Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Multilinear commutators of vector-valued intrinsic square functions on vector-valued generalized weighted Morrey spaces

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Abstract
In this paper, we will obtain the strong type and weak type estimates for vector-valued analogs of intrinsic square functions in the generalized weighted Morrey spaces $M_w^{p,\varphi}(l_2)$. We study the boundedness of intrinsic square functions including the Lusin area integral, the Littlewood-Paley g-function and g_λ^* -function, and their multilinear commutators on vector-valued generalized weighted Morrey spaces $M_w^{p,\varphi}(l_2)$. In all the cases the conditions for the boundedness are given either in terms of Zygmund-type integral inequalities on $\varphi(x,r)$ without assuming any monotonicity property of $\varphi(x,r)$ on r.

MSC: 42B25; 42B35

Keywords: intrinsic square functions; vector-valued generalized weighted Morrey spaces; vector-valued inequalities; A_p weights; multilinear commutators; BMO

1 Introduction

It is well known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderon [1, 2] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. Let K be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well-known result of Coifman *et al.* [3] states that the commutator operator $[b,K]f = K(bf) - bKf$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [4–8]).

The classical Morrey spaces were originally introduced by Morrey in [9] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [7–10]. Recently, Komori and Shirai [11] first defined the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Also, Guliyev [12, 13] introduced the generalized weighted Morrey spaces $M_w^{p,\varphi}$ and studied the boundedness of the sublinear operators and their higher order commutators generated by Calderón-Zygmund operators and Riesz potentials in these spaces (see, also [14–16]).

The intrinsic square functions were first introduced by Wilson in [17, 18]. They are defined as follows. For $0 < \alpha \leq 1$, let C_α be the family of functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that ϕ's support is contained in $\{x : |x| \le 1\}$, $\int_{\mathbb{R}^n} \phi(x)\,dx = 0$, and for $x, x' \in \mathbb{R}^n$,

$$\left|\phi(x) - \phi\left(x'\right)\right| \le \left|x - x'\right|^{\alpha}.$$

For $(y,t) \in \mathbb{R}^{n+1}_+$ and $f \in L^{1,\mathrm{loc}}(\mathbb{R}^n)$, set

$$A_\alpha f(t,y) \equiv \sup_{\phi \in C_\alpha} \left|f * \phi_t(y)\right|,$$

where $\phi_t(y) = t^{-n}\phi(\frac{y}{t})$. Then we define the varying-aperture intrinsic square (intrinsic Lusin) function of f by the formula

$$G_{\alpha,\beta}(f)(x) = \left(\iint_{\Gamma_\beta(x)} \left(A_\alpha f(t,y)\right)^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

where $\Gamma_\beta(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \beta t\}$. Denote $G_{\alpha,1}(f) = G_\alpha(f)$.

This function is independent of any particular kernel, such as Poisson kernel. It dominates pointwise the classical square function (Lusin area integral) and its real-variable generalizations. Although the function $G_{\alpha,\beta}(f)$ depends on kernels with uniform compact support, there is pointwise relation between $G_{\alpha,\beta}(f)$ with different β:

$$G_{\alpha,\beta}(f)(x) \le \beta^{\frac{3n}{2}+\alpha} G_\alpha(f)(x).$$

We can see details in [17].

The intrinsic Littlewood-Paley g-function and the intrinsic g^*_λ function are defined, respectively, by

$$g_\alpha f(x) = \left(\int_0^\infty \left(A_\alpha f(y,t)\right)^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

$$g^*_{\lambda,\alpha} f(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \left(A_\alpha f(y,t)\right)^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}.$$

When we say that f maps into l_2, we mean that $\vec{f}(x) = (f_j)_{j=1}^\infty$, where each f_j is Lebesgue measurable and, for almost every $x \in \mathbb{R}^n$

$$\left\|\vec{f}(x)\right\|_{l_2} = \left(\sum_{j=1}^\infty \left|f_j(x)\right|^2\right)^{1/2}.$$

Let $\vec{f} = (f_1, f_2, \ldots)$ be a sequence of locally integrable functions on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, Wilson [18] also defined the vector-valued intrinsic square functions of $\vec{f}$ by $\|G_\alpha \vec{f}(x)\|_{l_2}$ and proved the following result.

Theorem A *Let* $1 \le p < \infty$, $0 < \alpha \le 1$, *and* $w \in A_p$. *Then the operators* G_α *and* $g^*_{\lambda,\alpha}$ *are bounded from* $L^p_w(l_2)$ *into itself for* $p > 1$ *and from* $L^1_w(l_2)$ *to* $WL^1_w(l_2)$.

Moreover, in [19], Lerner showed sharp L^p_w norm inequalities for the intrinsic square functions in terms of the A_p characteristic constant of w for all $1 < p < \infty$. Also Huang and

Liu [20] studied the boundedness of intrinsic square functions on weighted Hardy spaces. Moreover, they characterized the weighted Hardy spaces by intrinsic square functions. In [21] and [22], Wang and Liu obtained some weak type estimates on weighted Hardy spaces. In [23], Wang considered intrinsic functions and the commutators generated with BMO functions on weighted Morrey spaces. Let $\vec{b} = (b_1, \dots, b_m)$ and b_j, $j = 1, \dots, m$ be locally integrable function on $\mathbb{R}^n$. Setting

$$A_{\alpha,\vec{b}}f(t,y) \equiv \sup_{\phi\in C_\alpha}\left|\int_{\mathbb{R}^n}\prod_{j=1}^{m}\left[b_j(x)-b_j(z)\right]\phi_t(y-z)f(z)\,dz\right|,$$

the multilinear commutators are defined by

$$[\vec{b}, G_\alpha]f(x) = \left(\iint_{\Gamma(x)}\left(A_{\alpha,\vec{b}}f(t,y)\right)^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

$$[\vec{b}, \mathrm{g}_\alpha]f(x) = \left(\int_0^\infty\left(A_{\alpha,\vec{b}}f(t,y)\right)^2\frac{dt}{t}\right)^{\frac{1}{2}},$$

and

$$\left[\vec{b}, \mathrm{g}^*_{\lambda,\alpha}\right]f(x) = \left(\iint_{\mathbb{R}^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left(A_{\alpha,\vec{b}}f(t,y)\right)^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}.$$

In [23], Wang proved the following result.

Theorem B *Let $1 < p < \infty$, $0 < \alpha \leq 1$, $w \in A_p$, and $b \in BMO(\mathbb{R}^n)$. Then the commutator operators $[b, G_\alpha]$ and $[b, \mathrm{g}^*_{\lambda,\alpha}]$ are bounded from $L^p_w(l_2)$ into itself.*

Analogously the following result may be proved.

Theorem B′

Let $1 < p < \infty$, $0 < \alpha \leq 1$, $w \in A_p$. Let also $\vec{b} = (b_1, \dots, b_m)$ and $b_j \in BMO(\mathbb{R}^n)$, $j = 1, \dots, m$. Then the multilinear commutator operators $[\vec{b}, G_\alpha]$ and $[\vec{b}, \mathrm{g}^*_{\lambda,\alpha}]$ are bounded from $L^p_w(l_2)$ into itself.

In this paper, we will consider the boundedness of the operators G_α, g_α, $\mathrm{g}^*_{\lambda,\alpha}$ and their multilinear commutators on vector-valued generalized weighted Morrey spaces. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times \mathbb{R}_+$ and w be non-negative measurable function on $\mathbb{R}^n$. For any $\vec{f} \in L^{p,\mathrm{loc}}_w(l_2)$, we denote by $M^{p,\varphi}_w(l_2)$ the vector-valued generalized weighted Morrey spaces, if

$$\|\vec{f}\|_{M^{p,\varphi}_w(l_2)} = \sup_{x\in\mathbb{R}^n, r>0}\varphi(x,r)^{-1}w\big(B(x,r)\big)^{-\frac{1}{p}}\left\|\,\|\vec{f}(\cdot)\|_{l_2}\right\|_{L^p_w(B(x,r))} < \infty.$$

When $w \equiv 1$, then $M^{p,\varphi}_w(l_2)$ coincide the vector-valued generalized Morrey spaces $M^{p,\varphi}(l_2)$. There are many papers discussed the conditions on $\varphi(x,r)$ to obtain the boundedness of operators on the generalized Morrey spaces. For example, in [24] (see, also [10]), by

Guliyev the following condition was imposed on the pair (φ_1, φ_2):

$$\int_r^\infty \varphi_1(x,t)\,\frac{dt}{t} \le C\varphi_2(x,r), \tag{1.1}$$

where $C > 0$ does not depend on x and r. Under the above condition, they obtained the boundedness of Calderón-Zygmund singular integral operators from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{p,\varphi_2}(\mathbb{R}^n)$. Also, in [25] and [26], Guliyev *et al.* introduced a weaker condition: If $1 \le p < \infty$, there exists a constant $C > 0$, such that, for any $x \in \mathbb{R}^n$ and $r > 0$,

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\,dt \le C\varphi_2(x,r). \tag{1.2}$$

If the pair (φ_1, φ_2) satisfies condition (1.1), then (φ_1, φ_2) satisfied condition (1.2). But the opposite is not true. We can see Remark 4.7 in [26] for details.

Recently, in [12, 13] (see, also [14–16]), Guliyev introduced a weighted condition: If $1 \le p < \infty$, there exists a constant $C > 0$, such that, for any $x \in \mathbb{R}^n$ and $t > 0$,

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s)w(B(x,s))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}}\,\frac{dt}{t} \le C\varphi_2(x,r). \tag{1.3}$$

In this paper, we will obtain the boundedness of the vector-valued intrinsic function, the intrinsic Littlewood-Paley g function, the intrinsic g_λ^* function and their multilinear commutators on vector-valued generalized weighted Morrey spaces when $w \in A_p$ and the pair (φ_1, φ_2) satisfies condition (1.3) or the following inequalities:

$$\int_r^\infty \ln^m\left(e + \frac{t}{r}\right)\frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s)w(B(x,s))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}}\,\frac{dt}{t} \le C\varphi_2(x,r), \tag{1.4}$$

where C does not depend on x and r. Our main results in this paper are stated as follows.

Theorem 1.1 *Let* $1 \le p < \infty$, $0 < \alpha \le 1$, $w \in A_p$, *and* (φ_1, φ_2) *satisfy condition* (1.3). *Then the operator* G_α *is bounded from* $M_w^{p,\varphi_1}(l_2)$ *to* $M_w^{p,\varphi_2}(l_2)$ *for* $p > 1$ *and from* $M_w^{1,\varphi_1}(l_2)$ *to* $WM_w^{1,\varphi_2}(l_2)$.

Theorem 1.2 *Let* $1 \le p < \infty$, $0 < \alpha \le 1$, $w \in A_p$, $\lambda > 3 + \frac{\alpha}{n}$, *and* (φ_1, φ_2) *satisfy condition* (1.3). *Then the operator* $\mathrm{g}_{\lambda,\alpha}^*$ *is bounded from* $M_w^{p,\varphi_1}(l_2)$ *to* $M_w^{p,\varphi_2}(l_2)$ *for* $p > 1$ *and from* $M_w^{1,\varphi_1}(l_2)$ *to* $WM_w^{1,\varphi_2}(l_2)$.

Theorem 1.3 *Let* $1 < p < \infty$, $0 < \alpha \le 1$, $w \in A_p$, *and* (φ_1, φ_2) *satisfy condition* (1.4). *Let also* $\vec{b} = (b_1, \ldots, b_m)$ *and* $b_j \in BMO(\mathbb{R}^n)$, $j = 1, \ldots, m$. *Then* $[\vec{b}, G_\alpha]$ *is bounded from* $M_w^{p,\varphi_1}(l_2)$ *to* $M_w^{p,\varphi_2}(l_2)$.

Theorem 1.4 *Let* $1 < p < \infty$, $0 < \alpha \le 1$, $w \in A_p$, *and* (φ_1, φ_2) *satisfy condition* (1.4). *Let also* $\vec{b} = (b_1, \ldots, b_m)$ *and* $b_j \in BMO(\mathbb{R}^n)$, $j = 1, \ldots, m$. *Then for* $\lambda > 3 + \frac{\alpha}{n}$, $[\vec{b}, \mathrm{g}_{\lambda,\alpha}^*]$ *is bounded from* $M_w^{p,\varphi_1}(l_2)$ *to* $M_w^{p,\varphi_2}(l_2)$.

In [17], the author proved that the functions $G_\alpha f$ and $\mathrm{g}_\alpha f$ are pointwise comparable. Thus, as a consequence of Theorem 1.1 and Theorem 1.3, we have the following results.

Corollary 1.5 *Let $1 \le p < \infty$, $0 < \alpha \le 1$, $w \in A_p$, and (φ_1, φ_2) satisfy condition* (1.3), *then* g_α *is bounded from $M_w^{p,\varphi_1}(l_2)$ to $M_w^{p,\varphi_2}(l_2)$ for $p > 1$ and from $M_w^{1,\varphi_1}(l_2)$ to $WM_w^{1,\varphi_2}(l_2)$.*

Corollary 1.6 *Let $1 < p < \infty$, $0 < \alpha \le 1$, $w \in A_p$, and (φ_1, φ_2) satisfy condition* (1.4). *Let also $\vec{b} = (b_1, \dots, b_m)$ and $b_j \in BMO(\mathbb{R}^n)$, $j = 1, \dots, m$. Then $[\vec{b}, \mathrm{g}_\alpha]$ is bounded from $M_w^{p,\varphi_1}(l_2)$ to $M_w^{p,\varphi_2}(l_2)$.*

Remark 1.7 Note that, in the scalar valued case and for $m = 1$, $w \equiv 1$ Theorems 1.1-1.4 and Corollaries 1.5-1.6 was proved in [27]. Also, in the scalar valued case and $m = 1$, $w \equiv A_p$, and $\varphi_1(x,r) = \varphi_2(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, $0 < \kappa < 1$ Theorems 1.1-1.4 and Corollaries 1.5-1.6 were proved by Wang in [23, 28]. If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, then the vector-valued generalized weighed Morrey space $M_w^{p,\varphi}(l_2)$ coincides with the vector-valued weighed Morrey space $L_w^{p,\kappa}(l_2)$ and the pair $(w(B(x,r))^{\frac{\kappa-1}{p}}, w(B(x,r))^{\frac{\kappa-1}{p}})$ satisfies the two conditions (1.3) and (1.4). Indeed, by Lemma 3.1 there exist $C > 0$ and $\delta > 0$ such that for all $x \in \mathbb{R}^n$ and $t > r$:

$$w\big(B(x,t)\big) \ge C\left(\frac{t}{r}\right)^{n\delta} w\big(B(x,r)\big).$$

Then

$$\begin{aligned}
\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} w(B(x,s))^{\frac{\kappa}{p}}}{w(B(x,t))^{1/p}} \frac{dt}{t} &\le \int_r^\infty \ln^m\left(e + \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<s<\infty} w(B(x,s))^{\frac{\kappa}{p}}}{w(B(x,t))^{1/p}} \frac{dt}{t} \\
&= \int_r^\infty \ln^m\left(e + \frac{t}{r}\right) w\big(B(x,t)\big)^{\frac{\kappa-1}{p}} \frac{dt}{t} \\
&\lesssim \int_r^\infty \ln^m\left(e + \frac{t}{r}\right) \left(\left(\frac{t}{r}\right)^{n\delta} w\big(B(x,r)\big)\right)^{\frac{\kappa-1}{p}} \frac{dt}{t} \\
&= w\big(B(x,r)\big)^{\frac{\kappa-1}{p}} \int_r^\infty \ln^m\left(e + \frac{t}{r}\right) \left(\frac{t}{r}\right)^{n\delta\frac{\kappa-1}{p}} \frac{dt}{t} \\
&= w\big(B(x,r)\big)^{\frac{\kappa-1}{p}} \int_1^\infty \ln^m(e+\tau) \tau^{n\delta\frac{\kappa-1}{p}} \frac{d\tau}{\tau} \\
&\approx w\big(B(x,r)\big)^{\frac{\kappa-1}{p}}.
\end{aligned}$$

Throughout this paper, we use the notation $A \lesssim B$ to express that there is a positive constant C independent of all essential variables such that $A \le CB$. Moreover, C may be different from place to place.

2 Vector-valued generalized weighted Morrey spaces

The classical Morrey spaces $M^{p,\lambda}$ were originally introduced by Morrey in [9] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [29, 30].

We denote by $M^{p,\lambda}(l_2) \equiv M^{p,\lambda}(\mathbb{R}^n, l_2)$ the vector-valued Morrey space, the space of all vector-valued functions $\vec{f} \in L^{p,\mathrm{loc}}(l_2)$ with finite quasinorm

$$\|\vec{f}\|_{M^{p,\lambda}(l_2)} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|\vec{f}\|_{L^p(B(x,r),l_2)},$$

where $1 \le p < \infty$ and $0 \le \lambda \le n$.

Note that $M^{p,0}(l_2) = L^p(l_2)$ and $M^{p,n}(l_2) = L^\infty(l_2)$. If $\lambda < 0$ or $\lambda > n$, then $M^{p,\lambda}(l_2) = \Theta$, where Θ is the set of all vector-valued functions equivalent to 0 on $\mathbb{R}^n$.

We define the vector-valued generalized weighed Morrey spaces as follows.

Definition 2.1 Let $1 \le p < \infty$, φ be a positive measurable vector-valued function on $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_w^{p,\varphi}(l_2)$ the vector-valued generalized weighted Morrey space, the space of all vector-valued functions $\vec{f} \in L_w^{p,\mathrm{loc}}(l_2)$ with finite norm

$$\|\vec{f}\|_{M_w^{p,\varphi}(l_2)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w\big(B(x,r)\big)^{-\frac{1}{p}} \|f\|_{L_w^p(B(x,r),l_2)},$$

where $L_w^p(B(x,r),l_2)$ denotes the vector-valued weighted L^p-space of measurable functions f for which

$$\|\vec{f}\|_{L_w^p(B(x,r))} \equiv \|\vec{f}\chi_{B(x,r)}\|_{L_w^p(\mathbb{R}^n)} = \left(\int_{B(x,r)} \|\vec{f}(y)\|_{l_2}^p w(y)\,dy\right)^{\frac{1}{p}}.$$

Furthermore, by $WM_w^{p,\varphi}(l_2)$ we denote the vector-valued weak generalized weighted Morrey space of all functions $f \in WL_w^{p,\mathrm{loc}}(l_2)$ for which

$$\|\vec{f}\|_{WM_w^{p,\varphi}(l_2)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w\big(B(x,r)\big)^{-\frac{1}{p}} \|\vec{f}\|_{WL_w^p(B(x,r),l_2)} < \infty,$$

where $WL_w^p(B(x,r),l_2)$ denotes the weak L_w^p-space of measurable functions f for which

$$\|\vec{f}\|_{WL_w^p(B(x,r),l_2)} \equiv \|\vec{f}\chi_{B(x,r)}\|_{WL_w^p(l_2)} = \sup_{t>0} t\left(\int_{\{y\in B(x,r):\|\vec{f}(y)\|_{l_2}>t\}} w(y)\,dy\right)^{\frac{1}{p}}.$$

Remark 2.2

(1) If $w \equiv 1$, then $M_1^{p,\varphi}(l_2) = M^{p,\varphi}(l_2)$ is the vector-valued generalized Morrey space.
(2) If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, then $M_w^{p,\varphi}(l_2) = L_w^{p,\kappa}(l_2)$ is the vector-valued weighted Morrey space.
(3) If $\varphi(x,r) \equiv \nu(B(x,r))^{\frac{\kappa}{p}} w(B(x,r))^{-\frac{1}{p}}$, then $M_w^{p,\varphi}(l_2) = L_{\nu,w}^{p,\kappa}(l_2)$ is the vector-valued two weighted Morrey space.
(4) If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M_w^{p,\varphi}(l_2) = L^{p,\lambda}(l_2)$ is the vector-valued Morrey space and $WM_w^{p,\varphi}(l_2) = WL^{p,\lambda}(l_2)$ is the vector-valued weak Morrey space.
(5) If $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, then $M_w^{p,\varphi}(l_2) = L_w^p(l_2)$ is the vector-valued weighted Lebesgue space.

3 Preliminaries and some lemmas

By a weight function, briefly weight, we mean a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E) = \int_E w(x)\,dx$, and denote the Lebesgue measure of E by $|E|$ and the characteristic function of E by χ_E. Given a weight w, we say that w satisfies the doubling condition if there exists a constant $D > 0$ such that for any ball B, we have $w(2B) \le Dw(B)$. When w satisfies this condition, we write for brevity $w \in \Delta_2$.

If w is a weight function, we denote by $L^p_w(l_2) \equiv L^p_w(\mathbb{R}^n, l_2)$ the vector-valued weighted Lebesgue space defined by finiteness of the norm

$$\|\vec{f}\|_{L^p_w(l_2)} = \left(\int_{\mathbb{R}^n} \|\vec{f}(x)\|^p_{l_2} w(x)\,dx\right)^{\frac{1}{p}} < \infty, \quad \text{if } 1 \le p < \infty$$

and by $\|\vec{f}\|_{L^\infty_w(l_2)} = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \|\vec{f}(x)\|_{l_2} w(x)$ if $p = \infty$.

We recall that a weight function w is in the Muckenhoupt class A_p [31], $1 < p < \infty$, if

$$\begin{aligned}[w]_{A_p} &:= \sup_B [w]_{A_p(B)} \\ &= \sup_B \left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}\,dx\right)^{p-1} < \infty,\end{aligned}$$

where the sup is taken with respect to all the balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, for all balls B, by Hölder's inequality

$$[w]^{1/p}_{A_p(B)} = |B|^{-1}\|w\|^{1/p}_{L^1(B)} \left\|w^{-1/p}\right\|_{L^{p'}(B)} \ge 1.$$

For $p = 1$, the class A_1 is defined by the condition $Mw(x) \le Cw(x)$ with $[w]_{A_1} = \sup_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)}$, and for $p = \infty$, $A_\infty = \bigcup_{1\le p<\infty} A_p$ and $[w]_{A_\infty} = \inf_{1\le p<\infty} [w]_{A_p}$.

Lemma 3.1 ([32])

(1) *If $w \in A_p$ for some $1 \le p < \infty$, then $w \in \Delta_2$. Moreover, for all $\lambda > 1$*

$$w(\lambda B) \le \lambda^{np} [w]_{A_p} w(B).$$

(2) *If $w \in A_\infty$, then $w \in \Delta_2$. Moreover, for all $\lambda > 1$*

$$w(\lambda B) \le 2^{\lambda^n} [w]_{A_\infty} w(B).$$

(3) *If $w \in A_p$ for some $1 \le p \le \infty$, then there exist $C > 0$ and $\delta > 0$ such that for any ball B and a measurable set $S \subset B$,*

$$\frac{w(S)}{w(B)} \le C\left(\frac{|S|}{|B|}\right)^\delta.$$

We are going to use the following result on the boundedness of the Hardy operator:

$$(Hg)(t) := \frac{1}{t}\int_0^t g(r)\,d\mu(r), \quad 0 < t < \infty,$$

where μ is a non-negative Borel measure on $(0, \infty)$.

Theorem 3.2 ([33]) *The inequality*

$$\operatorname*{ess\,sup}_{t>0} \omega(t) Hg(t) \le c \operatorname*{ess\,sup}_{t>0} v(t) g(t)$$

holds for all functions g non-negative and non-increasing on $(0,\infty)$ *if and only if*

$$A := \sup_{t>0} \frac{\omega(t)}{t} \int_0^t \frac{d\mu(r)}{\operatorname{ess\,sup}_{0<s<r} \nu(s)} < \infty,$$

and $c \approx A$.

We also need the following statement on the boundedness of the Hardy type operator:

$$(H_1 g)(t) := \frac{1}{t} \int_0^t \ln^m \left(e + \frac{t}{r}\right) g(r)\, d\mu(r), \quad 0 < t < \infty,$$

where μ is a non-negative Borel measure on $(0,\infty)$.

Theorem 3.3 *The inequality*

$$\operatorname*{ess\,sup}_{t>0} \omega(t) H_1 g(t) \le c \operatorname*{ess\,sup}_{t>0} \nu(t) g(t)$$

holds for all functions g non-negative and non-increasing on $(0,\infty)$ *if and only if*

$$A_1 := \sup_{t>0} \frac{\omega(t)}{t} \int_0^t \ln^m \left(e + \frac{t}{r}\right) \frac{d\mu(r)}{\operatorname{ess\,sup}_{0<s<r} \nu(s)} < \infty,$$

and $c \approx A_1$.

Note that Theorem 3.3 can be proved analogously to Theorem 4.3 in [34].

Definition 3.4 $BMO(\mathbb{R}^n)$ is the Banach space modulo constants with the norm $\|\cdot\|_*$ defined by

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|b(y) - b_{B(x,r)}\right| dy < \infty,$$

where $b \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y)\, dy.$$

Lemma 3.5 ([35], Theorem 5, p.236) *Let* $w \in A_\infty$. *Then the norm* $\|\cdot\|_*$ *is equivalent to the norm*

$$\|b\|_{*,w} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{w(B(x,r))} \int_{B(x,r)} \left|b(y) - b_{B(x,r),w}\right| w(y)\, dy,$$

where

$$b_{B(x,r),w} = \frac{1}{w(B(x,r))} \int_{B(x,r)} b(y) w(y)\, dy.$$

Remark 3.6 (1) The John-Nirenberg inequality: there are constants $C_1, C_2 > 0$, such that for all $b \in BMO(\mathbb{R}^n)$ and $\beta > 0$

$$\left|\left\{x \in B : \left|b(x) - b_B\right| > \beta\right\}\right| \le C_1 |B| e^{-C_2 \beta / \|b\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) For $1 \le p < \infty$ the John-Nirenberg inequality implies that

$$\|b\|_* \approx \sup_B \left(\frac{1}{|B|} \int_B \left| b(y) - b_B \right|^p dy \right)^{\frac{1}{p}} \tag{3.1}$$

and for $1 \le p < \infty$ and $w \in A_\infty$

$$\|b\|_* \approx \sup_B \left(\frac{1}{w(B)} \int_B \left| b(y) - b_B \right|^p w(y)\, dy \right)^{\frac{1}{p}}. \tag{3.2}$$

Note that by the John-Nirenberg inequality and Lemma 3.1 (part 3) it follows that

$$w\left(\left\{x \in B : \left| b(x) - b_B \right| > \beta \right\}\right) \le C_1^\delta w(B) e^{-C_2 \beta\delta/\|b\|_*}$$

for some $\delta > 0$. Hence

$$\begin{aligned} \int_B \left| b(y) - b_B \right|^p w(y)\, dy &= p \int_0^\infty \beta^{p-1} w\left(\left\{x \in B : \left| b(x) - b_B \right| > \beta \right\}\right) d\beta \\ &\le p C_1^\delta w(B) \int_0^\infty \beta^{p-1} e^{-C_2 \beta\delta/\|b\|_*}\, d\beta = C_3 w(B) \|b\|_*^p, \end{aligned}$$

where $C_3 > 0$ depends only on C_1^δ, C_2, p, and δ, which implies (3.2).

Also (3.1) is a particular case of (3.2) with $w \equiv 1$.

The following lemma was proved in [13].

Lemma 3.7 (i) *Let $w \in A_\infty$ and $b \in BMO(\mathbb{R}^n)$. Let also $1 \le p < \infty$, $x \in \mathbb{R}^n$, $k > 0$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{w(B(x,r_1))} \int_{B(x,r_1)} \left| b(y) - b_{B(x,r_2),w} \right|^{kp} w(y)\, dy \right)^{\frac{1}{p}} \le C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right)^k \|b\|_*^k,$$

where $C > 0$ is independent of f, w, x, r_1, and r_2.

(ii) *Let $w \in A_p$ and $b \in BMO(\mathbb{R}^n)$. Let also $1 < p < \infty$, $x \in \mathbb{R}^n$, $k > 0$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{w^{1-p'}(B(x,r_1))} \int_{B(x,r_1)} \left| b(y) - b_{B(x,r_2),w} \right|^{kp'} w(y)^{1-p'}\, dy \right)^{\frac{1}{p'}} \le C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right)^k \|b\|_*^k,$$

where $C > 0$ is independent of f, w, x, r_1, and r_2.

4 Proofs of main theorems

Before proving the main theorems, we need the following lemmas.

Lemma 4.1 [23] *For $j \in \mathbb{Z}_+$, denote*

$$G_{\alpha,2^j}(f)(x) = \left(\int_0^\infty \int_{|x-y| \le 2^j t} \left(A_\alpha f(y,t) \right)^2 \frac{dy\, dt}{t^{n+1}} \right)^{\frac{1}{2}}.$$

Let $0 < \alpha \le 1$, $1 < p < \infty$, and $w \in A_p$. Then any $j \in \mathbb{Z}_+$, we have

$$\left\| G_{\alpha,2^j}(f) \right\|_{L_w^p} \lesssim 2^{j(\frac{3n}{2}+\alpha)} \left\| G_\alpha(f) \right\|_{L_w^p}.$$

This lemma is easy by the following inequality, which is proved in [17]:

$$G_{\alpha,\beta}(f)(x) \le \beta^{\frac{3n}{2}+\alpha} G_\alpha(f)(x).$$

By a similar argument to [2], we can get the following lemma.

Lemma 4.2 *Let $1 < p < \infty$, $0 < \alpha \le 1$, and $w \in A_p$, then the multilinear commutator $[\vec{b}, G_\alpha]$ is bounded from $L_w^p(l_2)$ to itself whenever $\vec{b} = (b_1, \dots, b_m)$ and $b_j \in BMO(\mathbb{R}^n)$, $j = 1, \dots, m$.*

Now we are in a position to prove the theorems.

Lemma 4.3 *Let $1 \le p < \infty$, $0 < \alpha \le 1$, and $w \in A_p$.*

Then, for $p > 1$, the inequality

$$\|G_\alpha \vec{f}\|_{L_w^p(B,l_2)} \lesssim \big(w(B)\big)^{\frac{1}{p}} \int_{2r}^{\infty} \|\vec{f}\|_{L_w^p(B(x_0,t),l_2)} \big(w\big(B(x_0,t)\big)\big)^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B = B(x_0, r)$ and for all $\vec{f} \in L_w^{p,\mathrm{loc}}(l_2)$.

Moreover, for $p = 1$ the inequality

$$\|G_\alpha \vec{f}\|_{WL_w^1(B,l_2)} \lesssim w(B) \int_{2r}^{\infty} \|\vec{f}\|_{L_w^1(B(x_0,t),l_2)} \big(w\big(B(x_0,t)\big)\big)^{-1} \frac{dt}{t}$$

holds for any ball $B = B(x_0, r)$ and for all $\vec{f} \in L_w^{1,\mathrm{loc}}(l_2)$.

Proof The main ideas of these proofs come from [13]. For arbitrary $x \in \mathbb{R}^n$, set $B = B(x_0, r)$, $2B \equiv B(x_0, 2r)$. We decompose $\vec{f} = \vec{f}_0 + \vec{f}_\infty$, where $\vec{f}_0(y) = \vec{f}(y)\chi_{2B}(y)$, $\vec{f}_\infty(y) = \vec{f}(y) - \vec{f}_0(y)$. Then

$$\|G_\alpha \vec{f}\|_{L_w^p(B(x_0,r),l_2)} \le \|G_\alpha \vec{f}_0\|_{L_w^p(B(x_0,r),l_2)} + \|G_\alpha \vec{f}_\infty\|_{L^p(B(x_0,r),l_2)} := I + II.$$

First, let us estimate I. By Theorem A, we obtain

$$I \le \|G_\alpha \vec{f}_0\|_{L_w^p(l_2)} \lesssim \|\vec{f}_0\|_{L_w^p(l_2)} = \|\vec{f}\|_{L_w^p(2B,l_2)}. \tag{4.1}$$

On the other hand,

$$\begin{aligned} \|\vec{f}\|_{L_w^p(2B,l_2)} &\approx |B| \|\vec{f}\|_{L_w^p(2B,l_2)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \\ &\le |B| \int_{2r}^{\infty} \|\vec{f}\|_{L_w^p(B(x_0,t),l_2)} \frac{dt}{t^{n+1}} \\ &\lesssim w(B)^{\frac{1}{p}} \|w^{-1/p}\|_{L_{p'}(B)} \int_{2r}^{\infty} \|\vec{f}\|_{L_w^p(B(x_0,t),l_2)} \frac{dt}{t^{n+1}} \\ &\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|\vec{f}\|_{L_w^p(B(x_0,t),l_2)} \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\lesssim [\omega]_{A_p}^{1/p} w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|\vec{f}\|_{L_w^p(B(x_0,t),l_2)} \big(w\big(B(x_0,t)\big)\big)^{-\frac{1}{p}} \frac{dt}{t}. \end{aligned} \tag{4.2}$$

Therefore from (4.1) and (4.2) we get

$$I \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|\vec{f}\|_{L_w^p(B(x_0,t),l_2)} \big(w\big(B(x_0,t)\big)\big)^{-\frac{1}{p}} \frac{dt}{t}. \tag{4.3}$$

Then let us estimate II:

$$\big\|\vec{f} * \phi_t(y)\big\|_{l_2} = \left\| t^{-n} \int_{|y-z|\le t} \phi\left(\frac{y-z}{t}\right) \vec{f}_\infty(z)\, dz \right\|_{l_2} \le t^{-n} \int_{|y-z|\le t} \big\|\vec{f}_\infty(z)\big\|_{l_2}\, dz.$$

Since $x \in B(x_0,r)$, $(y,t) \in \Gamma(x)$, we have $|z-x| \le |z-y| + |y-x| \le 2t$, and

$$r \le |z-x_0| - |x_0-x| \le |x-z| \le |x-y| + |y-z| \le 2t.$$

So, we obtain

$$\begin{aligned}
\big\|G_\alpha \vec{f}_\infty(x)\big\|_{l_2} &\le \left(\iint_{\Gamma(x)} \left(t^{-n} \int_{|y-z|\le t} \big\|\vec{f}_\infty(z)\big\|_{l_2}\, dz \right)^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{2}} \\
&\le \left(\int_{t>r/2} \int_{|x-y|<t} \left(\int_{|x-z|\le 2t} \big\|\vec{f}_\infty(z)\big\|_{l_2}\, dz \right)^2 \frac{dy\,dt}{t^{3n+1}} \right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{t>r/2} \left(\int_{|z-x|\le 2t} \big\|\vec{f}_\infty(z)\big\|_{l_2}\, dz \right)^2 \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}}.
\end{aligned}$$

By Minkowski's and Hölder's inequalities and $|z-x| \ge |z-x_0| - |x_0-x| \ge \frac{1}{2}|z-x_0|$, we have

$$\begin{aligned}
\big\|G_\alpha \vec{f}_\infty(x)\big\|_{l_2} &\lesssim \int_{\mathbb{R}^n} \left(\int_{t>\frac{|z-x|}{2}} \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}} \big\|\vec{f}_\infty(z)\big\|_{l_2}\, dz \\
&\lesssim \int_{|z-x_0|>2r} \frac{\|\vec{f}(z)\|_{l_2}}{|z-x|^n}\, dz \lesssim \int_{|z-x_0|>2r} \frac{\|\vec{f}(z)\|_{l_2}}{|z-x_0|^n}\, dz \\
&= \int_{|z-x_0|>2r} \big\|\vec{f}(z)\big\|_{l_2} \int_{|z-x_0|}^{+\infty} \frac{dt}{t^{n+1}}\, dz \\
&= \int_{2r}^{+\infty} \int_{2r<|z-x_0|<t} \big\|\vec{f}(z)\big\|_{l_2}\, dz \frac{dt}{t^{n+1}} \\
&\lesssim \int_{2r}^{\infty} \big\| \big\|\vec{f}(z)\big\|_{l_2} \big\|_{L_w^p(B(x_0,t))} \big\|w^{-1}\big\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim \int_{2r}^{\infty} \|\vec{f}\|_{L_w^p(B(x_0,t),l_2)} \big(w\big(B(x_0,t)\big)\big)^{-\frac{1}{p}} \frac{dt}{t}.
\end{aligned}$$

Thus,

$$\|G_\alpha \vec{f}_\infty\|_{L_w^p(B,l_2)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|\vec{f}\|_{L_w^p(B(x_0,t),l_2)} \big(w\big(B(x_0,t)\big)\big)^{-\frac{1}{p}} \frac{dt}{t}. \tag{4.4}$$

By combining (4.3) and (4.4), we have

$$\|G_\alpha \vec{f}\|_{L_w^p(B,l_2)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|\vec{f}\|_{L_w^p(B(x_0,t),l_2)} \big(w\big(B(x_0,t)\big)\big)^{-\frac{1}{p}} \frac{dt}{t}. \qquad \square$$

Proof of Theorem 1.1 By Lemma 4.3 and Theorem 3.2 we have for $p > 1$

$$\begin{aligned}
\|G_\alpha\vec{f}\|_{M_w^{p,\varphi_2}(l_2)} &\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1}\int_r^\infty \|\vec{f}\|_{L_w^p(B(x_0,t),l_2)}\big(w\big(B(x_0,t)\big)\big)^{-\frac{1}{p}}\frac{dt}{t} \\
&= \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1}\int_0^{r^{-1}} \|\vec{f}\|_{L_w^p(B(x_0,t^{-1}),l_2)}\big(w\big(B\big(x_0,t^{-1}\big)\big)\big)^{-\frac{1}{p}}\frac{dt}{t} \\
&= \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2\big(x_0,r^{-1}\big)^{-1} r\frac{1}{r}\int_0^{r} \|\vec{f}\|_{L_w^p(B(x_0,t^{-1}),l_2)}\big(w\big(B\big(x_0,t^{-1}\big)\big)\big)^{-\frac{1}{p}}\frac{dt}{t} \\
&\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_1\big(x_0,r^{-1}\big)^{-1}\big(w\big(B\big(x_0,r^{-1}\big)\big)\big)^{-\frac{1}{p}}\|\vec{f}\|_{L_w^p(B(x_0,r^{-1}),l_2)} \\
&= \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_1(x_0,r)^{-1}\big(w\big(B(x_0,r)\big)\big)^{-\frac{1}{p}}\|\vec{f}\|_{L_w^p(B(x_0,r),l_2)} = \|\vec{f}\|_{M_w^{p,\varphi_1}(l_2)}
\end{aligned}$$

and for $p = 1$

$$\begin{aligned}
\|G_\alpha\vec{f}\|_{WM_w^{1,\varphi_2}(l_2)} &\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1}\int_r^\infty \|\vec{f}\|_{L_w^1(B(x_0,t),l_2)}\big(w\big(B(x_0,t)\big)\big)^{-1}\frac{dt}{t} \\
&= \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1}\int_0^{r^{-1}} \|\vec{f}\|_{L_w^1(B(x_0,t^{-1}),l_2)}\big(w\big(B\big(x_0,t^{-1}\big)\big)\big)^{-1}\frac{dt}{t} \\
&= \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2\big(x_0,r^{-1}\big)^{-1} r\frac{1}{r}\int_0^{r} \|\vec{f}\|_{L_w^1(B(x_0,t^{-1}),l_2)}\big(w\big(B\big(x_0,t^{-1}\big)\big)\big)^{-1}\frac{dt}{t} \\
&\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_1\big(x_0,r^{-1}\big)^{-1}\big(w\big(B\big(x_0,r^{-1}\big)\big)\big)^{-1}\|\vec{f}\|_{L_w^1(B(x_0,r^{-1}),l_2)} \\
&= \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_1(x_0,r)^{-1}\big(w\big(B(x_0,r)\big)\big)^{-1}\|\vec{f}\|_{L_w^1(B(x_0,r),l_2)} = \|\vec{f}\|_{M_w^{1,\varphi_1}(l_2)}. \qquad \square
\end{aligned}$$

Lemma 4.4 *Let $1 \le p < \infty$, $0 < \alpha \le 1$, $\lambda > 3 + \frac{\alpha}{n}$, and $w \in A_p$. Then, for $p > 1$, the inequality*

$$\big\|g_{\lambda,\alpha}^*(\vec{f})\big\|_{L_w^p(B,l_2)} \lesssim \big(w(B)\big)^{\frac{1}{p}}\int_{2r}^\infty \|\vec{f}\|_{L_w^p(B(x_0,t),l_2)}\big(w\big(B(x_0,t)\big)\big)^{-\frac{1}{p}}\frac{dt}{t}$$

holds for any ball $B = B(x_0,r)$ and for all $\vec{f} \in L_w^{p,\mathrm{loc}}(l_2)$.

Moreover, for $p = 1$ the inequality

$$\big\|g_{\lambda,\alpha}^*(\vec{f})\big\|_{WL_w^1(B,l_2)} \lesssim w(B)\int_{2r}^\infty \|\vec{f}\|_{L_w^1(B(x_0,t),l_2)}\big(w\big(B(x_0,t)\big)\big)^{-1}\frac{dt}{t}$$

holds for any ball $B = B(x_0,r)$ and for all $\vec{f} \in L_w^{1,\mathrm{loc}}(l_2)$.

Proof From the definition of $g_{\lambda,\alpha}^*(f)$, we readily see that

$$\begin{aligned}
\big\|g_{\lambda,\alpha}^*(\vec{f})(x)\big\|_{l_2} &= \bigg\|\bigg(\int_0^\infty\int_{\mathbb{R}^n}\bigg(\frac{t}{t+|x-y|}\bigg)^{n\lambda}\big(A_\alpha\vec{f}(y,t)\big)^2\frac{dy\,dt}{t^{n+1}}\bigg)^{l/2}\bigg\|_{l_2} \\
&\le \bigg\|\bigg(\int_0^\infty\int_{|x-y|<t}\bigg(\frac{t}{t+|x-y|}\bigg)^{n\lambda}\big(A_\alpha\vec{f}(y,t)\big)^2\frac{dy\,dt}{t^{n+1}}\bigg)^{l/2}\bigg\|_{l_2}
\end{aligned}$$

$$\begin{aligned}&+\left\|\left(\int_0^\infty\int_{|x-y|\geq t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\left(A_\alpha\vec{f}(y,t)\right)^2\frac{dy\,dt}{t^{n+1}}\right)^{l/2}\right\|_{l_2}\\ &:= III+IV.\end{aligned}$$

First, let us estimate III:

$$III\leq\left\|\left(\int_0^\infty\int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\left(A_\alpha\vec{f}(y,t)\right)^2\frac{dy\,dt}{t^{n+1}}\right)^{l/2}\right\|_{l_2}\leq\left\|G_\alpha\vec{f}(x)\right\|_{l_2}.$$

Now, let us estimate IV:

$$\begin{aligned}IV&\leq\left\|\left(\sum_{j=1}^\infty\int_0^\infty\int_{2^{j-1}t\leq|x-y|\leq 2^jt}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\left(A_\alpha\vec{f}(y,t)\right)^2\frac{dy\,dt}{t^{n+1}}\right)^{l/2}\right\|_{l_2}\\ &\lesssim\left\|\left(\sum_{j=1}^\infty\int_0^\infty\int_{2^{j-1}t\leq|x-y|\leq 2^jt}2^{-jn\lambda}\left(A_\alpha\vec{f}(y,t)\right)^2\frac{dy\,dt}{t^{n+1}}\right)^{l/2}\right\|_{l_2}\\ &\lesssim\sum_{j=1}^\infty 2^{-jn\lambda}\left\|\left(\int_0^\infty\int_{|x-y|\leq 2^jt}\left(A_\alpha\vec{f}(y,t)\right)^2\frac{dy\,dt}{t^{n+1}}\right)^{l/2}\right\|_{l_2}:=\sum_{j=1}^\infty 2^{-jn\lambda}\left\|G_{\alpha,2^j}(\vec{f})(x)\right\|_{l_2}.\end{aligned}$$

Thus,

$$\left\|\mathrm{g}^*_{\lambda,\alpha}(\vec{f})\right\|_{L^p_w(B,l_2)}\leq\|G_\alpha\vec{f}\|_{L^p_w(B,l_2)}+\sum_{j=1}^\infty 2^{-\frac{jn\lambda}{2}}\left\|G_{\alpha,2^j}(\vec{f})\right\|_{L^p_w(B,l_2)}. \tag{4.5}$$

By Lemma 4.3, we have

$$\|G_\alpha\vec{f}\|_{L^p_w(B,l_2)}\lesssim\left(w(B)\right)^{\frac{1}{p}}\int_{2r}^\infty\|\vec{f}\|_{L^p_w(B(x_0,t))}\left(w\left(B(x_0,t)\right)\right)^{-\frac{1}{p}}\frac{dt}{t}. \tag{4.6}$$

In the following, we will estimate $\|G_{\alpha,2^j}(\vec{f})\|_{L^p_w(B,l_2)}$. We divide $\|G_{\alpha,2^j}(\vec{f})\|_{L^p_w(B,l_2)}$ into two parts,

$$\left\|G_{\alpha,2^j}(\vec{f})\right\|_{L^p_w(B,l_2)}\leq\left\|G_{\alpha,2^j}(\vec{f}_0)\right\|_{L^p_w(B,l_2)}+\left\|G_{\alpha,2^j}(\vec{f}_\infty)\right\|_{L^p_w(B,l_2)}, \tag{4.7}$$

where $\vec{f}_0(y)=\vec{f}(y)\chi_{2B}(y)$, $\vec{f}_\infty(y)=\vec{f}(y)-\vec{f}_\infty(y)$. For the first part, by Lemma 4.1,

$$\begin{aligned}\left\|G_{\alpha,2^j}(\vec{f}_0)\right\|_{L^p_w(B,l_2)}&\lesssim 2^{j(\frac{3n}{2}+\alpha)}\left\|G_\alpha(\vec{f}_0)\right\|_{L^p_w(l_2)}\lesssim 2^{j(\frac{3n}{2}+\alpha)}\|f\|_{L^p_w(B,l_2)}\\ &\lesssim 2^{j(\frac{3n}{2}+\alpha)}w(B)^{\frac{1}{p}}\int_{2r}^\infty\|\vec{f}\|_{L^p_w(B(x_0,t),l_2)}\left(w\left(B(x_0,t)\right)\right)^{-\frac{1}{p}}\frac{dt}{t}.\end{aligned} \tag{4.8}$$

For the second part,

$$\begin{aligned}\left\|G_{\alpha,2^j}(\vec{f}_\infty)(x)\right\|_{l_2}&=\left\|\left(\int_0^\infty\int_{|x-y|\leq 2^jt}\left(A_\alpha\vec{f}(y,t)\right)^2\frac{dy\,dt}{t^{n+1}}\right)^{l/2}\right\|_{l_2}\\ &=\left\|\left(\int_0^\infty\int_{|x-y|\leq 2^jt}\left(\sup_{\phi\in C_\alpha}\left|\vec{f}*\phi_t(y)\right|\right)^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}\right\|_{l_2}\\ &\leq\left(\int_0^\infty\int_{|x-y|\leq 2^jt}\left(\int_{|z-y|\leq t}\left\|\vec{f}_\infty(z)\right\|_{l_2}dz\right)^2\frac{dy\,dt}{t^{3n+1}}\right)^{\frac{1}{2}}.\end{aligned}$$

Since $|x-z| \le |y-z| + |x-y| \le 2^{j+1}t$, we get

$$\begin{aligned}\bigl\|G_{\alpha,2^j}(\vec{f}_\infty)(x)\bigr\|_{l_2} &\le \left(\int_0^\infty \int_{|x-y|\le 2^j t}\left(\int_{|x-z|\le 2^{j+1}t}\bigl\|\vec{f}_\infty(z)\bigr\|_{l_2}\,dz\right)^2 \frac{dy\,dt}{t^{3n+1}}\right)^{\frac{1}{2}} \\ &\le \left(\int_0^\infty \left(\int_{|z-x|\le 2^{j+1}t}\bigl\|\vec{f}_\infty(z)\bigr\|_{l_2}\,dz\right)^2 \frac{2^{jn}dt}{t^{2n+1}}\right)^{\frac{1}{2}} \\ &\le 2^{\frac{jn}{2}}\int_{\mathbb{R}^n}\left(\int_{t\ge\frac{|x-z|}{2^{j+1}}}\bigl\|\vec{f}_\infty(z)\bigr\|_{l_2}^2\,\frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}}dz \\ &\le 2^{\frac{3jn}{2}}\int_{|x_0-z|>2r}\frac{\|\vec{f}(z)\|_{l_2}}{|x-z|^n}\,dz.\end{aligned}$$

For $|z-x| \ge |x_0-z| - |x-x_0| \ge |x_0-z| - \frac{1}{2}|x_0-z| = \frac{1}{2}|x_0-z|$, so by Fubini's theorem and Hölder's inequality, we obtain

$$\begin{aligned}\bigl\|G_{\alpha,2^j}(\vec{f}_\infty)(x)\bigr\|_{l_2} &\le 2^{\frac{3jn}{2}}\int_{|x_0-z|>2r}\frac{\|\vec{f}(z)\|_{l_2}}{|x_0-z|^n}\,dz \\ &= 2^{\frac{3jn}{2}}\int_{|x_0-z|>2r}\bigl\|\vec{f}(z)\bigr\|_{l_2}\int_{|x_0-z|}^\infty \frac{dt}{t^{n+1}}\,dz \\ &\le 2^{\frac{3jn}{2}}\int_{2r}^\infty\int_{|x_0-z|<t}\bigl\|\vec{f}(z)\bigr\|_{l_2}\,dz\,\frac{dt}{t^{n+1}} \\ &\le 2^{\frac{3jn}{2}}\int_{2r}^\infty\bigl\|\bigl\|\vec{f}(\cdot)\bigr\|_{l_2}\bigr\|_{L^1(B(x_0,t))}\frac{dt}{t^{n+1}} \\ &\le 2^{\frac{3jn}{2}}\int_{2r}^\infty\bigl\|\bigl\|\vec{f}(\cdot)\bigr\|_{l_2}\bigr\|_{L^p_w(B(x_0,t))}\bigl\|w^{-1}\bigr\|_{L_{p'}(B(x_0,t))}\frac{dt}{t^{n+1}} \\ &\le 2^{\frac{3jn}{2}}\int_{2r}^\infty\|\vec{f}\|_{L^p_w(B(x_0,t),l_2)}\bigl(w\bigl(B(x_0,t)\bigr)\bigr)^{-\frac{1}{p}}\,\frac{dt}{t}.\end{aligned}$$

So,

$$\bigl\|G_{\alpha,2^j}(\vec{f}_\infty)\bigr\|_{L^p_w(B,l_2)} \le 2^{\frac{3jn}{2}}w(B)^{\frac{1}{p}}\int_{2r}^\infty\|\vec{f}\|_{L^p_w(B(x_0,t),l_2)}\bigl(w\bigl(B(x_0,t)\bigr)\bigr)^{-\frac{1}{p}}\,\frac{dt}{t}. \tag{4.9}$$

Combining (4.7), (4.8), and (4.9), we have

$$\bigl\|G_{\alpha,2^j}(\vec{f})\bigr\|_{L^p_w(B,l_2)} \lesssim 2^{j(\frac{3n}{2}+\alpha)}w(B)^{\frac{1}{p}}\int_{2r}^\infty\|\vec{f}\|_{L^p_w(B(x_0,t),l_2)}\bigl(w\bigl(B(x_0,t)\bigr)\bigr)^{-\frac{1}{p}}\,\frac{dt}{t}. \tag{4.10}$$

Thus,

$$\bigl\|g^*_{\lambda,\alpha}(\vec{f})\bigr\|_{L^p_w(B,l_2)} \le \|G_\alpha\vec{f}\|_{L^p_w(B,l_2)} + \sum_{j=1}^\infty 2^{-\frac{jn\lambda}{2}}\bigl\|G_{\alpha,2^j}(\vec{f})\bigr\|_{L^p_w(B,l_2)}. \tag{4.11}$$

Since $\lambda > 3 + \frac{\alpha}{n}$, by (4.6), (4.10), and (4.11), we have the desired lemma. $\square$

Proof of Theorem 1.2 From inequality (4.5) we have

$$\big\|g^*_{\lambda,\alpha}(\vec{f})\big\|_{M^{p,\varphi_2}_w(l_2)} \le \|G_\alpha\vec{f}\|_{M^{p,\varphi_2}_w(l_2)} + \sum_{j=1}^{\infty} 2^{-\frac{jn\lambda}{2}} \big\|G_{\alpha,2^j}(\vec{f})\big\|_{M^{p,\varphi_2}_w(l_2)}. \tag{4.12}$$

By Theorem 1.1, we have

$$\|G_\alpha\vec{f}\|_{M^{p,\varphi_2}_w(l_2)} \lesssim \|\vec{f}\|_{M^{p,\varphi_1}_w(l_2)}. \tag{4.13}$$

In the following, we will estimate $\|G_{\alpha,2^j}(\vec{f})\|_{M^{p,\varphi_2}_w(l_2)}$. Thus, by substitution of variables and Theorem 3.2, we get

$$\begin{aligned}
&\big\|G_{\alpha,2^j}(\vec{f})\big\|_{M^{p,\varphi_2}_w(l_2)} \\
&\quad\lesssim 2^{j(\frac{3n}{2}+\alpha)} \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1} \int_r^\infty \|\vec{f}\|_{L^p_w(B(x_0,t),l_2)} \big(w\big(B(x_0,t)\big)\big)^{-\frac{1}{p}} \frac{dt}{t} \\
&\quad= 2^{j(\frac{3n}{2}+\alpha)} \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2\big(x_0,r^{-1}\big)^{-1} r\frac{1}{r} \int_0^r \|\vec{f}\|_{L^p_w(B(x_0,t^{-1}),l_2)} \big(w\big(B\big(x_0,t^{-1}\big)\big)\big)^{-\frac{1}{p}} \frac{dt}{t} \\
&\quad\lesssim 2^{j(\frac{3n}{2}+\alpha)} \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_1\big(x_0,r^{-1}\big)^{-1} \big(w\big(B\big(x_0,r^{-1}\big)\big)\big)^{-\frac{1}{p}} \|\vec{f}\|_{L^p_w(B(x_0,r^{-1}),l_2)} \\
&\quad= 2^{j(\frac{3n}{2}+\alpha)} \|\vec{f}\|_{M^{p,\varphi_1}_w(l_2)}.
\end{aligned} \tag{4.14}$$

Since $\lambda > 3 + \frac{\alpha}{n}$, by (4.12), (4.13), and (4.14), we have the desired theorem. □

Lemma 4.5 *Let* $1 < p < \infty$, $0 < \alpha \le 1$, $w \in A_p$, $\vec{b} = (b_1,\dots,b_m)$, *and* $b_i \in BMO(\mathbb{R}^n)$, $i = 1,\dots,m$. *Then the inequality*

$$\big\|[\vec{b},G_\alpha]\vec{f}\big\|_{L^p_w(B,l_2)} \lesssim \big(w(B)\big)^{\frac{1}{p}} \int_{2r}^\infty \ln^m\left(e+\frac{t}{r}\right) \|\vec{f}\|_{L^p_w(B(x_0,\tau),l_2)} \big(w\big(B(x_0,\tau)\big)\big)^{-\frac{1}{p}} \frac{d\tau}{\tau}$$

holds for any ball $B = B(x_0,r)$ *and for all* $f \in L^{p,\mathrm{loc}}_w(l_2)$.

Proof We decompose $\vec{f} = \vec{f}_0 + \vec{f}_\infty$, where $\vec{f}_0 = \vec{f}\chi_{2B}$ and $\vec{f}_\infty = \vec{f} - \vec{f}_0$. Then

$$\big\|[\vec{b},G_\alpha]\vec{f}\big\|_{L^p_w(B,l_2)} \le \big\|[\vec{b},G_\alpha]\vec{f}_0\big\|_{L^p_w(B,l_2)} + \big\|[\vec{b},G_\alpha]\vec{f}_\infty\big\|_{L^p_w(B,l_2)}.$$

Denote by $\|\vec{b}\|_* = \prod_{i=1}^m \|b_j\|_*$. By Lemma 4.2, we have

$$\begin{aligned}
\big\|[\vec{b},G_\alpha]\vec{f}_0\big\|_{L^p_w(B,l_2)} &\lesssim \|\vec{b}\|_*\|\vec{f}_0\|_{L^p_w(l_2)} = \|\vec{b}\|_*\|\vec{f}\|_{L^p_w(2B,l_2)} \\
&\lesssim \|\vec{b}\|_* w(B)^{\frac{1}{p}} \int_{2r}^\infty \|\vec{f}\|_{L^p_w(B(x_0,\tau),l_2)} \big(w\big(B(x_0,\tau)\big)\big)^{-\frac{1}{p}} \frac{d\tau}{\tau}.
\end{aligned} \tag{4.15}$$

For the term $\|[\vec{b},G_\alpha]\vec{f}_\infty\|_{L^p_w(B,l_2)}$, without loss of generality, we can assume $m = 2$. Thus, the operator $[\vec{b},G_\alpha]\vec{f}_\infty$ can be divided into four parts:

$$\begin{aligned}
&\big|[\vec{b},G_\alpha]\vec{f}_\infty(x)\big| \\
&\quad\le \left(\iint_{\Gamma(x)} \sup_{\phi\in C_\alpha} \left|\int_{\mathbb{R}^n} \phi_t(y-z)\big(b_1(z)-(b_1)_{B,\omega}\big)\big(b_2(z)-(b_2)_{B,\omega}\big)\vec{f}_\infty(z)\,dz\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2}
\end{aligned}$$

$$+ \big|b_1(x) - (b_1)_{B,\omega}\big| \left(\iint_{\Gamma(x)} \sup_{\phi \in C_\alpha} \left| \int_{\mathbb{R}^n} \phi_t(y-z)\big(b_2(z) - (b_2)_{B,\omega}\big)\vec{f}_\infty(z)\,dz \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2}$$

$$+ \big|b_2(x) - (b_2)_{B,\omega}\big| \left(\iint_{\Gamma(x)} \sup_{\phi \in C_\alpha} \left| \int_{\mathbb{R}^n} \phi_t(y-z)\big(b_1(z) - (b_1)_{B,\omega}\big)\vec{f}_\infty(z)\,dz \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2}$$

$$\le \big|\big(b_1(x) - (b_1)_{B,\omega}\big)\big(b_2(x) - (b_2)_{B,\omega}\big)\big| \left(\iint_{\Gamma(x)} \sup_{\phi \in C_\alpha} \left| \int_{\mathbb{R}^n} \phi_t(y-z)\vec{f}_\infty(z)\,dz \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2}$$

$$\equiv I_1(x) + I_2(x) + I_3(x) + I_4(x).$$

For $x \in B$ we have

$$\begin{aligned}
\big\|[\vec{b}, G_\alpha]\vec{f}_\infty(x)\big\|_{l_2} &\le \big\|I_1(x)\big\|_{l_2} + \big\|I_2(x)\big\|_{l_2} + \big\|I_3(x)\big\|_{l_2} + \big\|I_4(x)\big\|_{l_2} \\
&\le \int_{^{\complement}(2B)} \big|b_1(z) - (b_1)_{B,\omega}\big|\big|b_2(z) - (b_2)_{B,\omega}\big| \frac{\|\vec{f}(z)\|_{l_2}}{|x_0 - z|^n}\,dz \\
&\quad + \big|b_1(x) - (b_1)_{B,\omega}\big| \int_{^{\complement}(2B)} \big|b_2(z) - (b_2)_{B,\omega}\big| \frac{\|\vec{f}(z)\|_{l_2}}{|x_0 - z|^n}\,dz \\
&\quad + \big|b_2(x) - (b_2)_{B,\omega}\big| \int_{^{\complement}(2B)} \big|b_1(z) - (b_1)_{B,\omega}\big| \frac{\|\vec{f}(z)\|_{l_2}}{|x_0 - z|^n}\,dz \\
&\quad + \big|\big(b_1(x) - (b_1)_{B,\omega}\big)\big|\big|\big(b_2(x) - (b_2)_{B,\omega}\big)\big| \int_{^{\complement}(2B)} \frac{\|\vec{f}(z)\|_{l_2}}{|x_0 - z|^n}\,dz.
\end{aligned}$$

Then

$$\begin{aligned}
&\big\|[\vec{b}, G_\alpha]\vec{f}_\infty\big\|_{L^p_w(B,l_2)} \\
&\quad\le \left(\int_B \left(\int_{^{\complement}(2B)} \frac{\prod_{i=1}^2 |b_i(z) - (b_i)_{B,\omega}|}{|x_0 - z|^n} \big\|\vec{f}(z)\big\|_{l_2}\,dz \right)^p \omega(x)\,dx \right)^{1/p} \\
&\qquad + \left(\int_B \big|b_1(x) - (b_1)_{B,\omega}\big| \left(\int_{^{\complement}(2B)} \frac{|b_2(z) - (b_2)_{B,\omega}|}{|x_0 - z|^n} \big\|\vec{f}(z)\big\|_{l_2}\,dz \right)^p \omega(x)\,dx \right)^{1/p} \\
&\qquad + \left(\int_B \big|b_2(x) - (b_2)_{B,\omega}\big| \left(\int_{^{\complement}(2B)} \frac{|b_1(z) - (b_1)_{B,\omega}|}{|x_0 - z|^n} \big\|\vec{f}(z)\big\|_{l_2}\,dz \right)^p \omega(x)\,dx \right)^{1/p} \\
&\qquad + \left(\int_B \left(\int_{^{\complement}(2B)} \frac{\prod_{i=1}^2 |b_i(x) - (b_i)_{B,\omega}|}{|x_0 - z|^n} \big\|\vec{f}(z)\big\|_{l_2}\,dz \right)^p \omega(x)\,dx \right)^{1/p} \\
&\quad\equiv I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Let us estimate I_1:

$$\begin{aligned}
I_1 &= \omega(B)^{1/p} \int_{^{\complement}(2B)} \frac{\prod_{i=1}^2 |b_i(z) - (b_i)_{B,\omega}|}{|x_0 - z|^n} \big\|\vec{f}(z)\big\|_{l_2}\,dz \\
&\approx \omega(B)^{1/p} \int_{^{\complement}(2B)} \prod_{i=1}^2 \big|b_i(z) - (b_i)_{B,\omega}\big| \big\|\vec{f}(z)\big\|_{l_2} \int_{|x_0 - z|}^\infty \frac{d\tau}{\tau^{n+1}}\,dz
\end{aligned}$$

$$\begin{aligned}
&\approx \omega(B)^{1/p} \int_{2r}^{\infty} \int_{2r\leq|x_0-z|<\tau} \prod_{i=1}^{2} \big|b_i(z)-(b_i)_{B,\omega}\big| \big\|\vec{f}(z)\big\|_{l_2} \, dz \, \frac{d\tau}{\tau^{n+1}} \\
&\lesssim \omega(B)^{1/p} \int_{2r}^{\infty} \int_{B(x_0,\tau)} \prod_{i=1}^{2} \big|b_i(z)-(b_i)_{B,\omega}\big| \big\|\vec{f}(z)\big\|_{l_2} \, dz \, \frac{d\tau}{\tau^{n+1}}.
\end{aligned}$$

Applying Hölder's inequality and by Lemma 3.7, we get

$$\begin{aligned}
I_1 &\leq \omega(B)^{\frac{1}{p}} \int_{2r}^{\infty} \prod_{i=1}^{2} \left(\int_{B(x_0,\tau)} \big|b_i(z)-(b_i)_{B,\omega}\big|^{2p'} \omega(z)^{1-2p'} \, dz \right)^{\frac{1}{2p'}} \|f\|_{L_w^p(B(x_0,\tau),l_2)} \frac{d\tau}{\tau^{n+1}} \\
&\leq \prod_{i=1}^{2} \|b_j\|_* \omega(B)^{1/p} \int_{2r}^{\infty} \left(1+\ln\frac{\tau}{r}\right)^2 \big\|\omega^{-1/p}\big\|_{L_w^{p'}(B(x_0,\tau),l_2)} \|f\|_{L_w^p(B(x_0,\tau),l_2)} \frac{d\tau}{\tau^{n+1}} \\
&\leq \|\vec{b}\|_* \omega(B)^{1/p} \int_{2r}^{\infty} \ln^2\left(e+\frac{\tau}{r}\right) \|f\|_{L_w^p(B(x_0,\tau),l_2)} \omega\big(B(x_0,\tau)\big)^{-1/p} \frac{d\tau}{\tau}.
\end{aligned}$$

Let us estimate I_2:

$$\begin{aligned}
I_2 &= \left(\int_B \big|b_1(x)-(b_1)_{B,\omega}\big|^p \omega(x)\, dx \right)^{1/p} \int_{^{\complement}(2B)} \frac{|b_2(z)-(b_2)_{B,\omega}|}{|x_0-z|^n} \big\|\vec{f}(z)\big\|_{l_2} \, dz \\
&\lesssim \|b_1\|_* \omega(B)^{1/p} \left[\int_{^{\complement}(2B)} \big|b_2(z)-(b_2)_{B,\omega}\big| \big\|\vec{f}(z)\big\|_{l_2} \int_{|x_0-z|}^{\infty} \frac{d\tau}{\tau^{n+1}} \, dz \right] \\
&\approx \|b_1\|_* \omega(B)^{1/p} \int_{2r}^{\infty} \int_{2r\leq|x_0-z|\leq\tau} \big|b_2(z)-(b_2)_{B,\omega}\big| \big\|\vec{f}(z)\big\|_{l_2} \, dz \, \frac{d\tau}{\tau^{n+1}} \\
&\lesssim \|b_1\|_* \omega(B)^{1/p} \int_{2r}^{\infty} \int_{B(x_0,\tau)} \big|b_2(z)-(b_2)_{B,\omega}\big| \big\|\vec{f}(z)\big\|_{l_2} \, dz \, \frac{d\tau}{\tau^{n+1}}.
\end{aligned}$$

Applying Hölder's inequality and by Lemma 3.7, we get

$$\begin{aligned}
I_2 &\leq \|b_1\|_* \omega(B)^{\frac{1}{p}} \int_{2r}^{\infty} \left(\int_{B(x_0,\tau)} \big|b_2(z)-(b_2)_{B,\omega}\big|^{p'} \omega(z)^{1-p'} \, dz \right)^{\frac{1}{p'}} \|f\|_{L_w^p(B(x_0,\tau),l_2)} \frac{d\tau}{\tau^{n+1}} \\
&\leq \prod_{i=1}^{2} \|b_j\|_* \omega(B)^{1/p} \int_{2r}^{\infty} \left(1+\ln\frac{\tau}{r}\right) \big\|\omega^{-1/p}\big\|_{L_w^{p}(B(x_0,\tau),l_2)} \|f\|_{L_w^p(B(x_0,\tau),l_2)} \frac{d\tau}{\tau^{n+1}} \\
&\leq \|\vec{b}\|_* \omega(B)^{1/p} \int_{2r}^{\infty} \ln^2\left(e+\frac{\tau}{r}\right) \|f\|_{L_w^p(B(x_0,\tau),l_2)} \omega\big(B(x_0,\tau)\big)^{-1/p} \frac{d\tau}{\tau}.
\end{aligned}$$

In the same way, we shall get the result of I_3:

$$I_3 \leq \|\vec{b}\|_* \omega(B)^{1/p} \int_{2r}^{\infty} \ln^2\left(e+\frac{\tau}{r}\right) \|f\|_{L_w^p(B(x_0,\tau),l_2)} \omega\big(B(x_0,\tau)\big)^{-1/p} \frac{d\tau}{\tau}.$$

In order to estimate I_4 note that

$$\begin{aligned}
I_4 &= \left(\int_B \prod_{i=1}^{2} \big|b_i(x)-(b_i)_{B,\omega}\big|^p \omega(x)\, dx \right)^{1/p} \int_{^{\complement}(2B)} \frac{\|\vec{f}(z)\|_{l_2}}{|x_0-z|^n} \, dz \\
&\leq \prod_{i=1}^{2} \left(\int_B \big|b_i(x)-(b_i)_{B,\omega}\big|^{2p} \omega(x)\, dx \right)^{1/2p} \int_{^{\complement}(2B)} \frac{\|\vec{f}(z)\|_{l_2}}{|x_0-z|^n} \, dz.
\end{aligned}$$

By Lemma 3.7, we get

$$I_4 \le \|\vec{b}\|_* \omega(B)^{1/p} \int_{^{\complement}(2B)} \frac{\|\vec{f}(z)\|_{l_2}}{|x_0 - z|^n}\, dz.$$

Applying Hölder's inequality, we get

$$\begin{aligned}
&\int_{^{\complement}(2B)} \frac{\|\vec{f}(z)\|_{l_2}}{|x_0 - z|^n}\, dz \\
&\quad \le \int_{2r}^{\infty} \|f\|_{L^p_w(B(x_0,\tau),l_2)} \big\|\omega^{-1/p}\big\|_{L^{p'}_w(B(x_0,\tau),l_2)} \frac{d\tau}{\tau^{n+1}} \\
&\quad \le [\omega]^{1/p}_{A_p} \int_{2r}^{\infty} \|f\|_{L^p_w(B(x_0,\tau),l_2)} \omega\big(B(x_0,\tau)\big)^{-1/p} \frac{d\tau}{\tau}.
\end{aligned} \tag{4.16}$$

Thus by (4.16)

$$I_4 \le \|\vec{b}\|_* \omega(B)^{1/p} \int_{2r}^{\infty} \|f\|_{L^p_w(B(x_0,\tau),l_2)} \omega\big(B(x_0,\tau)\big)^{-1/p} \frac{d\tau}{\tau}.$$

Summing up I_1 and I_4, for all $p \in [1,\infty)$ we get

$$\begin{aligned}
\big\|[\vec{b}, G_\alpha]\vec{f}_\infty\big\|_{L^p_w(B(x_0,\tau),l_2)} &\le \|\vec{b}\|_* \omega(B)^{1/p} \int_{2r}^{\infty} \ln^2\left(e + \frac{\tau}{r}\right) \|f\|_{L^p_w(B(x_0,\tau),l_2)} \\
&\qquad \times \omega\big(B(x_0,\tau)\big) \omega\big(B(x_0,\tau)\big)^{-1/p} \frac{d\tau}{\tau}.
\end{aligned} \tag{4.17}$$

Finally, from (4.2), (4.15), and (4.17) we get

$$\begin{aligned}
&\big\|[\vec{b}, G_\alpha]\vec{f}\big\|_{L^p_w(B(x_0,\tau),l_2)} \\
&\quad \lesssim \|\vec{b}\|_* \omega(B)^{1/p} \int_{2r}^{\infty} \ln^2\left(e + \frac{\tau}{r}\right) \|f\|_{L^p_w(B(x_0,\tau),l_2)} \omega\big(B(x_0,\tau)\big)^{-1/p} \frac{d\tau}{\tau}.
\end{aligned}$$

□

Proof of Theorem 1.3 By substitution of variables, we obtain

$$\begin{aligned}
&\big\|[\vec{b}, G_\alpha]\vec{f}\big\|_{M^{p,\varphi_2}_\omega(l_2)} \\
&\quad \lesssim \|\vec{b}\|_* \sup_{x_0 \in \mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1} \int_{2r}^{\infty} \ln^m\left(e + \frac{\tau}{r}\right) \|\vec{f}\|_{L^p_w(B(x_0,\tau),l_2)} w\big(B(x_0,\tau)\big)^{-1/p} \frac{d\tau}{\tau} \\
&\quad \lesssim \|\vec{b}\|_* \sup_{x_0 \in \mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1} \int_0^{r^{-1}} \ln^m\left(e + \frac{1}{\tau r}\right) \|\vec{f}\|_{L^p_w(B(x_0,\tau^{-1}),l_2)} w\big(B\big(x_0,\tau^{-1}\big)\big)^{-\frac{1}{p}} \frac{d\tau}{\tau} \\
&\quad = \|\vec{b}\|_* \sup_{x \in \mathbb{R}^n, r>0} \varphi_2\big(x_0,r^{-1}\big)^{-1} r \frac{1}{r} \int_0^{r} \ln^m\left(e + \frac{r}{\tau}\right) \|\vec{f}\|_{L^p_w(B(x_0,\tau^{-1}),l_2)} w\big(B\big(x_0,\tau^{-1}\big)\big)^{-\frac{1}{p}} \frac{d\tau}{\tau} \\
&\quad \lesssim \|\vec{b}\|_* \sup_{x_0 \in \mathbb{R}^n, r>0} \varphi_1\big(x_0,r^{-1}\big)^{-1} w\big(B\big(x_0,r^{-1}\big)\big)^{-\frac{1}{p}} \|\vec{f}\|_{L^p_w(B(x_0,r^{-1}),l_2)} \\
&\quad = \|\vec{b}\|_* \sup_{x_0 \in \mathbb{R}^n, r>0} \varphi_1(x_0,r)^{-1} w\big(B(x_0,r)\big)^{-\frac{1}{p}} \|\vec{f}\|_{L^p_w(B(x_0,r),l_2)} \\
&\quad = \|\vec{b}\|_* \|\vec{f}\|_{M^{p,\varphi_1}_w(l_2)}.
\end{aligned}$$

By using an argument similar to the above proofs and that of Theorem 1.2, we can also show the boundedness of $[\vec{b}, g^{*}_{\lambda,\alpha}]$. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This work was carried out in collaboration between all authors. VSG raised these interesting problems in the research. VSG and MNO proved the theorems, interpreted the results and wrote the article. All authors defined the research theme, and read and approved the manuscript.

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Acknowledgements
The research of V Guliyev was partially supported by the grant of Ahi Evran University Scientific Research Projects (PYO.FEN.4003.13.003) and (PYO.FEN.4003-2.13.007). We thank both referees for some good suggestions, which helped to improve the final version of this paper.

Received: 19 December 2013 Accepted: 30 June 2014 Published: 22 Jul 2014

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10.1186/1029-242X-2014-258
Cite this article as: Guliyev and Omarova: **Multilinear commutators of vector-valued intrinsic square functions on vector-valued generalized weighted Morrey spaces.** *Journal of Inequalities and Applications* 2014, 2014:258