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Approximation properties of bivariate extension of q-Szász-Mirakjan-Kantorovich operators

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Abstract

In the present paper, a bivariate generalization of the q-Szász-Mirakjan-Kantorovich operators is constructed by q_R-integral and these operators' weighted A-statistical approximation properties are established. Also, we estimate the rate of pointwise convergence of the proposed operators by modulus of continuity.

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1 Introduction

In [1] the Kantorovich type generalization of the Szász-Mirakjan operators is defined by

$$K_n(f,x) := ne^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_{k/n}^{(k+1)/n} f(t)\,dt, \quad f \in C[0,\infty), 0 \le x < \infty.$$

In [2], for each positive integer n, Aral and Gupta defined q-type generalization of Szász-Mirakjan operators as

$$S_n^q(f;x) := E_q\left(-[n]_q\frac{x}{b_n}\right)\sum_{k=0}^{\infty} f\left(\frac{[k]_q b_n}{[n]_q}\right)\frac{([n]_q x)^k}{[k]_q!(b_n)^k},$$

where $0 < q < 1$, $f \in C[0,\infty)$, $0 \le x < \frac{b_n}{(1-q)[n]_q}$, b_n is a sequence of positive numbers such that $\lim_{x\to\infty} b_n = \infty$.

Recently, q-type generalization of Szász-Mirakjan operators, which was different from that in [2], was introduced, and the convergence properties of these operators were studied by Mahmudov [3]. Weighted statistical approximation properties of the modified q-Szász-Mirakjan operators were obtained in [4]. Also, Durrmeyer and Kantorovich-type generalizations of the linear positive operators based on q-integers were studied by some authors. The Bernstein-Durrmeyer operators related to the q-Bernstein operators were studied by Derriennic [5]. Gupta [6] introduced and studied approximation properties of q-Durrmeyer operators. The generalizations of the q-Baskakov-Kantorovich operators were constructed and weighted statistical approximation properties of these operators were examined in [7] and [8]. The q-extensions of the Szász-Mirakjan, Szász-Mirakjan-Kantorovich, Szász-Schurer and Szász-Schurer-Kantorovich operators were given shortly in [8]. Generalized Szász Durrmeyer operators were studied in [9]. With the help of q_R-integral, Örkcü and Doğru [10] introduced a Kantorovich-type modification of the q-Szász-Mirakjan operators as follows:

$$K_n^q(f;x) := [n]_q E_q\left(-[n]_q \frac{x}{q}\right) \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{[k]_q! q^k} \int_{q[k]_q/[n]_q}^{[k+1]_q/[n]_q} f(t)\, d_q^R t, \tag{1.1}$$

where $q \in (0,1)$, $0 \leq x < \frac{q}{1-q^n}$, $f \in C[0,\infty)$.

The paper of Mursaleen *et al.* [11] is one of the latest references on approximation by q-analogue. They investigated approximation properties for new q-Lagrange polynomials. Also, the q-analogue of Bernstein-Schurer-Stancu operators were introduced in [12].

On the other hand, Stancu [13] first introduced linear positive operators in two and several dimensional variables. Afterward, Barbosu [14] introduced a generalization of two-dimensional Bernstein operators based on q-integers and called them bivariate q-Bernstein operators. In recent years, many results have been obtained in the Korovkin-type approximation theory via A-statistical convergence for functions of several variables (for instance, [15–17]).

In this study, we construct a bivariate generalization of the Szász-Mirakjan-Kantorovich operators based on q-integers and obtain the weighted A-statistical approximation properties of these operators.

Now we recall some definitions about q-integers. For any non-negative integer r, the q-integer of the number r is defined by

$$[r]_q = \begin{cases} 1 + q + \cdots + q^{r-1} & \text{if } q \neq 1, \\ r & \text{if } q = 1, \end{cases}$$

where q is a positive real number. The q-factorial is defined as

$$[r]_q! = \begin{cases} [1]_q[2]_q \cdots [r]_q & \text{if } r = 1, 2, \ldots, \\ 1 & \text{if } r = 0. \end{cases}$$

Two q-analogues of the exponential function e^x are given as

$$E_q(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!}, \quad x \in \mathbb{R},$$

$$\varepsilon_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \quad |x| < \frac{1}{1-q}.$$

The following relation between q-exponential functions $E_q(x)$ and $\varepsilon_q(x)$ holds

$$E_q(x)\varepsilon_q(-x) = 1, \quad |x| < \frac{1}{1-q}. \tag{1.2}$$

In the fundamental books about q-calculus (see [18, 19]), the q-integral of the function f over the interval $[0,b]$ is defined by

$$\int_0^b f(t)\,d_qt = b(1-q)\sum_{j=0}^{\infty} f\left(bq^j\right)q^j, \quad 0 < q < 1.$$

If f is integrable over $[0,b]$, then

$$\lim_{q\to 1^-}\int_0^b f(t)\,d_qt = \int_0^b f(t)\,dt.$$

A generally accepted definition for q-integral over an interval $[a,b]$ is

$$\int_a^b f(t)\,d_qt = \int_0^b f(t)\,d_qt - \int_0^a f(t)\,d_qt.$$

In order to generalize and spread the existing inequalities, Marinkovic *et al.* considered a new type of q-integral. So, the problems that ensue from the general definition of q-integral were overcome. The Riemann-type q-integral [20] in the interval $[a,b]$ was defined as

$$\int_a^b f(t)\,d_q^Rt = (1-q)(b-a)\sum_{j=0}^{\infty} f\left(a+(b-a)q^j\right)q^j, \quad 0 < q < 1.$$

This definition includes only a point inside the interval of the integration.

2 Construction of the bivariate operators

The aim of this part is to construct a bivariate extension of the operators defined by (1.1).

For $n \in \mathbb{N}$, $0 < q_1, q_2 < 1$ and $0 \leq x < \frac{q_1}{1-q_1^n}$, $0 \leq y < \frac{q_2}{1-q_2^n}$, the bivariate extension of the operators $K_n^q(f;x)$ is as follows:

$$\begin{aligned} K_n^{q_1,q_2}(f;x,y) &= [n]_{q_1}[n]_{q_2}E_{q_1}\left(-[n]_{q_1}\frac{x}{q_1}\right)E_{q_2}\left(-[n]_{q_2}\frac{y}{q_2}\right) \\ &\quad\times \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{([n]_{q_1}x)^k}{[k]_{q_1}!q_1^k}\frac{([n]_{q_2}y)^l}{[l]_{q_2}!q_2^l} \\ &\quad\times \int_{q_2[l]_{q_2}/[n]_{q_2}}^{[l+1]_{q_2}/[n]_{q_2}}\int_{q_1[k]_{q_1}/[n]_{q_1}}^{[k+1]_{q_1}/[n]_{q_1}} f(t,s)\,d_{q_1}^Rt\,d_{q_2}^Rs, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \int_c^d\int_a^b f(t,s)\,d_{q_1}^Rt\,d_{q_2}^Rs &= (1-q_1)(1-q_2)(b-a)(c-d) \\ &\quad\times \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} f\left(a+(b-a)q_1^i, c+(c-d)q_2^j\right)q_1^iq_2^j. \end{aligned} \tag{2.2}$$

Also, f is a q_R-integrable function, so the series in (2.2) converges. It is clear that the operators given in (2.1) are linear and positive.

First, let us give the following lemma.

Lemma 1 *Let $e_{ij} = x^i y^j$, $(i,j) \in \mathbb{N}_0 \times \mathbb{N}_0$ with $i + j \leq 2$ be the two-dimensional test functions. Then the following results hold for the operators given by* (2.1):

(i) $K_n^{q_1,q_2}(e_{00}; x, y) = 1$;

(ii) $K_n^{q_1,q_2}(e_{10}; x, y) = x + \dfrac{1}{[2]_{q_1}} \dfrac{1}{[n]_{q_1}}$;

(iii) $K_n^{q_1,q_2}(e_{01}; x, y) = y + \dfrac{1}{[2]_{q_2}} \dfrac{1}{[n]_{q_2}}$;

(iv) $K_n^{q_1,q_2}(e_{20}; x, y) = q_1 x^2 + \left(q_1 + \dfrac{2}{[2]_{q_1}}\right) \dfrac{1}{[n]_{q_1}} x + \dfrac{1}{[3]_{q_1}} \dfrac{1}{[n]_{q_1}^2}$;

(v) $K_n^{q_1,q_2}(e_{02}; x, y) = q_2 y^2 + \left(q_2 + \dfrac{2}{[2]_{q_2}}\right) \dfrac{1}{[n]_{q_2}} y + \dfrac{1}{[3]_{q_2}} \dfrac{1}{[n]_{q_2}^2}$.

Proof From $\int_{q_2[l]_{q_2}/[n]_{q_2}}^{[l+1]_{q_2}/[n]_{q_2}} \int_{q_1[k]_{q_1}/[n]_{q_1}}^{[k+1]_{q_1}/[n]_{q_1}} d_{q_1}^R t \, d_{q_2}^R s = \frac{1}{[n]_{q_1}[n]_{q_2}}$ and the equality in (1.2), we have

$$\begin{aligned} K_n^{q_1,q_2}(e_{00}; x, y) &= E_{q_1}\left(-[n]_{q_1} \frac{x}{q_1}\right) E_{q_2}\left(-[n]_{q_2} \frac{y}{q_2}\right) \\ &\quad \times \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{([n]_{q_1} x)^k}{[k]_{q_1}! q_1^k} \frac{([n]_{q_2} y)^l}{[l]_{q_2}! q_2^l} \\ &= 1. \end{aligned}$$

Since $\int_{q_2[l]_{q_2}/[n]_{q_2}}^{[l+1]_{q_2}/[n]_{q_2}} \int_{q_1[k]_{q_1}/[n]_{q_1}}^{[k+1]_{q_1}/[n]_{q_1}} t \, d_{q_1}^R t \, d_{q_2}^R s = \frac{1}{[n]_{q_1}[n]_{q_2}} (\frac{q_1[k]_{q_1}}{[n]_{q_1}} + \frac{1}{[2]_{q_1}} \frac{1}{[n]_{q_1}})$, we get from the linearity of $K_n^{q_1,q_2}$ that

$$\begin{aligned} K_n^{q_1,q_2}(e_{10}; x, y) &= E_{q_1}\left(-[n]_{q_1} \frac{x}{q_1}\right) E_{q_2}\left(-[n]_{q_2} \frac{y}{q_2}\right) \\ &\quad \times \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{q_1 [k]_{q_1}}{[n]_{q_1}} \frac{([n]_{q_1} x)^k}{[k]_{q_1}! q_1^k} \frac{([n]_{q_2} y)^l}{[l]_{q_2}! q_2^l} \\ &\quad + E_{q_1}\left(-[n]_{q_1} \frac{x}{q_1}\right) E_{q_2}\left(-[n]_{q_2} \frac{y}{q_2}\right) \\ &\quad \times \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{[2]_{q_1}} \frac{1}{[n]_{q_1}} \frac{([n]_{q_1} x)^k}{[k]_{q_1}! q_1^k} \frac{([n]_{q_2} y)^l}{[l]_{q_2}! q_2^l}. \end{aligned}$$

Then we have from the definition of q-factorial and $K_n^{q_1,q_2}(e_{00}; x, y) = 1$

$$\begin{aligned} K_n^{q_1,q_2}(e_{10}; x, y) &= E_{q_1}\left(-[n]_{q_1} \frac{x}{q_1}\right) E_{q_2}\left(-[n]_{q_2} \frac{y}{q_2}\right) \\ &\quad \times \sum_{l=0}^{\infty} \sum_{k=1}^{\infty} \frac{x([n]_{q_1} x)^{k-1}}{[k-1]_{q_1}! q_1^{k-1}} \frac{([n]_{q_2} y)^l}{[l]_{q_2}! q_2^l} + \frac{1}{[2]_{q_1}} \frac{1}{[n]_{q_1}} \\ &= x + \frac{1}{[2]_{q_1}} \frac{1}{[n]_{q_1}}. \end{aligned}$$

Similarly, we write that

$$K_n^{q_1,q_2}(e_{01};x,y) = y + \frac{1}{[2]_{q_2}}\frac{1}{[n]_{q_2}}.$$

Now we compute the value $K_n^{q_1,q_2}(e_{20};x,y)$. Applying the equalities $\int_{q_2[l]_{q_2}/[n]_{q_2}}^{[l+1]_{q_2}/[n]_{q_2}} \times \int_{q_1[k]_{q_1}/[n]_{q_1}}^{[k+1]_{q_1}/[n]_{q_1}} t^2\, d_{q_1}^R t\, d_{q_2}^R s = \frac{1}{[n]_{q_1}[n]_{q_2}}(\frac{q_1^2[k]_{q_1}^2}{[n]_{q_1}^2} + \frac{1}{[2]_{q_1}}\frac{2q_1[k]_{q_1}}{[n]_{q_1}^2} + \frac{1}{[3]_{q_1}}\frac{1}{[n]_{q_1}^2})$, $K_n^{q_1,q_2}(e_{10};x,y) = x + \frac{1}{[2]_{q_1}}\frac{1}{[n]_{q_1}}$ and $K_n^{q_1,q_2}(e_{00};x,y) = 1$, we obtain

$$\begin{aligned} K_n^{q_1,q_2}(e_{20};x,y) &= E_{q_1}\left(-[n]_{q_1}\frac{x}{q_1}\right)E_{q_2}\left(-[n]_{q_2}\frac{y}{q_2}\right) \\ &\quad\times \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{q_1^2[k]_{q_1}^2}{[n]_{q_1}^2}\frac{([n]_{q_1}x)^k}{[k]_{q_1}!q_1^k}\frac{([n]_{q_2}y)^l}{[l]_{q_2}!q_2^l} \\ &\quad+ \frac{2}{[2]_{q_1}}\frac{1}{[n]_{q_1}}x + \frac{1}{[3]_{q_1}}\frac{1}{[n]_{q_1}^2}. \end{aligned}$$

Next, using the fact that $[k]_{q_1} = q_1[k-1]_{q_1} + 1$, we obtain

$$\begin{aligned} K_n^{q_1,q_2}(e_{20};x,y) &= E_{q_1}\left(-[n]_{q_1}\frac{x}{q_1}\right)E_{q_2}\left(-[n]_{q_2}\frac{y}{q_2}\right) \\ &\quad\times \sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\frac{q_1[k-1]_{q_1}+1}{[n]_{q_1}^2}\frac{([n]_{q_1}x)^k}{[k-1]_{q_1}!q_1^{k-2}}\frac{([n]_{q_2}y)^l}{[l]_{q_2}!q_2^l} \\ &\quad+ \frac{2}{[2]_{q_1}}\frac{1}{[n]_{q_1}}x + \frac{1}{[3]_{q_1}}\frac{1}{[n]_{q_1}^2} \\ &= E_{q_1}\left(-[n]_{q_1}\frac{x}{q_1}\right)E_{q_2}\left(-[n]_{q_2}\frac{y}{q_2}\right) \\ &\quad\times \sum_{l=0}^{\infty}\sum_{k=2}^{\infty}q_1x^2\frac{([n]_{q_1}x)^{k-2}}{[k-2]_{q_1}!q_1^{k-2}}\frac{([n]_{q_2}y)^l}{[l]_{q_2}!q_2^l} \\ &\quad+ E_{q_1}\left(-[n]_{q_1}\frac{x}{q_1}\right)E_{q_2}\left(-[n]_{q_2}\frac{y}{q_2}\right) \\ &\quad\times \sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\frac{q_1}{[n]_{q_1}}x\frac{([n]_{q_1}x)^{k-1}}{[k-1]_{q_1}!q_1^{k-1}}\frac{([n]_{q_2}y)^l}{[l]_{q_2}!q_2^l} \\ &\quad+ \frac{2}{[2]_{q_1}}\frac{1}{[n]_{q_1}}x + \frac{1}{[3]_{q_1}}\frac{1}{[n]_{q_1}^2} \\ &= q_1x^2 + \left(q_1 + \frac{2}{[2]_{q_1}}\right)\frac{1}{[n]_{q_1}}x + \frac{1}{[3]_{q_1}}\frac{1}{[n]_{q_1}^2}. \end{aligned}$$

Similarly, we write

$$K_n^{q_1,q_2}(e_{02};x,y) = q_2y^2 + \left(q_2 + \frac{2}{[2]_{q_2}}\right)\frac{1}{[n]_{q_2}}y + \frac{1}{[3]_{q_2}}\frac{1}{[n]_{q_2}^2},$$

which completes the proof. □

3 *A*-Statistical approximation properties

The Korovkin-type theorem for functions of two variables was proved by Volkov [21]. The theorem on weighted approximation for functions of several variables was proved by Gadjiev in [22].

Let B_ω be the space of real-valued functions defined on $\mathbb{R}^2$ and satisfying the bounded condition $|f(x,y)| \leq M_f \omega(x,y)$, where $\omega(x,y) \geq 1$ for all $(x,y) \in \mathbb{R}^2$ is called a weight function if it is continuous on $\mathbb{R}^2$ and $\lim_{\sqrt{x^2+y^2}\to\infty} \omega(x,y) = \infty$. We denote by C_ω the space of all continuous functions in the B_ω with the norm

$$\|f\|_\omega = \sup_{(x,y)\in\mathbb{R}^2} \frac{|f(x,y)|}{\omega(x,y)}.$$

Theorem 1 [22] *Let $\omega_1(x,y)$ and $\omega_2(x,y)$ be weight functions satisfying*

$$\lim_{\sqrt{x^2+y^2}\to\infty} \frac{\omega_1(x,y)}{\omega_2(x,y)} = 0.$$

Assume that T_n is a sequence of linear positive operators acting from C_{ω_1} to B_{ω_2}. Then, for all $f \in C_{\omega_1}$,

$$\lim_{n\to\infty} \|T_n f - f\|_{\omega_2} = 0$$

if and only if

$$\lim_{n\to\infty} \|T_n F_\nu - F_\nu\|_{\omega_1} = 0 \quad (\nu = 0,1,2,3),$$

where $F_0(x,y) = \frac{\omega_1(x)}{1+x^2+y^2}$, $F_1(x,y) = \frac{x\omega_1(x)}{1+x^2+y^2}$, $F_2(x,y) = \frac{y\omega_1(x)}{1+x^2+y^2}$, $F_3(x,y) = \frac{(x^2+y^2)\omega_1(x)}{1+x^2+y^2}$.

In [16], using the concept of A-statistical convergence, Erkuş and Duman investigated a Korovkin-type approximation result for a sequence of positive linear operators defined on the space of all continuous real-valued functions on any compact subset of the real m-dimensional space.

Now we recall the concepts of regularity of a summability matrix and A-statistical convergence. Let $A := (a_{nk})$ be an infinite summability matrix. For a given sequence $x := (x_k)$, the A-transform of x, denoted by $Ax := ((Ax)_n)$, is defined as $(Ax)_n := \sum_{k=1}^{\infty} a_{nk} x_k$ provided the series converges for each n. A is said to be regular if $\lim_n (Ax)_n = L$ whenever $\lim x = L$ [23]. Suppose that A is a non-negative regular summability matrix. Then x is A-statistically convergent to L if for every $\varepsilon > 0$, $\lim_n \sum_{k:|x_k - L| \geq \varepsilon} a_{nk} = 0$, and we write $st_A\text{-}\lim x = L$ [24]. Actually, x is A-statistically convergent to L if and only if, for every $\varepsilon > 0$, $\delta_A(k \in \mathbb{N} : |x_k - L| \geq \varepsilon) = 0$, where $\delta_A(K)$ denotes the A-density of the subset K of the natural numbers and is given by $\delta_A(K) := \lim_n \sum_{k=1}^{\infty} a_{nk} \chi_K(k)$ provided the limit exists, where χ_K is the characteristic function of K. If $A = C_1$, the Cesáro matrix of order one, then A-statistical convergence reduces to the statistical convergence [25]. Also, taking $A = I$, the identity matrix, A-statistical convergence coincides with the ordinary convergence.

We consider $\omega_1(x,y) = 1+x^2+y^2$ and $\omega_2(x,y) = (1+x^2+y^2)^{1+\alpha}$ for $\alpha > 0$, $(x,y) \in \mathbb{R}_0^2$, where $\mathbb{R}_0^2 := \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$.

We obtain statistical approximation properties of the operator defined by (2.1) with the help of Korovkin-type theorem given in [26]. Let $(q_{1,n})$ and $(q_{2,n})$ be two sequences in the interval $(0,1)$ so that

$$\begin{aligned} &st_A\text{-}\lim_n q_{1,n}^n = 1 \quad \text{and} \quad st_A\text{-}\lim_n q_{2,n}^n = 1, \\ &st_A\text{-}\lim_n \frac{1}{[n]_{q_{1,n}}} = 0 \quad \text{and} \quad st_A\text{-}\lim_n \frac{1}{[n]_{q_{2,n}}} = 0. \end{aligned} \tag{3.1}$$

Theorem 2 *Let $A = (a_{nk})$ be a nonnegative regular summability matrix, and let $(q_{1,n})$ and $(q_{2,n})$ be two sequences satisfying* (3.1). *Then, for any function $f \in C_{\omega_1}(\mathbb{R}_0^2)$ and q_R-integrable function, for $\alpha > 0$, we have*

$$st_A\text{-}\lim_n \left\| K_n^{q_{1,n},q_{2,n}} f - f \right\|_{\omega_2} = 0.$$

Proof Let $\tilde{K}_n^{q_{1,n},q_{2,n}}$ be defined as

$$\tilde{K}_n^{q_{1,n},q_{2,n}}(f;x,y) = \begin{cases} K_n^{q_{1,n},q_{2,n}}(f;x,y), & 0 \leq x < \frac{q_{1,n}}{1-q_{1,n}^n}, 0 \leq y < \frac{q_{2,n}}{1-q_{2,n}^n}, \\ f(x,y), & x \geq \frac{q_{1,n}}{1-q_{1,n}^n}, y \geq \frac{q_{2,n}}{1-q_{2,n}^n}. \end{cases}$$

From Lemma 1, since $|K_n^{q_{1,n},q_{2,n}}(1+t^2+s^2;x,y)| \leq c(1+x^2+y^2)^{1+\alpha}$ for $x \in [0, \frac{q_{1,n}}{1-q_{1,n}^n})$ and $y \in [0, \frac{q_{2,n}}{1-q_{2,n}^n})$, $\{\tilde{K}_n^{q_{1,n},q_{2,n}}(f;\cdot)\}$ is a sequence of linear positive operators acting from $C_{\omega_1}(\mathbb{R}_0^2)$ to $B_{\omega_2}(\mathbb{R}_0^2)$.

From (i) of Lemma 1, it is clear that

$$st_A\text{-}\lim_n \left\| \tilde{K}_n^{q_{1,n},q_{2,n}}(e_{00};\cdot) - e_{00} \right\|_{\omega_1} = 0$$

holds. By (ii) of Lemma 1, we get

$$\begin{aligned} &\sup_{0 \leq x < \frac{q_{1,n}}{1-q_{1,n}^n}, 0 \leq y < \frac{q_{2,n}}{1-q_{2,n}^n}} \frac{|K_n^{q_{1,n},q_{2,n}}(e_{10};\cdot) - e_{10}|}{1+x^2+y^2} \\ &\quad = \sup_{0 \leq x < \frac{q_{1,n}}{1-q_{1,n}^n}, 0 \leq y < \frac{q_{2,n}}{1-q_{2,n}^n}} \frac{|x + \frac{1}{[2]_{q_{1,n}}} \frac{1}{[n]_{q_{1,n}}} - x|}{1+x^2+y^2} \\ &\quad = \sup_{0 \leq x < \frac{q_{1,n}}{1-q_{1,n}^n}, 0 \leq y < \frac{q_{2,n}}{1-q_{2,n}^n}} \frac{1}{1+x^2+y^2} \frac{1}{[2]_{q_{1,n}}} \frac{1}{[n]_{q_{1,n}}} \\ &\quad = \frac{1}{[2]_{q_{1,n}}} \frac{1}{[n]_{q_{1,n}}}. \end{aligned}$$

Since $st_A\text{-}\lim_n \frac{1}{[n]_{q_{1,n}}} = 0$, $st_A\text{-}\lim_n \|\tilde{K}_n^{q_{1,n},q_{2,n}}(e_{10};\cdot) - e_{10}\|_{\omega_1} = 0$. Similarly, since $st_A\text{-}\lim_n \frac{1}{[n]_{q_{2,n}}} = 0$, $st_A\text{-}\lim_n \|\tilde{K}_n^{q_{1,n},q_{2,n}}(e_{01};\cdot) - e_{01}\|_{\omega_1} = 0$. Also, we have from (iv) of Lemma 1

$$\begin{aligned}
&\sup_{0\le x<\frac{q_{1,n}}{1-q_{1,n}^n},0\le y<\frac{q_{2,n}}{1-q_{2,n}^n}} \frac{|K_n^{q_{1,n},q_{2,n}}(e_{20};\cdot) - e_{20}|}{1+x^2+y^2} \\
&\quad= \sup_{0\le x<\frac{q_{1,n}}{1-q_{1,n}^n},0\le y<\frac{q_{2,n}}{1-q_{2,n}^n}} \frac{|q_{1,n}x^2 + (q_{1,n} + \frac{2}{[2]_{q_{1,n}}})\frac{1}{[n]_{q_{1,n}}}x + \frac{1}{[3]_{q_{1,n}}}\frac{1}{[n]_{q_{1,n}}^2} - x^2|}{1+x^2+y^2} \\
&\quad\le (1-q_{1,n}) + \left(\frac{q_{1,n}}{2} + \frac{1}{[2]_{q_{1,n}}}\right)\frac{1}{[n]_{q_{1,n}}} + \frac{1}{[3]_{q_{1,n}}}\frac{1}{[n]_{q_{1,n}}^2}.
\end{aligned}$$

So, we can write

$$\left\|\tilde{K}_n^{q_{1,n},q_{2,n}}(e_{20};\cdot) - e_{20}\right\|_{\omega_1} \le (1-q_{1,n}) + \left(\frac{q_{1,n}}{2} + \frac{1}{[2]_{q_{1,n}}}\right)\frac{1}{[n]_{q_{1,n}}} + \frac{1}{[3]_{q_{1,n}}}\frac{1}{[n]_{q_{1,n}}^2}. \tag{3.2}$$

Since $st_A\text{-}\lim_n(1-q_{1,n}) = 0$, $st_A\text{-}\lim_n(\frac{q_{1,n}}{2} + \frac{1}{[2]_{q_{1,n}}})\frac{1}{[n]_{q_{1,n}}} = 0$ and $st_A\text{-}\lim_n \frac{1}{[3]_{q_{1,n}}}\frac{1}{[n]_{q_{1,n}}^2} = 0$, for each $\varepsilon > 0$, we define the following sets.

$$\begin{aligned}
&D := \left\{k : \left\|\tilde{K}_n^{q_{1,n},q_{2,n}}(e_{20};\cdot) - e_2\right\|_{\omega_1} \ge \varepsilon\right\}, \qquad D_1 := \left\{k : 1 - q_{1,k} \ge \frac{\varepsilon}{3}\right\}, \\
&D_2 := \left\{k : \left(\frac{q_{1,k}}{2} + \frac{1}{[2]_{q_{1,k}}}\right)\frac{1}{[n]_{q_{1,k}}} \ge \frac{\varepsilon}{3}\right\}, \qquad D_3 := \left\{k : \frac{1}{[3]_{q_{1,k}}}\frac{1}{[n]_{q_{1,k}}^2} \ge \frac{\varepsilon}{3}\right\}.
\end{aligned}$$

By (3.2), it is clear that $D \subseteq D_1 \cup D_2 \cup D_3$, which implies that for all $n \in \mathbb{N}$,

$$\sum_{k\in D} a_{nk} \le \sum_{k\in D_1} a_{nk} + \sum_{k\in D_2} a_{nk} + \sum_{k\in D_3} a_{nk}.$$

Taking limit as $n \to \infty$, we have

$$st_A\text{-}\lim_n \left\|\tilde{K}_n^{q_{1,n},q_{2,n}}(e_{20};\cdot) - e_{20}\right\|_{\omega_1} = 0.$$

Similarly, since $st_A\text{-}\lim_n(1-q_{2,n}) = 0$, $st_A\text{-}\lim_n(\frac{q_{2,n}}{2} + \frac{1}{[2]_{q_{2,n}}})\frac{1}{[n]_{q_{2,n}}} = 0$ and $st_A\text{-}\lim_n \frac{1}{[3]_{q_{2,n}}} \times \frac{1}{[n]_{q_{2,n}}^2} = 0$, we write $st_A\text{-}\lim_n \|\tilde{K}_n^{q_{1,n},q_{2,n}}(e_{02};\cdot) - e_{02}\|_{\omega_1} = 0$. So, the proof is completed. □

If we define the function $\varphi_{x,y}(t,s) = (t-x)^2 + (s-y)^2$, $(x,y) \in [0, \frac{q_1}{1-q_1^n}) \times [0, \frac{q_2}{1-q_2^n})$, then by Lemma 1 one gets the following result

$$\begin{aligned}
K_n^{q_1,q_2}\left(\varphi_{x,y}(t,s);x,y\right) &= (q_1-1)x^2 + (q_2-1)y^2 + \frac{q_1}{[n]_{q_1}}x + \frac{q_2}{[n]_{q_2}}y \\
&\quad+ \frac{1}{[3]_{q_1}}\frac{1}{[n]_{q_1}^2} + \frac{1}{[3]_{q_2}}\frac{1}{[n]_{q_2}^2}.
\end{aligned}$$

We use the modulus of continuity $\omega(f,\delta)$ defined as follows:

$$\omega(f,\delta) := \sup\left\{\left|f(t,s) - f(x,y)\right| : (t,s),(x,y) \in \mathbb{R}_0^2 \text{ and } \sqrt{(t-x)^2 + (s-y)^2} \le \delta\right\},$$

where $\delta > 0$ and $f \in C_B(\mathbb{R}_0^2)$ the space of all bounded and continuous functions on $\mathbb{R}_0^2$. Observe that, for all $f \in C_B(\mathbb{R}_0^2)$ and $\lambda, \delta > 0$, we have

$$\omega(f, \lambda\delta) \leq \bigl(1 + [\lambda]\bigr)\omega(f, \delta), \tag{3.3}$$

where $[\lambda]$ is defined to be the greatest integer less than or equal to λ.

By the definition of modulus of continuity, we have

$$\bigl|f(t,s) - f(x,y)\bigr| \leq \omega\bigl(f, \sqrt{(t-x)^2 + (s-y)^2}\bigr),$$

and by (3.3), for any $\delta > 0$,

$$\bigl|f(t,s) - f(x,y)\bigr| \leq \left(1 + \left[\frac{\sqrt{(t-x)^2 + (s-y)^2}}{\delta}\right]\right)\omega(f, \delta),$$

which implies that

$$\bigl|f(t,s) - f(x,y)\bigr| \leq \left(1 + \frac{(t-x)^2 + (s-y)^2}{\delta^2}\right)\omega(f, \delta). \tag{3.4}$$

Using the linearity and positivity of the operators $K_n^{q_1,q_2}$, we get from (3.4) and $K_n^{q_1,q_2}(e_{00}; x,y) = 1$ that, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\bigl|K_n^{q_1,q_2}(f;x,y) - f(x,y)\bigr| &\leq K_n^{q_1,q_2}\bigl(\bigl|f(t,s) - f(x,y)\bigr|;x,y\bigr) + \bigl|f(x,y)\bigr|\bigl|K_n^{q_1,q_2}(e_{00};x,y) - e_{00}\bigr| \\
&\leq K_n^{q_1,q_2}\left(\left(1 + \frac{(t-x)^2 + (s-y)^2}{\delta^2}\right)\omega(f,\delta);x,y\right) \\
&= \left(1 + \frac{1}{\delta^2}K_n^{q_1,q_2}\bigl(\varphi_{x,y}(t,s);x,y\bigr)\right)\omega(f,\delta).
\end{aligned}$$

Now, if we replace $q_{1,n}$ and $q_{2,n}$ by sequences $(q_{1,n})$ and $(q_{2,n})$ to be two sequences satisfying (3.1), and we take $\delta := \delta_n(x,y) = \sqrt{K_n^{q_{1,n},q_{2,n}}(\varphi_{x,y}(t,s);x,y)}$, $0 \leq x < \frac{q_{1,n}}{1-q_{1,n}^n}$, $0 \leq y < \frac{q_{2,n}}{1-q_{2,n}^n}$, then we can write

$$\bigl|K_n^{q_1,q_2}(f;x,y) - f(x,y)\bigr| \leq 2\omega(f,\delta).$$

Competing interests
The author did not provide this information.

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