(z_2), \tag{1}$$

where $f : (D_{R_1} \cup [R_1,\infty)) \times (D_{R_2} \cup [R_2,\infty)) \to \mathbb{C}$ is a uniformly continuous function bounded on $[0,\infty) \times [0,\infty)$, $a_n = [n]_{q_1}^{\beta-1}$, $b_n = [n]_{q_1}^{\beta}$, $a_m = [m]_{q_2}^{\beta-1}$, $b_m = [m]_{q_2}^{\beta}$ for $n,m \in \mathbb{N}$, $q_1, q_2 \in (0,1]$, $0 < \beta \leq \frac{2}{3}$.

$$p_{n,k}(z_1) = \frac{q_1^{k(k-1)/2} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{q_1} (a_n z_1)^k}{\prod_{s_1=0}^{n-1}(1 + q_1^{s_1} a_n z_1)} \quad \text{and} \quad p_{m,j}(z_2) = \frac{q_2^{j(j-1)/2} \left[\begin{smallmatrix} m \\ j \end{smallmatrix}\right]_{q_2} (a_m z_2)^j}{\prod_{s_2=0}^{m-1}(1 + q_2^{s_2} a_m z_2)}$$

for all $s_1 = 0,1,\ldots,n-1$, $s_2 = 0,1,\ldots,m-1$ and $z_1, z_2 \in \mathbb{C}$ with $z_1 \neq -\frac{1}{q_1^{s_1} a_n}$ and $z_2 \neq -\frac{1}{q_2^{s_2} a_m}$.

The complex bivariate q-Balázs-Szabados operators of the tensor product kind are well defined and linear, and these operators are analytic for all $n \geq n_0$, $m \geq m_0$, $|z_1| \leq r_1 < [n_0]_{q_1}^{1-\beta}$ and $|z_2| \leq r_2 < [m_0]_{q_2}^{1-\beta}$.

The aim of this paper is to obtain the exact degree of approximation of the complex bivariate q-Balázs-Szabados operators of the tensor product kind. The Voronovskaja-type theorem in the bivariate case is very different from the univariate case, so the exact degree of approximation of these operators can be obtained for $0 < \beta < \frac{1}{2}$.

Throughout this paper, we denote by $\|f\|_{r_1,r_2} = \max\{|f(z_1,z_2)| : (z_1,z_2) \in \bar{D}_{r_1} \times \bar{D}_{r_1}\}$ the uniform norm of the function f in the space of continuous functions on $\bar{D}_{r_1} \times \bar{D}_{r_1}$ and by $\|f\|_{B([0,\infty)\times[0,\infty))} = \sup\{|f(z_1,z_2)| : (z_1,z_2) \in [0,\infty) \times [0,\infty)\}$ the norm of the function f in the space of bounded functions on $[0,\infty) \times [0,\infty)$, where $D_r = \{z \in \mathbb{C} : |z| < r\}$ for $r > 0$.

The convergence results will be obtained under the condition that $f : (D_{R_1} \cup [R_1,\infty)) \times (D_{R_2} \cup [R_2,\infty)) \to \mathbb{C}$ is analytic in $D_{R_1} \times D_{R_2}$ for $r_1 < R_1$ and $r_2 < R_2$, which ensures the representation $f(z_1,z_2) = \sum_{k=0}^{\infty} f_k(z_2) z_1^k$, where $f_k(z_2) = \sum_{j=0}^{\infty} c_{k,j} z_2^j$ for all $(z_1,z_2) \in D_{R_1} \times D_{R_2}$.

2 Auxiliary results

Let $q = (q_n)$ be a sequence satisfying

$$\lim_{n\to\infty} q_n = 1 \quad \text{and} \quad \lim_{n\to\infty} q_n^n = c \quad (0 \leq c < 1). \tag{2}$$

We need the following lemmas in order to prove the main results for the operators (1).

Lemma 1 *Let $n_0, m_0 \geq 2$, $0 < \beta \leq \frac{2}{3}$, $\frac{1}{2} < r_1 < R_1 \leq \frac{[n_0]_{q_1}^{1-\beta}}{2}$ and $\frac{1}{2} < r_2 < R_2 \leq \frac{[m_0]_{q_1}^{1-\beta}}{2}$. If $f : (D_{R_1} \cup [R_1, \infty)) \times (D_{R_2} \cup [R_2, \infty)) \to \mathbb{C}$ is a uniformly continuous function bounded on $[0,\infty) \times [0,\infty)$ and analytic in $D_{R_1} \times D_{R_2}$ then we have the form*

$$R_{n,m}^{q_1,q_2}(f)(z_1,z_2) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} c_{k,j} R_{n,m}^{q_1,q_2}(e_{k,j})(z_1,z_2)$$

for all $(z_1,z_2) \in D_{r_1} \times D_{r_2}$, where $(e_{k,j})(z_1,z_2) = e_1^k(z_1)e_2^j(z_2)$ with $e_1^k(z_1) = z_1^k$, $e_2^j(z_2) = z_2^j$ for $k, j \in \mathbb{N}$.

Proof For any $s, r \in \mathbb{N}$, we define

$$f_{s,r}(z_1,z_2) = \sum_{k=0}^{s}\sum_{j=0}^{r} c_{k,j} e_{k,j}(z_1,z_2) \quad \text{if } |z_1| \leq r_1, |z_2| \leq r_2 \quad \text{and}$$

$$f_{s,r}(z_1,z_2) = f(z_1,z_2) \quad \text{if } (z_1,z_2) \in (r_1,\infty) \times (r_2,\infty).$$

From the hypothesis on f, it is clear that each $f_{s,r}$ is bounded on $[0,\infty) \times [0,\infty)$, which implies that

$$\left|R_{n,m}^{q_1,q_2}(f_{s,r})(z_1,z_2)\right| \leq \sum_{k=0}^{n}\sum_{j=0}^{m} \left|p_{n,k}(z_1)\right|\left|p_{m,j}(z_2)\right| M_{f_{s,r}} < \infty,$$

where $M_{f_{s,r}}$ is a constant depending on $f_{s,r}$, so all $R_{n,m}^{q_1,q_2}(f_{s,r})$ are well defined for all $n, m \in \mathbb{N}$, $n \geq n_0$, $m \geq m_0$, $r_1 < \frac{[n_0]_q^{1-\beta}}{2}$, $r_2 < \frac{[m_0]_q^{1-\beta}}{2}$ and $(z_1,z_2) \in D_{r_1} \times D_{r_2}$.

Defining

$$f_{s,r,k,j}(z_1,z_2) = c_{k,j} e_{k,j}(z_1,z_2) \quad \text{if } |z_1| \leq r_1, |z_2| \leq r_2 \quad \text{and}$$

$$f_{s,r,k,j}(z_1,z_2) = \frac{f(z_1,z_2)}{(s+1)(r+1)} \quad \text{if } (z_1,z_2) \in (r_1,\infty) \times (r_2,\infty).$$

It is clear that each $f_{s,r,k,j}$ is bounded on $[0,\infty) \times [0,\infty)$ and

$$f_{s,r}(z_1,z_2) = \sum_{k=0}^{s}\sum_{j=0}^{r} f_{s,r,k,j}(z_1,z_2).$$

From the linearity of $R_{n,m}^{q_1,q_2}$, we have

$$R_{n,m}^{q_1,q_2}(f_{s,r})(z_1,z_2) = \sum_{k=0}^{s}\sum_{j=0}^{r} c_{k,j} R_{n,m}^{q_1,q_2}(e_{k,j})(z_1,z_2).$$

It suffices to prove that

$$\lim_{s,r\to\infty} R_{n,m}^{q_1,q_2}(f_{s,r})(z_1,z_2) = R_{n,m}^{q_1,q_2}(f)(z_1,z_2)$$

for any fixed $n, m \in \mathbb{N}$, $n \geq n_0$, $m \geq m_0$, $|z_1| \leq r_1$ and $|z_2| \leq r_2$. Since

$$\|f_{s,r} - f\|_{B([0,\infty)\times[0,\infty))} \leq \|f_{s,r} - f\|_{r_1,r_2},$$

we can write

$$\begin{aligned}\left|R_{n,m}^{q_1,q_2}(f_{s,r})(z_1,z_2) - R_{n,m}^{q_1,q_2}(f)(z_1,z_2)\right| &\le M_{r_1,r_2,m,n}^{q_1,q_2}\|f_{s,r} - f\|_{B([0,\infty)\times[0,\infty))} \\ &\le M_{r_1,r_2,m,n}^{q_1,q_2}\|f_{s,r} - f\|_{r_1,r_2} \qquad (3)\end{aligned}$$

for $|z_1| \le r_1$ and $|z_2| \le r_2$.

In equation (3), taking the limit as $s, r \to \infty$ and using $\lim_{s,r\to\infty}\|f_{s,r} - f\|_{r_1,r_2} = 0$, we get the result. □

Lemma 2 *Let $n_0, m_0 \ge 2$, $0 < \beta \le \frac{2}{3}$, $\frac{1}{2} < r_1 < R_1 \le \frac{[n_0]_{q_1}^{1-\beta}}{2}$ and $\frac{1}{2} < r_2 < R_2 \le \frac{[m_0]_{q_2}^{1-\beta}}{2}$. For all $n \ge n_0$, $m \ge m_0$, $|z_1| \le r_1$, $|z_2| \le r_2$ and $k = 0, 1, 2, \ldots$ the following inequality holds*:

$$\left|R_{n,m}^{q_1,q_2}(e_{k,j})(z_1,z_2)\right| \le k!j!(20r_1)^k(20r_2)^j.$$

Proof Using Lemma 4 in [19], the lemma is easily proved, so we omit the proof of the lemma. □

3 Main results

Let us denote by A_C the space of all uniformly continuous complex valued functions defined on $(D_{R_1} \cup [R_1,\infty)) \times (D_{R_2} \cup [R_2,\infty))$, bounded on $[0,\infty) \times [0,\infty)$ and analytic in $D_{R_1} \times D_{R_2}$ and for which there exist $M > 0$, $0 < A_1 < \frac{1}{20r_1}$ and $0 < A_2 < \frac{1}{20r_2}$ with $|c_{k,j}| \le M\frac{A_1^kA_2^j}{k!j!}$ for all $k, j = 0, 1, 2, \ldots$ (which implies $|f(z_1,z_2)| \le Me^{A_1|z_1|+A_2|z_2|}$ for all $(z_1,z_2) \in D_{R_1} \times D_{R_2}$).

We have the following upper estimate.

Theorem 1 *Let $q_1 = (q_{1,n})$ and $q_2 = (q_{2,m})$ be sequences satisfying the conditions given in equation (2) and let $n_0, m_0 \ge 2$, $0 < \beta \le \frac{2}{3}$, $\frac{1}{2} < r_1 < R_1 \le \frac{[n_0]_{q_1}^{1-\beta}}{2}$ and $\frac{1}{2} < r_2 < R_2 \le \frac{[m_0]_{q_2}^{1-\beta}}{2}$. If $f \in A_C$, then for all $n \ge n_0$, $m \ge m_0$, $|z_1| \le r_1$ and $|z_2| \le r_2$ the following inequality holds*:

$$\left|R_{n,m}^{q_1,q_2}(f)(z_1,z_2) - f(z_1,z_2)\right| \le \left(a_n + \frac{1}{b_n}\right)C^3(f) + \left(a_m + \frac{1}{b_m}\right)C^4(f),$$

where

$$C^3(f) = \max\left\{Mr_1r_2e^{2r_1A_1+r_2A_2}, 9Me^{r_2A_2}\sum_{k=1}^{\infty}(k-1)(20r_1A_1)^{k-1}\right\},$$

$$C^4(f) = \max\left\{\begin{array}{c}2M(r_2)^2e^{2r_2A_2}\sum_{k=0}^{\infty}(20r_1A_1)^k, \\ 9M\sum_{k=0}^{\infty}(20r_1A_1)^k\sum_{j=1}^{\infty}(j-1)(20r_2A_2)^{j-1}\end{array}\right\},$$

and also the series $\sum_{k=0}^{\infty}(20r_1A_1)^k$, $\sum_{k=1}^{\infty}(k-1)(20r_1A_1)^{k-1}$ and $\sum_{j=1}^{\infty}(j-1)(20r_2A_2)^{j-1}$ are convergent.

Proof Using Lemma 1, we can write

$$\left|R_{n,m}^{q_1,q_2}(f)(z_1,z_2) - f(z_1,z_2)\right| \le \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}|c_{k,j}|\left|R_{n,m}^{q_1,q_2}(e_{k,j})(z_1,z_2) - e_{k,j}(z_1,z_2)\right|. \qquad (4)$$

Taking into account Lemma 4 in [19] and the estimate given in the proof of Theorem 2 in [19], for all $|z_1| \le r_1$ and $|z_2| \le r_2$, we obtain

$$\begin{aligned}
&\left|R_{n,m}^{q_1,q_2}(e_{k,j})(z_1,z_2) - e_{k,j}(z_1,z_2)\right| \\
&\quad = \left|R_n^{q_1}\left(e_1^k\right)(z_1).R_m^{q_2}\left(e_2^j\right)(z_2) - z_1^k z_2^j\right| \\
&\quad \le \left|R_n^{q_1}\left(e_1^k\right)(z_1)\right|\left|R_m^{q_2}\left(e_2^j\right)(z_2) - z_2^j\right| + \left|z_2^j\right|\left|R_n^{q_1}\left(e_1^k\right)(z_1) - z_1^k\right| \\
&\quad \le (k!)(20r_1)^k\left\{2a_m(r_2)^2 j(2r_2)^{j-1} + \frac{9}{b_m}(j-1)(j!)(20r_2)^{j-1}\right\} \\
&\qquad + (r_2)^j\left\{2a_n(r_1)^2 k(2r_1)^{k-1} + \frac{9}{b_n}(k-1)(k!)(20r_1)^{k-1}\right\} \\
&\quad = 2a_n(r_1)^2(2r_1)^{k-1} j(r_2)^j + 2a_m(r_2)^2(k!)(20r_1)^k(2r_2)^{j-1} \\
&\qquad + \frac{9}{b_n}(k-1)(k!)(20r_1)^{k-1}(r_2)^j + \frac{9}{b_m}(k!)(20r_1)^k(j-1)(j!)(20r_2)^{j-1}.
\end{aligned} \tag{5}$$

Applying equation (5) in equation (4), we get

$$\begin{aligned}
&\left|R_{n,m}^{q_1,q_2}(f)(z_1,z_2) - f(z_1,z_2)\right| \\
&\quad \le a_n M r_1 r_2 e^{2r_1A_1 + r_2A_2} + 2a_m M(r_2)^2 e^{2r_2A_2}\sum_{k=0}^{\infty}(20r_1A_1)^k \\
&\qquad + \frac{9M}{b_n}e^{r_2A_2}\sum_{k=1}^{\infty}(k-1)(20r_1A_1)^{k-1} + \frac{9M}{b_m}\sum_{k=0}^{\infty}(20r_1A_1)^k\sum_{j=1}^{\infty}(j-1)(20r_2A_2)^{j-1}.
\end{aligned}$$

Choosing $C^3(f)$ and $C^4(f)$ as given in the theorem, we reach the desired result. □

For $f(z_1,z_2)$, we define the parametric extensions of the Voronovskaja formula by

$$\begin{aligned}
{}_{z_1}L_{n,q_1}(f)(z_1,z_2) := {}& R_n^{q_1}\left(f(\cdot,z_2)\right)(z_1) - f(z_1,z_2) - \psi_{n,q_1}^1(z_1)\frac{\partial f}{\partial z_1}(z_1,z_2) \\
& - \frac{1}{2}\psi_{n,q_1}^2(z_1)\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)
\end{aligned}$$

and

$$\begin{aligned}
{}_{z_2}L_{m,q_2}(f)(z_1,z_2) := {}& R_m^{q_2}\left(f(z_1,\cdot)\right)(z_2) - f(z_1,z_2) - \psi_{m,q_2}^1(z_2)\frac{\partial f}{\partial z_2}(z_1,z_2) \\
& - \frac{1}{2}\psi_{m,q_2}^2(z_2)\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2),
\end{aligned}$$

where $\psi_{k,q}^i(z) = R_k^q((t-z)^i;z)$ for $i = 1,2$ given in Lemma 6 in [19].

Their product (composition) gives

$$\begin{aligned}
&{}_{z_2}L_{m,q_2}(f)(z_1,z_2) \circ {}_{z_1}L_{n,q_1}(f)(z_1,z_2) \\
&\quad = R_m^{q_2}\left(R_n^{q_1}\left(f(\cdot,\cdot)\right)(z_1) - f(z_1,\cdot) - \psi_{n,q_1}^1(z_1)\frac{\partial f}{\partial z_1}(z_1,\cdot) - \psi_{n,q_1}^2(z_1)\frac{\partial^2 f}{\partial z_1^2}(z_1,\cdot)\right)(z_2)
\end{aligned}$$

$$
\begin{aligned}
&- \left[R_n^{q_1}\big(f(\cdot,z_2)\big)(z_1) - f(z_1,z_2) - \psi_{n,q_1}^1(z_1)\frac{\partial f}{\partial z_1}(z_1,z_2) - \psi_{n,q_1}^2(z_1)\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)\right] \\
&- \psi_{m,q_2}^1(z_2)\left[R_n^{q_1}\left(\frac{\partial f}{\partial z_2}(\cdot,z_2)\right)(z_1) - \frac{\partial f}{\partial z_2}(z_1,z_2) - \psi_{n,q_1}^1(z_1)\frac{\partial^2 f}{\partial z_2\,\partial z_1}(z_1,z_2)\right. \\
&\left. - \psi_{n,q_1}^2(z_1)\frac{\partial^3 f}{\partial z_2\,\partial z_1^2}(z_1,z_2)\right] \\
&- \psi_{m,q_2}^2(z_2)\left[R_n^{q_1}\left(\frac{\partial^2 f}{\partial z_2^2}(\cdot,z_2)\right)(z_1) - \frac{\partial^2 f}{\partial z_2^2}(z_1,z_2) - \psi_{n,q_1}^1(z_1)\frac{\partial^3 f}{\partial z_2^2\,\partial z_1}(z_1,z_2)\right. \\
&\left. - \psi_{n,q_1}^2(z_1)\frac{\partial^4 f}{\partial z_2^2\,\partial z_1^2}(z_1,z_2)\right] := E_1 - E_2 + E_3 - E_4. \qquad (6)
\end{aligned}
$$

After a simple calculation, we obtain the commutativity property,

$$
z_2L_{m,q_2}(f)(z_1,z_2) \circ z_1L_{n,q_1}(f)(z_1,z_2) = z_1L_{n,q_1}(f)(z_1,z_2) \circ z_2L_{m,q_2}(f)(z_1,z_2).
$$

In the following a Voronovskaja-type result for the operators (1) is presented. It will be the product of the parametric extensions generated by Voronovskaja's formula in the univariate case.

Theorem 2 *Let $q_1 = (q_{1,n})$ and $q_2 = (q_{2,m})$ be sequences satisfying the conditions given in equation* (2) *and let $n_0, m_0 \geq 2$, $0 < \beta \leq \frac{2}{3}$, $\frac{1}{2} < r_1 < R_1 \leq \frac{[n_0]_{q_1}^{1-\beta}}{2}$ and $\frac{1}{2} < r_2 < R_2 \leq \frac{[m_0]_{q_2}^{1-\beta}}{2}$. If $f \in A_C$, then for all $n \geq n_0$, $m \geq m_0$, $|z_1| \leq r_1$ and $|z_2| \leq r_2$ the following inequality holds*:

$$
\left|z_2L_{m,q_2}(f)(z_1,z_2) \circ z_1L_{n,q_1}(f)(z_1,z_2)\right| \leq C^5(f)\left[\left(a_n + \frac{1}{b_n}\right)^2 + \left(a_m + \frac{1}{b_m}\right)^2\right],
$$

where $C^5(f) = \frac{1}{2}\max\{C_{r_1,r_2}^1(f), C_{r_1,r_2}^2(f)\}$, C', and C'' are fixed constants,

$$
\begin{aligned}
C_{r_1,r_2}^1(f) = {}& Mr_1^3 e^{r_2A_2}\sum_{k=2}^{\infty}(k-2)(k-1)k(k+1)(20r_1A_1)^{k-3} \\
& \times \max\left\{C'e^{-r_2A_2}\sum_{j=0}^{\infty}(20r_2A_2)^j, C', C'r_2A_2, C''\big(1+r_2+r_2^2\big)r_2A_2^2\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
C_{r_1,r_2}^2(f) = {}& Mr_2^3 e^{r_1A_1}\sum_{j=2}^{\infty}(j-2)(j-1)j(j+1)(20r_2A_2)^{j-3} \\
& \times \max\left\{C'e^{-r_1A_1}\sum_{k=0}^{\infty}(20r_1A_1)^k, C', C'r_1A_1, C''\big(1+r_1+r_1^2\big)r_1A_1^2\right\}.
\end{aligned}
$$

Proof From the analyticity of f in $D_{R_1} \times D_{R_2}$, since all partial derivatives of f are analytic in $D_{R_1} \times D_{R_2}$, using Lemma 1, we can write

$$
\begin{aligned}
& R_n^{q_1}\big(f(\cdot,z_2)\big)(z_1) - f(z_1,z_2) - \psi_{n,q_1}^1(z_1)\frac{\partial f}{\partial z_1}(z_1,z_2) - \psi_{n,q_1}^2(z_1)\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2) \\
&\quad = \sum_{k=2}^{\infty} f_k(z_2)\big[R_n^{q_1}\big(e_1^k\big)(z_1) - e_1^k(z_1) - \psi_{n,q_1}^1(z_1)kz_1^{k-1} - \psi_{n,q_1}^2(z_1)k(k-1)z_1^{k-2}\big]. \qquad (7)
\end{aligned}
$$

Applying now $R_m^{q_2}$ to equation (7) with respect to z_2 and Lemma 1 in [19], we obtain

$$\begin{aligned}E_1 &= \sum_{k=2}^{\infty} R_m^{q_2}(f_k)(z_2)\bigl[R_n^{q_1}\bigl(e_1^k\bigr)(z_1) - e_1^k(z_1) - \psi_{n,q_1}^1(z_1)kz_1^{k-1} \\ &\quad - \psi_{n,q_1}^2(z_1)k(k-1)z_1^{k-2}\bigr] \\ &= \sum_{k=2}^{\infty}\sum_{j=0}^{\infty} c_{k,j}R_m^{q_2}\bigl(e_2^j\bigr)(z_2)\bigl[R_n^{q_1}\bigl(e_1^k\bigr)(z_1) - e_1^k(z_1) - \psi_{n,q_1}^1(z_1)kz_1^{k-1} \\ &\quad - \psi_{n,q_1}^2(z_1)k(k-1)z_1^{k-2}\bigr]. \qquad (8)\end{aligned}$$

In equation (8), passing now to absolute value for $|z_1| \le r_1$ and $|z_2| \le r_2$ and taking into account the Lemma 4 in [19] and the estimate given in the proof of Theorem 3 in [19], it follows that

$$\begin{aligned}|E_1| &\le \left(a_n + \frac{1}{b_n}\right)^2 \sum_{j=0}^{\infty}|c_{k,j}|j!(20r_2)^j \sum_{k=2}^{\infty} C(k-2)(k-1)k(k+1)k!(20r_1)^{k+3} \\ &\le \left(a_n + \frac{1}{b_n}\right)^2 MC'r_1^3\sum_{j=0}^{\infty}(20r_2A_2)^j\sum_{k=2}^{\infty}(k-2)(k-1)k(k+1)(20r_1A_1)^{k-3} \qquad (9)\end{aligned}$$

for $|z_1| \le r_1$ and $|z_2| \le r_2$.

Similarly, using the estimate given in the proof of Theorem 3 in [19] for $|z_1| \le r_1$ and $|z_2| \le r_2$ we have

$$\begin{aligned}|E_2| &\le \sum_{k=2}^{\infty}\bigl|f_k(z_2)\bigr|\bigl|R_n^{q_1}\bigl(e_1^k\bigr)(z_1) - e_1^k(z_1) - \psi_{n,q_1}^1(z_1)kz_1^{k-1} - \psi_{n,q_1}^2(z_1)k(k-1)z_1^{k-2}\bigr| \\ &\le \left(a_n + \frac{1}{b_n}\right)^2 \sum_{j=0}^{\infty}|c_{k,j}|r_2^j\sum_{k=2}^{\infty}C(k-2)(k-1)k(k+1)k!(20r_1)^{k+3} \\ &\le \left(a_n + \frac{1}{b_n}\right)^2 MC'r_1^3e^{r_2A_2}\sum_{k=2}^{\infty}(k-2)(k-1)k(k+1)(20r_1A_1)^{k-3}. \qquad (10)\end{aligned}$$

Using

$$\begin{aligned}R_n^{q_1}\left(\frac{\partial f}{\partial z_2}(\cdot, z_2)\right)(z_1) &= \sum_{k=0}^{\infty}\frac{\partial f_k}{\partial z_2}(z_2)R_n^{q_1}\bigl(e_1^k\bigr)(z_1) \\ &= \sum_{k=0}^{\infty}\sum_{j=1}^{\infty}c_{k,j}jz_2^{j-1}R_n^{q_1}\bigl(e_1^k\bigr)(z_1),\end{aligned}$$

we can write

$$\begin{aligned}E_3 &= \psi_{m,q_2}^1(z_2)\left[R_n^{q_1}\left(\frac{\partial f}{\partial z_2}(\cdot, z_2)\right)(z_1) - \frac{\partial f}{\partial z_2}(z_1,z_2) - \psi_{n,q_1}^1(z_1)\frac{\partial^2 f}{\partial z_2\,\partial z_1}(z_1,z_2) \right. \\ &\quad \left. - \psi_{n,q_1}^2(z_1)\frac{\partial^3 f}{\partial z_2\,\partial z_1^2}(z_1,z_2)\right]\end{aligned}$$

$$= \psi^1_{m,q_2}(z_2)\sum_{k=2}^{\infty}\sum_{j=1}^{\infty}c_{k,j}jz_2^{j-1}\bigl[R_n^{q_1}\bigl(e_1^k\bigr)(z_1) - e_1^k(z_1) - \psi^1_{n,q_1}(z_1)kz_1^{k-1} - \psi^2_{n,q_1}(z_1)k(k-1)z_1^{k-2}\bigr].$$

Considering Lemma 6 in [19] and the estimate given in the proof of Theorem 3 in [19], for $|z_1| \le r_1$ and $|z_2| \le r_2$, we obtain

$$\begin{aligned}
|E_3| &\le \left(a_n + \frac{1}{b_n}\right)^2 \bigl|\psi^1_{m,q_2}(z_2)\bigr| \sum_{k=2}^{\infty}\sum_{j=1}^{\infty}|c_{k,j}|jr_2^{j-1}C(k-2)(k-1)k(k+1)k!(20r_1)^{k+3} \\
&\le \left(a_n + \frac{1}{b_n}\right)^2 MC' r_1^3 r_2 A_2 e^{r_2A_2}\sum_{k=2}^{\infty}(k-2)(k-1)k(k+1)(20r_1A_1)^{k-3}
\end{aligned} \tag{11}$$

and also, using

$$\begin{aligned}
R_n\left(\frac{\partial^2 f}{\partial z_2^2}(\cdot, z_2)\right)(z_1) &= \sum_{k=0}^{\infty}\frac{\partial^2 f_k}{\partial z_2^2}(z_2)R_n\bigl(e_1^k\bigr)(z_1) \\
&= \sum_{k=0}^{\infty}\sum_{j=2}^{\infty}c_{k,j}j(j-1)z_2^{j-2}R_n\bigl(e_1^k\bigr)(z_1),
\end{aligned}$$

we can write

$$\begin{aligned}
E_4 &= \psi^2_{m,q_2}(z_2)\left[R_n\left(\frac{\partial^2 f}{\partial z_2^2}(\cdot, z_2)\right)(z_1) - \frac{\partial^2 f}{\partial z_2^2}(z_1,z_2) - \psi^1_{n,q_1}(z_1)\frac{\partial^3 f}{\partial z_1\,\partial z_2^2}(z_1,z_2) \right.\\
&\quad \left. - \psi^2_{n,q_1}(z_1)\frac{\partial^4 f}{\partial z_1^2\,\partial z_2^2}(z_1,z_2)\right] \\
&= \psi^2_{m,q_2}(z_2)\sum_{k=2}^{\infty}\sum_{j=2}^{\infty}c_{k,j}j(j-1)z_2^{j-2}\bigl[R_n^{q_1}\bigl(e_1^k\bigr)(z_1) - e_1^k(z_1) - \psi^1_{n,q_1}(z_1)kz_1^{k-1} \\
&\quad - \psi^2_{n,q_1}(z_1)k(k-1)z_1^{k-2}\bigr].
\end{aligned}$$

Taking into account Lemma 6 in [19] and the estimate given in the proof of Theorem 3 in [19], for $|z_1| \le r_1$ and $|z_2| \le r_2$ we get

$$\begin{aligned}
|E_4| &\le \left(a_n + \frac{1}{b_n}\right)^2 \bigl|\psi^2_{m,q_2}(z_2)\bigr| \sum_{j=2}^{\infty}j(j-1)r_2^{j-2} \\
&\quad \times \sum_{k=2}^{\infty}|c_{k,j}|C(k-2)(k-1)k(k+1)k!(20r_1)^{k+3} \\
&\le \left(a_n + \frac{1}{b_n}\right)^2 MC'' r_1^3\bigl(1 + r_2 + r_2^2\bigr)r_2A_2^2e^{A_2r_2} \\
&\quad \times \sum_{k=2}^{\infty}(k-2)(k-1)k(k+1)(20A_1r_1)^{k-3}
\end{aligned} \tag{12}$$

for $|z_1| \leq r_1$ and $|z_2| \leq r_2$. Using equations (9)-(12), we get

$$\begin{aligned}\left|z_2 L_{m,q_2}(f)(z_1,z_2) \circ z_1 L_{n,q_1}(f)(z_1,z_2)\right| &\leq |E_1| + |E_2| + |E_3| + |E_4| \\ &\leq C^1_{r_1,r_2}(f)\left(a_n + \frac{1}{b_n}\right)^2.\end{aligned}$$

If we estimate $|z_1 L_{n,q_1}(f)(z_1,z_2) \circ z_2 L_{m,q_2}(f)(z_1,z_2)|$, then by reason of the symmetry we get a similar order of approximation, simply interchanging above the places of n with m and r_1 with r_2.

In conclusion, using the commutativity property, we reach the result. □

Let us denote by $A_C^{(2)}$ the space of all complex valued functions where they and their first and second partial derivatives are uniformly continuous on $(D_{R_1} \cup [R_1,\infty)) \times (D_{R_2} \cup [R_2,\infty))$, bounded on $[0,\infty) \times [0,\infty)$ and analytic in $D_{R_1} \times D_{R_2}$, and there exist $M > 0$, $0 < A_1 < \frac{1}{20r_1}$, $0 < A_2 < \frac{1}{20r_2}$ with $|c_{k,j}| \leq M\frac{A_1^k A_2^j}{k!j!}$ (which implies $|f(z_1,z_2)| \leq Me^{A_1|z_1|+A_2|z_2|}$ for all $(z_1,z_2) \in D_{R_1} \times D_{R_2}$).

Theorems 1 and 2 will be used to find the exact degree in the approximation of $R_{n,n}^{q_1,q_2}(f)$. In this sense, we have the following lower estimate.

Theorem 3 *Let $q_1 = (q_{1,n})$ and $q_2 = (q_{2,n})$ be sequences satisfying the conditions given in equation* (2) *and let $n_0 \geq 2$, $0 < \beta < \frac{1}{2}$, $\frac{1}{2} < r_1 < R_1 \leq \frac{[n_0]_{q_1}^{1-\beta}}{2}$ and $\frac{1}{2} < r_2 < R_2 \leq \frac{[n_0]_{q_2}^{1-\beta}}{2}$. If $f \in A_C^{(2)}$ and f is not a solution of the complex partial differential equation*

$$K(f)(z_1,z_2) = z_1\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2) + z_2\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2) = 0,$$

then for all $n \geq n_0$ we have

$$\left\|R_{n,n}^{q_1,q_2}(f) - f\right\|_{r_1,r_2} \geq \frac{1}{36(1 + a_n b_n)}\left(a_n + \frac{1}{b_n}\right)\left\|K(f)\right\|_{r_1,r_2}.$$

Proof From equation (6), we can write

$$\begin{aligned}&R_{n,n}^{q_1,q_2}(f)(z_1,z_2) - f(z_1,z_2) \\ &\quad= 2\left(a_n + \frac{1}{b_n}\right)\left\{K_n(f)(z_1,z_2) + 2\left(a_n + \frac{1}{b_n}\right)\left[\frac{D_n(f)(z_1,z_2)}{4(a_n + \frac{1}{b_n})^2}\right]\right. \\ &\qquad\left. + E_n(f)(z_1,z_2) + F_n(f)(z_1,z_2) + G_n(f)(z_1,z_2)\right\},\end{aligned} \tag{13}$$

where

$$D_n(f)(z_1,z_2) = z_2 L_{n,q_2}(f)(z_1,z_2) \circ z_1 L_{n,q_1}(f)(z_1,z_2),$$

$$E_n(f)(z_1,z_2) = \frac{z_1 L_{n,q_1}(f)(z_1,z_2) + z_2 L_{n,q_2}(f)(z_1,z_2)}{2(a_n + \frac{1}{b_n})},$$

$$F_n(f)(z_1,z_2) = \sum_{h=1}^{4} F_n^h(f)(z_1,z_2)$$

with

$$\begin{aligned}
F_n^1(f)(z_1,z_2) &= \frac{b_n\psi_{n,q_1}^1(z_1)}{2(1+a_nb_n)}\left[R_n^{q_2}\left(\frac{\partial f}{\partial z_1}(z_1,\cdot)\right)(z_2)-\frac{\partial f}{\partial z_1}(z_1,z_2)\right],\\
F_n^2(f)(z_1,z_2) &= \frac{b_n\psi_{n,q_2}^1(z_2)}{2(1+a_nb_n)}\left[R_n^{q_1}\left(\frac{\partial f}{\partial z_2}(\cdot,z_2)\right)(z_1)-\frac{\partial f}{\partial z_2}(z_1,z_2)\right],\\
F_n^3(f)(z_1,z_2) &= \frac{b_n\psi_{n,q_1}^2(z_1)}{4(1+a_nb_n)}\left[R_n^{q_2}\left(\frac{\partial^2 f}{\partial z_1^2}(z_1,\cdot)\right)(z_2)-\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)\right],\\
F_n^4(f)(z_1,z_2) &= \frac{b_n\psi_{n,q_2}^2(z_2)}{4(1+a_nb_n)}\left[R_n^{q_1}\left(\frac{\partial^2 f}{\partial z_2^2}(\cdot,z_2)\right)(z_1)-\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2)\right],\\
G_n(f)(z_1,z_2) &= \frac{b_n\psi_{n,q_1}^1(z_1)}{2(1+a_nb_n)}\frac{\partial f}{\partial z_1}(z_1,z_2)+\frac{b_n\psi_{n,q_2}^1(z_2)}{2(1+a_nb_n)}\frac{\partial f}{\partial z_2}(z_1,z_2)\\
&\quad-\frac{b_n\psi_{n,q_2}^1(z_2)\psi_{n,q_1}^1(z_1)}{2(1+a_nb_n)}\frac{\partial^2 f}{\partial z_2\,\partial z_1}(z_1,z_2)\\
&\quad-\frac{b_n\psi_{n,q_2}^1(z_2)\psi_{n,q_1}^2(z_1)}{4(1+a_nb_n)}\frac{\partial^3 f}{\partial z_2\,\partial z_1^2}(z_1,z_2)\\
&\quad-\frac{b_n\psi_{n,q_2}^2(z_2)\psi_{n,q_1}^1(z_1)}{4(1+a_nb_n)}\frac{\partial^3 f}{\partial z_2^2\,\partial z_1}(z_1,z_2)\\
&\quad-\frac{b_n\psi_{n,q_2}^2(z_2)\psi_{n,q_1}^2(z_1)}{8(1+a_nb_n)}\frac{\partial^4 f}{\partial z_2^2\,\partial z_1^2}(z_1,z_2),
\end{aligned}$$

and

$$K_n(f)(z_1,z_2)=\frac{b_n}{4(1+a_nb_n)}\left\{\psi_{n,q_1}^2(z_1)\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)+\psi_{n,q_2}^2(z_2)\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2)\right\}.$$

Considering Theorems 2 and 3 in [19], we get

$$\lim_{n\to\infty}E_n(f)(z_1,z_2)=0\quad\text{and}\quad\lim_{n\to\infty}F_n(f)(z_1,z_2)=0.$$

Under the conditions of the theorem, since $\lim_{n\to\infty}a_n=0$, $\lim_{n\to\infty}\frac{1}{b_n}=0$, $\lim_{n\to\infty}a_n\times b_n=0$ for $0<\beta<\frac{1}{2}$, it is also clear that

$$\lim_{n\to\infty}G_n(f)(z_1,z_2)=0.$$

From Theorem 2, we obtain

$$\lim_{n\to\infty}\left\|2\left(a_n+\frac{1}{b_n}\right)\left[\frac{D_n(f)}{4(a_n+\frac{1}{b_n})^2}\right]+E_n(f)+F_n(f)+G_n(f)\right\|_{r_1,r_2}=0.$$

Using $\lim_{n\to\infty}a_nb_n=0$ for $0<\beta<\frac{1}{2}$ and $\frac{1}{1+a_n|z_1|}\geq\frac{2}{3}$, we get

$$\left\|K_n(f)\right\|_{r_1,r_2}\geq\frac{1}{18(1+a_nb_n)}|z_1|\left|\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)\right|. \tag{14}$$

Similarly, it follows that

$$\|K_n(f)\|_{r_1,r_2} \geq \frac{1}{18(1+a_nb_n)}|z_2|\left|\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2)\right|. \tag{15}$$

From equations (14) and (15), we can write

$$\|K_n(f)\|_{r_1,r_2} \geq \frac{1}{36(1+a_nb_n)}\|K(f)\|_{r_1,r_2}. \tag{16}$$

In equation (13), taking into account the inequalities

$$\|H+T\|_{r_1,r_2} \geq \big|\|H\|_{r_1,r_2} - \|T\|_{r_1,r_2}\big| \geq \|H\|_{r_1,r_2} - \|T\|_{r_1,r_2},$$

and equation (16), it follows that

$$\begin{aligned}\|R_{n,n}^{q_1,q_2}(f)-f\|_{r_1,r_2} &\geq 2\left(a_n+\frac{1}{b_n}\right)\Bigg\{\|K_n(f)\|_{r_1,r_2} \\ &\qquad - \left\|2\left(a_n+\frac{1}{b_n}\right)\left[\frac{D_n(f)}{4(a_n+\frac{1}{b_n})^2}\right]+E_n(f)+F_n(f)+G_n(f)\right\|_{r_1,r_2}\Bigg\} \\ &\geq \left(a_n+\frac{1}{b_n}\right)\|K_n(f)\|_{r_1,r_2} \\ &\geq \left(a_n+\frac{1}{b_n}\right)\frac{1}{36(1+a_nb_n)}\|K(f)\|_{r_1,r_2}\end{aligned}$$

for all $n \geq n_0$ with n_0 depending only f, r_1 and r_2. We used that by hypothesis we have $\|K(f)\|_{r_1,r_2} > 0$. □

Combining Theorem 2 with Theorem 3, we immediately obtain the following result giving the exact degree of the operators (1).

Corollary 1 *Suppose that the hypothesis in the statement of Theorem* 3 *holds. If the Taylor series of f contains at least one term of the form $c_{k,0}z_1^k$ with $c_{k,0} \neq 0$ and $k \geq 2$ or of the form $c_{0,j}z_2^j$ with $c_{0,j} \neq 0$ and $j \geq 2$, then for all $n \geq n_0$ we have*

$$\|R_{n,n}^{q_1,q_2}(f)-f\|_{r_1,r_2} \sim \left(a_n+\frac{1}{b_n}\right).$$

Proof It suffices to prove that, under the hypothesis on f, it cannot be a solution of the complex partial differential equation

$$z_1\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)+z_2\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2)=0, \quad |z_1|<R_1, |z_2|<R_2.$$

Indeed, suppose the contrary. Since a simple calculation gives

$$\begin{aligned}z_1\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)+z_2\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2) &= \sum_{k=1}^{\infty}c_{k+1,0}k(k+1)z_1^k+\sum_{k=1}^{\infty}c_{k+1,1}k(k+1)z_1^kz_2 \\ &\quad + 2\sum_{j=2}^{\infty}c_{2,j}z_1z_2^j+\sum_{k=2}^{\infty}\sum_{j=2}^{\infty}c_{k+1,j}k(k+1)z_1^kz_2^j,\end{aligned}$$

$$+\sum_{j=1}^{\infty}c_{0,j+1}j(j+1)z_2^j+\sum_{j=1}^{\infty}c_{1,j+1}j(j+1)z_1z_2^j$$
$$+2\sum_{k=2}^{\infty}c_{k,2}z_1^kz_2+\sum_{k=2}^{\infty}\sum_{j=2}^{\infty}c_{k,j+1}j(j+1)z_1^kz_2^j,$$

by setting equal to zero and by the identification of the coefficients, from the terms under the first and fifth sign $\sum$, we immediately get $c_{k+1,0}=c_{0,j+1}=0$, for all $k=1,2,\ldots$ and $j=1,2,\ldots$, which contradicts the hypothesis on f. Therefore the hypothesis and the lower estimate in Theorem 3 are satisfied, which completes the proof. □

Competing interests
I declare that I have no competing interests.

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