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On spectral properties of the modified convolution operator

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Abstract

We investigated the s-number of the modified convolution operator and obtained the following results

$$c_1 \sup_{Q\in G} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left|\int_Q \varphi(x)\,dx\right| \le \|\varphi\|_{M_p^q} \le c_3 \sup_{Q\in F} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left|\int_Q \varphi(s)\,ds\right|,$$

where $1 < p < 2 < q < \infty$, $p' = \frac{p}{p-1}$, G is a set of all segments Q from $[0,1]$, F is a set of all compacts from $[0,1]$, $|Q|$ is the measure of a set Q.

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1 Introduction

Let $1 \le p < \infty$, $0 < q \le \infty$. We denote by $\mathfrak{S}_{p,q}$ the space of all compact operators A, acting in the space $L_2[0,1]$ of all 1-periodic functions square integrable on $[0,1]$ for s-numbers such that the following quasinorm is finite

$$\|A\|_{\mathfrak{S}_{p,q}} = \left(\sum_{m=1}^{\infty} s_m^q(A) m^{q/p-1}\right)^{1/q},$$

if $q < \infty$, and

$$\|A\|_{\mathfrak{S}_{p,\infty}} = \sup_m m^{\frac{1}{p}} s_m \quad \text{if } q = \infty.$$

Recall that the sequence $s_m(A)$ (s-numbers of operator A) are numerated eigenvalues of the operator $\sqrt{A^*A}$.

We consider the convolution operator

$$(Af)(y) = \int_0^1 K(x-y)f(x)\,dx$$

acting in $L_2[0,1]$. Given a function $\varphi \in L_1[0,1]$, we consider also the modified convolution operator

$$(A_\varphi f)(y) = \int_0^1 (K\varphi)(x-y)f(x)\,dx.$$

We say that φ belongs to the space $M_{p_0,q_0}^{p_1,q_1}$, if for $A \in \mathfrak{S}_{p_0,q_0}$, $A_\varphi \in \mathfrak{S}_{p_1,q_1}$ and

$$\|A_\varphi\|_{\mathfrak{S}_{p_1,q_1}} \le c\|A\|_{\mathfrak{S}_{p_0,q_0}},$$

where $c > 0$ depends only on p_0, q_0, p_1, q_1.

This means that the linear operator R_φ defined by the equality $R_\varphi(A) = A_\varphi$ is bounded from $\mathfrak{S}_{p_0,q_0}$ to $\mathfrak{S}_{p_1,q_1}$. Let

$$\|\varphi\|_{M_{p_0,q_0}^{p_1,q_1}} = \|R_\varphi\|_{\mathfrak{S}_{p_0,q_0}\longrightarrow\mathfrak{S}_{p_1,q_1}}.$$

Given that the eigenvalues of the operator $K * f$ coincide with the Fourier coefficients of the kernel K with respect to the trigonometric system, in the case $p_0 = p_1 = q_0 = q_1 = p$ this problem reduces to the well-known problem of Fourier series multipliers. Let $K \in L_1([0,1])$ and $\{a_m(K)\}_{m\in Z}$ be the sequence of its Fourier coefficients with respect to the trigonometric system $\{e^{2\pi ikx}\}_{k\in Z}$. It is assumed that K is such that $\{a_m(K)\}_{m\in Z} \in l_p$, $1 \le p \le \infty$. Let $T_\varphi = \{a_m(K\varphi)\}_{m\in Z} \in l_p$. The problem is to determine conditions on the function φ ensuring the boundedness of the operator $T_\varphi : l_p \longrightarrow l_p$.

This problem was considered in the works of Stechkin [1], Hirschman [2], Edelstein [3], Birman and Solomyak [4], Karadzhov [5], and others.

We obtain sufficient conditions on a multiplier φ ensuring that it belongs to the space $M_{p_0,q_0}^{p_1,q_1}$. These conditions are expressed in terms of Lorentz and Besov spaces. We also construct examples showing the sharpness of the obtained constants for corresponding embedding theorems.

2 Main results

Let f be a μ measurable function which takes finite values almost everywhere and let

$$m(\sigma,f) = \mu\big(\{x : x \in [0,1], |f| > \sigma\}\big)$$

be its distribution function. The function

$$f^*(t) = \inf\{\sigma : m(\sigma,f) \le t\}$$

is a nonincreasing rearrangement of f.

We say that a function f belongs to the Lorentz space $L_{p,q}$ if f is measurable and

$$\|f\|_{L_{p,q}} = \left(\int_0^\infty \big(t^{\frac{1}{p}}f^*(t)\big)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty,$$

for $1 \le q < \infty$ and

$$\|f\|_{L_{p,\infty}} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty,$$

for $q = \infty$.

Theorem 1 *Let* $1 < p_0 < 2 \le p_1$, $1 \le q_1 \le q_0 \le \infty$, $\frac{1}{r} = \frac{1}{p_0} - \frac{1}{p_1}$, $\frac{1}{s} = \frac{1}{q_1} - \frac{1}{q_0}$ *and* $\varphi \in L_{r,s}[0,1]$. *If* $A \in \mathfrak{S}_{p_0,q_0}$, *then* $A_\varphi \in \mathfrak{S}_{p_1,q_1}$ *and*

$$\|A_\varphi\|_{\mathfrak{S}_{p_1,q_1}} \le c\|\varphi\|_{L_{r,s}}\|A\|_{\mathfrak{S}_{p_0,q_0}},$$

i.e. $L_{r,s}[0,1] \hookrightarrow M_{p_0,q_0}^{p_1,q_1}$.

In the following theorem the cases $p = p_o = q_0$, $q = p_1 = q_1$ are considered. The upper and the lower estimates of the norm $\|\varphi\|_{M_p^q}$ ($M_p^q := M_{p,p}^{q,q}$) are obtained.

Theorem 2 *Let* $1 < p < 2 < q < \infty$, $p' = \frac{p}{p-1}$. *Let G be a set of all segments Q from* $[0,1]$, *F be a set of all compacts from* $[0,1]$, *then*

$$c_1 \sup_{Q \in G} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left| \int_Q \varphi(x)\,dx \right| \le \|\varphi\|_{M_p^q} \le c_3 \sup_{Q \in F} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left| \int_Q \varphi(s)\,ds \right|,$$

where $|Q|$ *is the measure of a set Q.*

We shall define the class of generalized monotone functions for which the upper and the lower estimates coincide.

We say that function f is a generalized monotone function, if there exists a constant $c > 0$ such that for every $x \in (0,1]$ the inequality

$$|f(x)| \le \frac{c}{x} \left| \int_0^x f(y)\,dy \right|$$

holds. The class of such functions is denoted by $\mathfrak{N}$.

Corollary 1 *Let* $1 < p < 2 < q < \infty$. *If* $\varphi \in \mathfrak{N}$, *then* $\varphi \in M_p^q$ *if and only if*

$$\sup_{t>0} t^{\frac{1}{p}-\frac{1}{q}} \varphi^*(t) < \infty.$$

Moreover, $\|\varphi\|_{M_p^q} \sim \sup_{t>0} t^{\frac{1}{p}-\frac{1}{q}} \varphi^*(t)$.

In case parameters p_0, p_1 are both either less or greater than 2, we use the space of smooth functions.

Let $1 \le p < \infty$, $\alpha > 0$. We denote by $B_{p,q}^\alpha[0,1]$ the space of all measurable functions f on $[0,1]$ such that

$$\|f\|_{B_{p,q}^\alpha} = \left(\sum_{k=0}^{\infty} \left(2^{\alpha k}\|\Delta_k f\|_p\right)^q \right)^{\frac{1}{q}} < \infty$$

for $1 \le q < \infty$, and

$$\|f\|_{B_{p,\infty}^\alpha} = \sup_k 2^{\alpha k}\|\Delta_k f\|_p < \infty$$

for $q = \infty$. Here $\{a_m(f)\}_{m\in N}$ are the Fourier coefficients of the function f by trigonometric system $\{e^{2\pi ikx}\}_{k\in\mathbb{Z}}$, $\Delta_k f = \Delta_k f(x) = \sum_{[2^{k-1}]\le|m|<2^k} a_m(f)e^{2\pi imx}$, and $[2^{k-1}]$ is the integer part of 2^{k-1}.

This class is called the Nikol'skii-Besov space.

Theorem 3 *Let* $1 < p_0 \le p_1 < \infty$, $2 \notin [p_0, p_1]$, $1 < q_0 \le q_1 \le \infty$,

$$\alpha = \min_{x\in[\frac{1}{p_1},\frac{1}{p_0}]}\left|\frac{1}{2} - x\right|, \qquad \frac{1}{r} = \max_{x\in[\frac{1}{p_1},\frac{1}{p_0}]}\left|\frac{1}{2} - x\right|, \qquad 1 - \frac{1}{s} = \frac{1}{q_0} - \frac{1}{q_1}$$

and $\varphi \in B^{\alpha}_{r,s}[0,1]$.

If $A \in \mathfrak{S}_{p_0,q_0}$, *then* $A_\varphi \in \mathfrak{S}_{p_1,q_1}$ *and*

$$\|A_\varphi\|_{\mathfrak{S}_{p_1,q_1}} \le c\|\varphi\|_{B^{\alpha}_{r,s}}\|A\|_{\mathfrak{S}_{p_0,q_0}},$$

i.e., $B^{\alpha}_{r,s} \hookrightarrow M^{p_1,q_1}_{p_0,q_0}$.

In the case $p_0 = p_1 = q_0 = q_1$, Karadzhov's result (see [5]) follows from Theorem 3:

$$B^{\frac{1}{r}}_{r,1} \hookrightarrow M_p = M^{p,p}_{p,p}, \qquad \frac{1}{r} = \left|\frac{1}{p} - \frac{1}{2}\right|.$$

Now consider the case $1 \le q_1 < q_0 \le \infty$.

Theorem 4 *Let* $1 < p_0 < p_1 < \infty$, $1 \le q_1 < q_0 \le \infty$, $2 \notin (p_0, p_1)$, $\frac{1}{r} - \alpha = \frac{1}{p_0} - \frac{1}{p_1}$, $\frac{1}{s} = \frac{1}{q_1} - \frac{1}{q_0}$, $\alpha > \min_{x\in[\frac{1}{p_1},\frac{1}{p_0}]}|\frac{1}{2} - x|$.

Then $B^{\alpha}_{r,s}[0,1] \hookrightarrow M^{p_1,q_1}_{p_0,q_0}$.

3 Properties of $M^{p_1,q_1}_{p_0,q_0}$ class

To prove the properties of $M^{p_1,q_1}_{p_0,q_0}$ class we need the following lemma. We first define a discrete Lorentz space. l_{pq} is called a discrete Lorentz space whose elements are sequences of numbers $\xi = \{\xi_k\}^{\infty}_{k=\infty}$ with the only limit point 0 such that

$$\|\xi\|_{l_{pq}} = \left(\sum_{m=1}^{\infty}|\xi^*_m|^q m^{\frac{q}{p}-1}\right)^{\frac{1}{q}}, \quad 1 \le q < \infty$$

where $\{\xi^*_m\}^{\infty}_{m=1}$ nonincreasing rearrangement of the sequence $\{|\xi_k|\}^{\infty}_{k=\infty}$.

For $q = \infty$,

$$\|\xi\|_{l_{p\infty}} = \sup_m m^{\frac{1}{p}}\xi^*_m.$$

Lemma 1 (See [6]) *Let* $1 < r, p_0, p_1 < \infty$, $1 \le q_0, q_1, s \le \infty$. *Then*

$$\|a * b\|_{l_{p_1,q_1}} \le c\|b\|_{l_{r,s}}\|a\|_{l_{p_0,q_0}},$$

where $\frac{1}{p_1} + 1 = \frac{1}{r} + \frac{1}{p_0}$, $\frac{1}{q_1} = \frac{1}{s} + \frac{1}{q_0}$.

Let $\bar{X} = (X_0, X_1)$, where X_0, X_1 are Banach spaces, be a compatible pair. We define the functional $K(t, a)$ for $t > 0$ and $a \in X_0 + X_1$ by the following formula:

$$K(t,a) = \inf_{a=a_0+a_1} \left(\|a_0\|_{X_0} + t\|a_1\|_{X_1} \right).$$

We denote by $\bar{X}_{\theta,q,k}$ the space $\{a \in X_0 + X_1 : \|a\|_{\theta,q,k} = \Phi_{\theta,q}(K(t,a))\}$, where $\Phi_{\theta,q}$ is a functional defined on nonnegative functions φ by formula

$$\Phi_{\theta,q}\big(\varphi(t)\big) = \left(\int_0^\infty \big(t^{-\theta}\varphi(t)\big)^q \frac{dt}{t} \right)^{\frac{1}{q}}, \quad 1 \le q < \infty$$

and

$$\Phi_{\theta,\infty}\big(\varphi(t)\big) = \sup_{t>0} t^{-\theta}\varphi(t), \quad q = \infty.$$

Let $X_{\alpha_1^0,p_1^0}$ and $X_{\alpha_2^1,p_2^1}$ be the spaces obtained by the method of real interpolation of Banach pairs of spaces (X_0^1, X_1^1), (X_0^2, X_1^2) respectively.

Lemma 2 (See [7]) *Let $0 < \alpha_i, \beta_i < 1$, $1 \le p_i, q_i \le \infty$, $i = 0,1$, $\alpha_0 \ne \alpha_1$, $\beta_0 \ne \beta_1$. If T is a bilinear operator:*

$$T : X_{\alpha_0,p_0} \times Y_0 \longrightarrow Z_{\beta_0,q_0}$$

and

$$T : X_{\alpha_1,p_1} \times Y_1 \longrightarrow Z_{\beta_1,q_1}$$

then

$$T : X_{\alpha,p} \times Y_{\theta,r} \longrightarrow Z_{\beta,q}.$$

Here $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\beta = (1-\theta)\beta_0 + \theta\beta_1$, $\frac{1}{p} + \frac{1}{r} > 1$, $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r} + (1-\theta)(\frac{1}{q_0} - \frac{1}{p_0}) + \theta(\frac{1}{q_1} - \frac{1}{p_1})_+$, $x_+ = \max(x, 0)$.

Remark Since the s-numbers of convolution operator A coincide with the modules of the Fourier coefficients of the kernel K, the problem of estimating the s-numbers of "transformed" operator A_φ can be reduced to the study of the following inequality

$$\|a * b\|_{l_{p_1,q_1}} \le c\|a\|_{lp_0,q_0}, \tag{1}$$

and we have to describe the class of those functions φ with Fourier coefficients $b = \{b_m\}_{m\in Z}$, for which Inequality (1) holds.

Theorem 5

(1) Let $1 \le p_0, p_1 < \infty$, $1 \le q_0, q_1 \le \infty$, $\frac{1}{p_i} + \frac{1}{p_i'} = \frac{1}{q_i} + \frac{1}{q_i'} = 1$, $i = 0,1$. Then

$$M_{p_0,q_0}^{p_1,q_1} = M_{p_1',q_1'}^{p_0',q_0'}.$$

(2) Let $1 < p_0 < r_0 < p_1' < \infty$, $\frac{1}{p_1} + \frac{1}{p_1'} = 1$, $\frac{1}{p_1} - \frac{1}{p_0} = \frac{1}{r_1} - \frac{1}{r_0}$, *then*

$$M^{p_1,q_1}_{p_0,q_0} \hookrightarrow M^{r_1,t}_{r_0,s},$$

where $\frac{1}{t} - \frac{1}{s} = (\frac{1}{q_1} - \frac{1}{q_0})_+$.

Proof The proof of the first statement follows from Remark and from the fact that $\|(T_\varphi)^*\| = \|T_{\bar{\varphi}}\|$, where $\bar{\varphi}$ is a complex conjugate of the function φ. Now we prove (2).

Let $\varphi \in M^{p_1,q_1}_{p_0,q_0}$, then by (1) it follows that $\varphi \in M^{p_0',q_0'}_{p_1',q_1'}$, and

$$\|A_\varphi\|_{\mathfrak{S}_{p_1,q_1}} \le \|\varphi\|_{M^{p_1,q_1}_{p_0,q_0}} \|A\|_{\mathfrak{S}_{p_0,q_0}}, \quad \forall A \in \mathfrak{S}_{p_0,q_0},$$
$$\|A_\varphi\|_{\mathfrak{S}_{p_0',q_0'}} \le \|\varphi\|_{M^{p_1,q_1}_{p_0,q_0}} \|A\|_{\mathfrak{S}_{p_1',q_1'}}, \quad \forall A \in \mathfrak{S}_{p_1',q_1'},$$

where $\frac{1}{p_i} + \frac{1}{p_i'} = \frac{1}{q_i} + \frac{1}{q_i'} = 1$. According to Lemma 1, the operator $T(a,\varphi) = a * b$

$$T : l_{p_0,q_0} \times M^{p_1,q_1}_{p_0,q_0} \longrightarrow l_{p_1,q_1}$$

is bounded. Using (1) we have

$$T : l_{p_1',q_1'} \times M^{p_1,q_1}_{p_0,q_0} \longrightarrow l_{p_0',q_0'}.$$

Further, applying the theorem on bilinear interpolation (Lemma 2) we find that the operator

$$T : l_{r_0,s} \times M^{p_1,q_1}_{p_0,q_0} \longrightarrow l_{r_1,t}$$

is also bounded, i.e., $M^{p_1,q_1}_{p_0,q_0} \hookrightarrow M^{r_1,t}_{r_0,s}$, where

$$\frac{1}{r_1} = \frac{1-\theta}{p_1} + \frac{\theta}{p_0'}, \qquad \frac{1}{r_0} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1'}, \qquad \frac{1}{t} - \frac{1}{s} = \left(\frac{1}{q_1} - \frac{1}{q_0}\right)_+$$

for every $0 < \theta < 1$. Eliminating θ from this equation, we obtain that

$$\frac{1}{p_1} - \frac{1}{p_0} = \frac{1}{r_1} - \frac{1}{r_0},$$

and the condition $0 < \theta < 1$ implies the condition $1 < p_0 < r_0 < p_1' < \infty$, where $\frac{1}{p_1} + \frac{1}{p_1'} = 1$.

The proof is complete. □

By (2), in particular, the following proposition follows.

Let $1 < p < r < p' < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$M_{p,q} \hookrightarrow M_{r,t},$$

where $M_{p,q} = M^{p,q}_{p,q}$ and $q, t \in [1, \infty[$ are any.

4 Proof of main results

For a given pair $\bar{X} = (X_0, X_1)$ we consider the space $\Gamma(\bar{X})$ consisting of all functions f bounded and continuous in the strip

$$S = \{z : 0 \leq \operatorname{Re} z \leq 1\}$$

with values in $X_0 + X_1$. Moreover, f are analytic in the open strip

$$S_0 = \{z : 0 < \operatorname{Re} z < 1\}$$

and such that the mapping $t \longrightarrow f(j + it)$ $(j = 0, 1)$ is a continuous function on the real axis with values in X_j $(j = 0, 1)$ which tends to 0 for $|t| \longrightarrow \infty$. It is clear that $\Gamma(\bar{X})$ is a vector space. We endow Γ with the norm

$$\|f\|_\Gamma = \max\Bigl(\sup_t \bigl\|f(it)\bigr\|_{X_0}, \sup_t \bigl\|f(1 + it)\bigr\|_{X_1}\Bigr).$$

The space $\bar{X}_{[\theta]}$, $0 \leq \theta \leq 1$ consists of all elements $a \in X_0 + X_1$ such that $a = f(\theta)$ for some function $f \in \Gamma(\bar{X})$. The norm on $\bar{X}_{[\theta]}$ is equal to

$$\|a\|_{[\theta]} = \inf\bigl\{\|f\|_\Gamma : f(\theta) = a, f \in \Gamma\bigr\}.$$

In order to prove our main result, we need two lemmas in [8].

Lemma 3 (Bilinear interpolation, the complex method, see [8]) *Let T be a bilinear operator such that*

$$T : X_0 \times Y_0 \longrightarrow Z_0$$

and

$$T : X_1 \times Y_1 \longrightarrow Z_1.$$

Then

$$T : X_{[\theta]} \times Y_{[\theta]} \longrightarrow Z_{[\theta]},$$

where $X_{[\theta]}$, $Y_{[\theta]}$, $Z_{[\theta]}$ are the spaces obtained by the method of complex interpolation of Banach pairs of spaces (X_0, X_1), (Y_0, Y_1), (Z_0, Z_1) respectively.

Lemma 4 (Bilinear interpolation, the real method, see [8]) *Let T be a bilinear operator such that*

$$T : X_0 \times Y_0 \longrightarrow Z_0$$

and

$$T : X_1 \times Y_1 \longrightarrow Z_1$$

with the norms B_0, B_1 respectively. Then

$$T : X_{\theta,t_1} \times Y_{\theta,t_2} \longrightarrow Z_{\theta,s},$$

where $\frac{1}{s} + 1 = \frac{1}{t_1} + \frac{1}{t_2}$. *Moreover,*

$$\|T\| \le cB_0^{1-\theta} B_1^{\theta}.$$

Proof of Theorem 1 First we prove the inequality:

$$\|a * b\|_{l_{p_1,q_1}} \le c\|\varphi\|_{L_r} \|a\|_{l_{p_0,q_0}}, \tag{2}$$

where $b = \{b_k\}_{k\in\mathbb{Z}}$ are Fourier coefficients of the function φ.

If $r \le 2$, Inequality (2) follows by Lemma 1 and Hardy-Littlewood-Paley inequality [9]. Indeed, since $\varphi \in L_{rs}$, by the Hardy-Littlewood-Paley theorem, we have $b \in l_{r's}$ and the following inequality holds

$$\|b\|_{l_{r's}} \le c\|\varphi\|_{L_{rs}}.$$

Taking $s = r$, we get

$$\|b\|_{l_{r',r}} \le c\|\varphi\|_{L_r}.$$

Now let $2 < r < \infty$. Let $a \in l_2, f \sim \sum_{k\in Z} a_k e^{2\pi ikx}$, then by Parseval's equality we get

$$\|a * b\|_{l_2} = \|f\varphi\|_{L_2} \le \|f\|_{L_2} \|\varphi\|_{L_\infty} = \|\varphi\|_{L_\infty} \|a\|_{l_2},$$

i.e., $M_2 = L_\infty$. From Lemma 1, using Parseval's equality we have

$$\|a * b\|_{l_{p_1,q_1}} \le c\|\varphi\|_{L_2} \|a\|_{l_{p_0,q_0}},$$

where $\frac{1}{p_1} + 1 = \frac{1}{2} + \frac{1}{p_0}$, $\frac{1}{p_1} + \frac{1}{2} = \frac{1}{p_0}$, $\frac{1}{q_1} = \frac{1}{q_0} + \frac{1}{2}$.

Thus, for the bilinear operator $T(a, \varphi) = a * b$ we obtain

$$T : l_2 \times L_\infty \longrightarrow l_2,$$

$$T : l_{p_0,q_0} \times L_2 \longrightarrow l_{p_1,q_1}.$$

Applying the method of complex interpolation (Lemma 3), we obtain Inequality (2). Now we shall prove the inequality

$$\|a * b\|_{l_{p_1,q_1}} \le c\|\varphi\|_{L_{r,s}} \|a\|_{l_{p_0,q_0}}, \tag{3}$$

where $\frac{1}{s} = \frac{1}{q_1} - \frac{1}{q_0}$.

Let $q_0 = \infty$ and p_0 be fixed in Inequality (2). Taking $\frac{1}{q_1^i} = \frac{1}{r_i}$, $i = 0, 1$, choose parameters r_0, r_1, p_1^0, p_1^1 such that

$$\frac{1}{p_0} = \frac{1}{p_1^i} + \frac{1}{r_i}, \quad i = 0, 1. \tag{4}$$

Then from Inequality (2) we have

$$\|a*b\|_{l_{p_1^0,r_0}} \le c_1\|a\|_{l_{p_0,\infty}}\|\varphi\|_{L_{r_0}},$$
$$\|a*b\|_{l_{p_1^1,r_1}} \le c_2\|a\|_{l_{p_0,\infty}}\|\varphi\|_{L_{r_1}}.$$

Using Marcinkiewicz-Calderón interpolation theorem (see [8]), we get

$$\|a*b\|_{l_{p_1,s}} \le \left(c_1\|a\|_{l_{p_0,\infty}}\right)^{\theta}\left(c_2\|a\|_{l_{p_0,\infty}}\right)^{1-\theta}\|\varphi\|_{L_{r,s}} = c\|a\|_{l_{p_0,\infty}}\|\varphi\|_{L_{r,s}}, \tag{5}$$

where $\frac{1}{p_1} = \frac{1-\theta}{p_1^0} + \frac{\theta}{p_1^1}$, $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$, i.e., $\frac{1}{p_0} - \frac{1}{p_1} = \frac{1}{r}$.

Now we apply Lemma 2 with fixed parameters r, s and parameters p_1^i, p_0^i, $i = 0,1$ satisfying (4) and the inequality of type (5). We have:

$$(L_{r,s},L_{r,s})_{\theta,1} \times (l_{p_0^0,\infty}, l_{p_0^1,\infty})_{\theta,q_0} \longrightarrow (l_{p_1^0,s}, l_{p_1^1,s})_{\theta,q_1}$$

or

$$T : L_{r,s} \times l_{p_0,q_0} \longrightarrow l_{p_1,q_1},$$

where $\frac{1}{q_1} - \frac{1}{q_0} = \frac{1}{s} - \frac{1}{\infty}$, $\frac{1}{p_1} = \frac{1-\theta}{p_1^0} + \frac{\theta}{p_1^1}$, $\frac{1}{p_0} = \frac{1-\theta}{p_0^0} + \frac{\theta}{p_0^1}$, i.e., $\frac{1}{q_1} = \frac{1}{s} + \frac{1}{q_0}$, $\frac{1}{p_0} - \frac{1}{p_1} = \frac{1}{r}$.

Since the parameters p_1^i, p_0^i, $i = 0,1$ are arbitrary in Inequality (5), it guarantees the arbitrary of the corresponding parameters in Inequality (4).

Thus, the following inequality holds:

$$\|a*b\|_{l_{p_1,q_1}} \le c\|a\|_{l_{p_0,q_0}}\|\varphi\|_{L_{r,s}},$$

where $b = \{b_m\}_{m\in Z}$ are Fourier coefficients of the function φ and $\frac{1}{r} = \frac{1}{p_0} - \frac{1}{p_1}$, $\frac{1}{s} = \frac{1}{q_1} - \frac{1}{q_0}$. According to Remark, this inequality is equivalent to the statement of Theorem 1. □

Proof of Theorem 2 Let $\varphi \in M_p^q$ and Q be an arbitrary segment in $[0,1]$,

$$f_0(x) = \begin{cases} 1, & x \in Q, \\ 0, & x \notin Q. \end{cases}$$

Note that by Boas theorem [10] (see also [11]) we get

$$\|\widehat{f_0}\|_{l_p} \sim \|f_0\|_{L_{p',p}} = \left(\int_0^1 \left(t^{\frac{1}{p'}} f_0^*(t)\right)^p \frac{dt}{t}\right)^{\frac{1}{p}} = |Q|^{\frac{1}{p'}}. \tag{6}$$

Applying Theorem 5 from [12] and using (6), we obtain:

$$\|\varphi\|_{M_p^q} = \sup_{f\neq 0} \frac{\|\widehat{f\varphi}\|_{l_q}}{\|\widehat{f}\|_{l_p}} \ge \frac{\|\widehat{f_0\varphi}\|_{l_q}}{\|\widehat{f_0}\|_{l_p}}$$
$$\ge \frac{c}{|Q|^{\frac{1}{p'}}} \int_0^1 \left(t^{\frac{1}{q'}} \left(\sup_{|W|\ge t, W\in G} \frac{1}{|W|}\left|\int_W f_0(x)\varphi(x)\,dx\right|\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

$$\geq \frac{c}{|Q|^{\frac{1}{p'}}} \sup_{t>0} t^{\frac{1}{q'}} \left(\sup_{|W|\geq t, W\in G} \frac{1}{|W|} \left| \int_{W\cap Q} \varphi(x)\,dx \right| \right)$$
$$\geq \frac{c_1}{|Q|^{\frac{1}{p'}}} |Q|^{\frac{1}{q'}-1} \left| \int_Q \varphi(x)\,dx \right| = \frac{c_1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left| \int_Q \varphi(x)\,dx \right|.$$

Since the interval Q is arbitrary, we get

$$\|\varphi\|_{M_p^q} \geq c_1 \sup_{Q\in G} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left| \int_Q \varphi(x)\,dx \right|,$$

where constants c and c_1 depend only on parameters p and q.

The proof of obtaining an upper estimate follows from Theorem 1 and the embedding $l_{p,p} \hookrightarrow l_{p,q}$, for $p < q$.

Indeed, from Theorem 1 it follows

$$L_{r,\infty} \hookrightarrow M_p^q,$$

i.e.,

$$\|\varphi\|_{M_p^q} \leq c_2 \sup_{t>0} t^{\frac{1}{r}} \varphi^*(t) \leq c_3 \sup_{t>0} \frac{1}{t^{\frac{1}{p'}+\frac{1}{q}}} \int_0^t \varphi^*(s)\,ds$$
$$= c_3 \sup_{Q\in F} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \int_Q |\varphi(x)|\,dx \sim \sup_{Q\in F} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left| \int_Q \varphi(x)\,dx \right|. \qquad \square$$

Proof of Corollary 1 Let Q be an arbitrary compact from F.

From the condition of generalized monotonicity of the function φ we have

$$\frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left| \int_Q \varphi(y)\,dy \right| \leq \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \int_Q \frac{c}{y} \left| \int_0^y \varphi(x)\,dx \right| dy$$
$$\leq \frac{c}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \sup_{A\in G} \frac{1}{|A|^{\frac{1}{p'}+\frac{1}{q}}} \left| \int_A \varphi(x)\,dx \right| \int_Q \frac{dy}{y^{1-\frac{1}{q}-\frac{1}{p'}}}$$
$$\leq c_1 \sup_{A\in G} \frac{1}{|A|^{\frac{1}{p'}+\frac{1}{q}}} \left| \int_A \varphi(x)\,dx \right|.$$

Taking into account that $Q \in F$ is arbitrary, we have

$$\sup_{Q\in F} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left| \int_Q \varphi(x)\,dx \right| \leq c \sup_{Q\in G} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left| \int_Q \varphi(x)\,dx \right|.$$

Thus, from Theorem 2 we get

$$\|\varphi\|_{M_p^q} \sim \sup_{Q\in F} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left| \int_Q \varphi(x)\,dx \right| \sim \sup_{t>0} t^{\frac{1}{p}-\frac{1}{q}} \varphi^*(t). \qquad \square$$

Proof of Theorem 3 Let $2 < p_0 \leq p_1 < \infty$. For a sequence of numbers $a = \{a_m\}_{m\in Z}$ and a function $\varphi \in L_1[0,1]$ we consider the mapping T of the form $T(a,\varphi) = a * b$, where $b =$

$\{b_m\}_{m\in Z}$ is the sequence of Fourier coefficients on the trigonometric system of functions φ. This map is bilinear and from Karadzhov's theorem [5] and Remark it follows that it is bounded from $l_1 \times B^{\frac{1}{2}}_{2,1}$ to l_1.

Since $M_2 = L_\infty$, the mapping

$$T_\varphi : l_2 \times L_\infty \longrightarrow l_2$$

is also bounded. Thus, for the operator T, the following is true

$$T : l_1 \times B^{\frac{1}{2}}_{2,1} \longrightarrow l_1,$$
$$T : l_2 \times L_\infty \longrightarrow l_2.$$

Then, by Lemma 4 on bilinear interpolation, we get

$$(l_1, l_2)_{\theta,q} \times \left(B^{\frac{1}{2}}_{2,1}, L_\infty\right)_{\theta,1} \longrightarrow (l_1, l_2)_{\theta,q},$$

i.e., the operator T is bounded from $l_{p,q} \times (B^{\frac{1}{2}}_{2,1}, L_\infty)_{\theta,1}$ to $l_{p,q}$. In the paper [5] it is shown that $B^{\frac{1}{r}}_{r,1} \hookrightarrow (B^{\frac{1}{2}}_{2,1}, L_\infty)_{\theta,1}$, where $\frac{1}{r} = \frac{1-\theta}{2}$. Thus, taking into account Theorem 5, we will get

$$T : l_{p,q} \times B^{\frac{1}{r}}_{r,1} \longrightarrow l_{p,q}, \tag{7}$$

for $2 < p < \infty$, $1 \le q \le \infty$, $\frac{1}{r} = \frac{1}{2} - \frac{1}{p}$.

From Minkowski's inequality and Parseval's equality we get

$$T : l_1 \times L_2 \longrightarrow l_2.$$

Thus, for the operator T, the following is true

$$T : l_{p,q} \times B^{\frac{1}{r}}_{r,1} \longrightarrow l_{p,q},$$
$$T : l_1 \times L_2 \longrightarrow l_2.$$

Then, by Lemma 3 we get

$$(l_{p,q}, l_1)_{[\theta]} \times \left(B^{\frac{1}{r}}_{r,1}, L_2\right)_{[\theta]} \longrightarrow (l_{p,q}, l_2)_{[\theta]}$$

i.e., T is a bounded mapping from $l_{p_0,q_0} \times B^{\alpha}_{r,s}$ to l_{p_1,q_1}, where $2 < p_0 \le p_1 < \infty$, $\frac{1}{r} = \frac{1}{2} - \frac{1}{p_1}$, $\alpha = \frac{1}{2} - \frac{1}{p_0}$. The arbitrary choice of parameters guarantees the arbitrary of the parameters available in the theorem. □

The case $1 < p_0 < p_1 < 2$ follows from the statements proved above and the property $M^{p_1,q_1}_{p_0,q_0} = M^{p_0',q_0'}_{p_1',q_1'}$.

Proof of Theorem 4 Let $1 < p_0 < p_1 \le 2$. Let us consider the bilinear mapping $T(a,\varphi) = a * b$, where $b = \{b_m\}_{m\in Z}$ is the sequence of Fourier coefficients of the function φ. The mapping

$$T : l_{p_0,q_0} \times L_2 \longrightarrow l_{p_1,q_1} \tag{8}$$

is bounded according to Theorem 1. Here $\frac{1}{p_0} - \frac{1}{p_1} = \frac{1}{2}$, $\frac{1}{q_1} - \frac{1}{q_0} = \frac{1}{2}$, $1 < q_1 < 2 < q_0$, $1 < p_0 < 2 < p_1$. The result of Theorem 3, in the case $q_0 = q_1 = 1$, $p_0 = p_1 = p$ can be written as

$$T : l_{p,1} \times B^{1/t}_{t,1} \longrightarrow l_{p,1}, \qquad \frac{1}{t} = \frac{1}{p} - \frac{1}{2}. \tag{9}$$

Applying Lemma 3 on the bilinear interpolation to (8) and (9), and taking into account the properties of the embedding of the spaces $l_{p,q}$ and $B^{\alpha}_{p,q}$, we have:

$$T : l_{p_0,1} \times B^{\alpha}_{r,1} \longrightarrow l_{p_1,\infty}, \tag{10}$$

where parameters r, α, p_0, p_1 satisfy the following conditions:

$$1 < p_0 < p_1 \leq 2, \qquad \frac{1}{r} - \alpha = \frac{1}{p_0} - \frac{1}{p_1}, \qquad \alpha > \frac{1}{p_1} - \frac{1}{2}. \tag{11}$$

Let in (11) parameter r be fixed. Using Lemma 4 on bilinear interpolation and taking into account that

$$\left(B^{\alpha_0}_{r,1}, B^{\alpha_1}_{r,1}\right)_{\theta,h} = B^{\alpha}_{r,h}, \quad \texttt{with}\, \alpha = (1-\theta)\alpha^0 + \alpha^1,$$

we get

$$T : l_{p_0,h_1} \times B^{\alpha}_{r,h_2} \longrightarrow l_{p,h_3},$$

where $\frac{1}{h_1} + 1 = \frac{1}{h_2} + \frac{1}{h_3}$, $\alpha > \frac{1}{p_1} - \frac{1}{2} = \min_{x \in [\frac{1}{p_1}, \frac{1}{p_0}]} |\frac{1}{2} - x|$, $\frac{1}{r} - \alpha = \frac{1}{p_0} - \frac{1}{p_1}$.

Therefore, with fixed $a \in l_{p_0,\infty}$ and r we obtain that

$$P_a : B^{\alpha_i}_{r,1} \longrightarrow l_{p_1^i,\infty},$$

and

$$\|P_a\|_{B^{\alpha_i}_{r,1} \longrightarrow l_{p_1^i,\infty}} \leq c_i \|a\|_{l_{p_0,\infty}},$$

where $\frac{1}{r} - \alpha_i = \frac{1}{p_0} - \frac{1}{p_1^i}$, $\alpha^i > \frac{1}{p_1^i} - \frac{1}{2}$, $i = 0, 1$.

Using Marcinkiewicz-Calderón interpolation theorem we have

$$P_a : B^{\alpha}_{r,s} \longrightarrow l_{p_1,s},$$

and

$$\|P_a\|_{B^{\alpha}_{r,s} \longrightarrow l_{p_1,s}} \leq c\|a\|_{l_{p_0,\infty}}.$$

Thus

$$T : l_{p_0,\infty} \times B^{\alpha}_{r,s} \longrightarrow l_{p_1,s}.$$

To complete the proof we fix the function φ and the parameters r, s, α and we choose the parameters p_0^i, p_1^i, $i = 0,1$ satisfying (11). We use Lemma 2 to get $B^{\alpha}_{r,s}[0,1] \hookrightarrow M^{p_1,q_1}_{p_0,q_0}$. □

The case $2 \le p_0 < p_1 < \infty$, as in the proof of Theorem 3, will follow from $M^{p_1,q_1}_{p_0,q_0} = M^{p_0',q_0'}_{p_1',q_1'}$.

5 Examples demonstrating the sharpness of the results

Proposition 1 *Let* $1 < p_0 < 2 \le p_1$, $\frac{1}{r} = \frac{1}{p_0} - \frac{1}{p_1}$, $\frac{1}{s} = (\frac{1}{q_1} - \frac{1}{q_0})_+$. *If* $q_1 < q_0$, *then for any* $\varepsilon > 0$ *there exists* $\varphi_1 \in L_{r,s+\varepsilon}$ *such that* $\varphi_1 \notin M^{p_1,q_1}_{p_0,q_0}$, *if* $q_1 \ge q_0$ *there exists* $\varphi_2 \in L_{r-\varepsilon,\infty}$ *such that* $\varphi_2 \notin M^{p_1,q_1}_{p_0,q_0}$.

Proof Let ε be an arbitrary positive number, and numbers β_1, β_2 be such that

$$\beta_1 > \frac{1}{s+\varepsilon}, \qquad \beta_2 > \frac{1}{q_0}, \qquad \beta_1 + \beta_2 < \frac{1}{s} + \frac{1}{q_0} = \frac{1}{q_1}.$$

Let

$$b_k = \frac{1}{(|k|+1)^{1/r'} \ln^{\beta_1}(|k|+1)},$$
$$a_k = \frac{1}{(|k|+1)^{1/p_0} \ln^{\beta_2}(|k|+1)},$$

and

$$\varphi_1 \sim \sum_{k=-\infty}^{+\infty} b_k e^{2\pi ikx}.$$

Then for $m \neq 0$

$$\begin{aligned}(a*b)_m &= \sum_{k=-\infty}^{+\infty} \frac{1}{(|k|+1)^{1/r'} \ln^{\beta_1}(|k|+1)(|k-m|+1)^{1/p_0} \ln^{\beta_2}(|k-m|+1)} \\ &\sim \int_{-\infty}^{+\infty} \frac{dx}{|x|^{1/r'} |\ln|x||^{\beta_1} |x-m|^{1/p_0} |\ln|x-m||^{\beta_2}},\end{aligned}$$

$$\begin{aligned}&\int_{-\infty}^{+\infty} \frac{dx}{|x|^{1/r'} |\ln|x||^{\beta_1} |x-m|^{1/p_0} |\ln|x-m||^{\beta_2}} \\ &\quad = |m|^{-(\frac{1}{r'}+\frac{1}{p_0})+1} \cdot \int_{-\infty}^{+\infty} \frac{dy}{|y|^{1/r'} |\ln|y| + \ln|m||^{\beta_1} |y-1|^{1/p_0} |\ln|y-1| + \ln|m||^{\beta_2}} \\ &\quad = |m|^{-(\frac{1}{r'}+\frac{1}{p_0})+1} \bigl|\ln|m|\bigr|^{-\beta_1-\beta_2} \cdot \int_{-\infty}^{+\infty} \frac{dy}{|y|^{1/r'} |\frac{\ln|y|}{\ln|m|} + 1|^{\beta_1} |y-1|^{1/p_0} |\frac{\ln|y-1|}{\ln|m|} + 1|^{\beta_2}} \\ &\quad \ge |m|^{-(\frac{1}{r'}+\frac{1}{p_0})+1} \bigl|\ln|m|\bigr|^{-\beta_1-\beta_2} \cdot \int_{-\infty}^{+\infty} \frac{dy}{|y|^{1/r'} |\ln|y| + 1|^{\beta_1} |y-1|^{1/p_0} |\ln|y-1| + 1|^{\beta_2}}.\end{aligned}$$

Thus, $(a*b)_m \ge c(|m|+1)^{-(\frac{1}{r'}+\frac{1}{p_0})+1} |\ln(|m|+2)|^{-\beta_1-\beta_2}$. Since

$$\sum_{m=0}^{+\infty} \Bigl(\bigl((m+1)^{-(\frac{1}{r'}+\frac{1}{p_0})+1} \bigl|\ln\bigl(|m|+2\bigr)\bigr|^{-\beta_1-\beta_2}\bigr)^{q_1} \bigl(|m|+1\bigr)^{(\frac{q_1}{p_1}-1)}\Bigr) = \infty,$$

$a * b \notin l_{p_1,q_1}$, and therefore $\varphi_1 \notin M_{p_0,q_0}^{p_1,q_1}$. Since Fourier coefficients of φ_1 are the sequence $\{b_k\}_{k\in Z}$ it follows that $\varphi_1 \in L_{r,s}$.

To prove the second part of the proposition, we take $s = \infty$. Let numbers α_1 and α_2 be such that

$$\alpha_1 > \frac{1}{(r-\varepsilon)'} = 1 - \frac{1}{r-\varepsilon}, \qquad \alpha_2 > \frac{1}{p_0}, \qquad \alpha_1 + \alpha_2 < 1 - \frac{1}{r} + \frac{1}{p_0}$$

(note that the last inequality does not contradict the previous two). Choosing

$$b_k = \frac{1}{(|k|+1)^{\alpha_1}}, \qquad a_k = \frac{1}{(|k|+1)^{\alpha_2}}, \qquad \varphi_2 \sim \sum_{k=-\infty}^{+\infty} b_k e^{2\pi ikx},$$

we can show that

$$a * b \sim \bigl\{ \bigl(|k|+1\bigr)^{-\alpha_1-\alpha_2+1} \bigr\}_{k\in Z}.$$

Hence $a * b \notin l_{p_1,q_1}$, and therefore $\varphi_2 \notin M_{p_0q_0}^{p_1q_1}$. At the same time taking into account the monotonicity of the sequence $\{b_k\}_{k\in\mathbb{Z}}$ and Hardy-Littlewood theorem, we have that $\varphi_2 \in L_{r-\varepsilon,\infty}$. The statement is proved. □

Theorem 6 *Let* $1 < p_0 < p_1 < 2$, $1 < q_1 \leq q_0$, $\frac{1}{r} - \alpha = \frac{1}{p_0} - \frac{1}{p_1}$, $\frac{1}{s} = \frac{1}{q_1} - \frac{1}{q_0}$. *Then for any* $\varepsilon > 0$ *there exist* $\varphi_1 \in B_{r,\infty}^{\alpha-\varepsilon} \cap B_{r-\varepsilon,\infty}^{\alpha}$ *and* $\varphi_2 \in B_{r,s+\varepsilon}^{\alpha}$ *such that* $\varphi_1 \notin M_{p_0,q_0}^{p_1,q_1}$, $\varphi_2 \notin M_{p_0,q_0}^{p_1,q_1}$.

Proof Let $s < \infty$ and numbers β_1, β_2 be such that

$$\beta_1 > \frac{1}{s+\varepsilon}, \qquad \beta_2 > \frac{1}{q_0}, \qquad \beta_1 + \beta_2 < \frac{1}{s} + \frac{1}{q_0} = \frac{1}{q_1}.$$

Let $b = \{b_k\}_{k\in Z}$ and $a = \{a_k\}_{k\in Z}$, where

$$b_k = \frac{1}{(|k|+1)^{\alpha-\frac{1}{r}+1} \ln^{\beta_1}(|k|+2)^{\beta_1}},$$
$$a_k = \frac{1}{(|k|+1)^{k/p_0} \ln^{\beta_2}(|k|+2)^{\beta_2}}.$$

It is obvious that $a \in l_{p_0,q_0}$, and $\varphi_2 \sim \sum_{k=-\infty}^{+\infty} b_k e^{2\pi ikx}$ belongs to $B_{r,s+\varepsilon}^{\alpha}$.

It is easy to show that

$$(a * b)_m \geq c\bigl(|m|+1\bigr)^{\alpha-\frac{1}{r}+\frac{1}{p_0}} \bigl(\ln\bigl(|m|+2\bigr)\bigr)^{\beta_1+\beta_2}$$

and consequently, $a * b \notin l_{p_1,q_1}$. Therefore $\varphi_2 \notin M_{p_0,q_0}^{p_1,q_1}$.

To construct the function φ_1, it is sufficient to consider the sequences

$$b = \left\{ \frac{1}{(|m|+1)^{\gamma_1}} \right\}_{m\in Z}, \qquad a = \left\{ \frac{1}{(|m|+1)^{\gamma_2}} \right\}_{m\in Z},$$

where

$$\gamma_1 > \max\left(\alpha - \varepsilon - \frac{1}{r} + 1, \alpha - \frac{1}{r+\varepsilon} + 1\right), \qquad \gamma_2 > \frac{1}{p_0}$$

and

$$\gamma_1 + \gamma_2 < \alpha - \frac{1}{r} + \frac{1}{p_0}.$$

$\varphi_1 \sim \sum_{k=-\infty}^{+\infty} b_k e^{2\pi ikx}$. The proof that $\varphi_1 \in B^{\alpha-\varepsilon}_{r,\infty} \cap B^{\alpha}_{r-\varepsilon,\infty}$, $\varphi_1 \notin M^{p_1,q_1}_{p_0,q_0}$ is similar to the proof of the first part. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to writing this article. All authors read and approved the final manuscript.

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