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On Hilbert type inequalities

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Abstract
In the present paper we establish new inequalities similar to the extensions of Hilbert's double-series inequality and also give their integral analogues. Our results provide some new estimates to these types of inequalities.
MSC: 26D15

Keywords: Hilbert's inequality; Pachpatte's inequality; Hölder's inequality

1 Introduction

In recent years several authors have given considerable attention to Hilbert's double-series inequality together with its integral version, inverse version, and various generalizations (see [1–9]). In this paper, we establish multivariable sum inequalities for the extensions of Hilbert's inequality and also obtain their integral forms. Our results provide some new estimates to these types of inequalities.

The well-known classical extension of Hilbert's double-series theorem can be stated as follows [10, p.253].

Theorem A *If $p_1, p_2 > 1$ are real numbers such that $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$ and $0 < \lambda = 2 - \frac{1}{p_1} - \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} \leq 1$, where, as usual, q_1 and q_2 are the conjugate exponents of p_1 and p_2 respectively, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} \leq K\left(\sum_{m=1}^{\infty} a_m^p\right)^{1/p_1}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/p_2}, \tag{1.1}$$

where $K = K(p_1, p_2)$ depends on p_1 and p_2 only.

In 2000, Pachpatte [11] established a new inequality similar to inequality (1.1) as follows:

Theorem A′ *Let p, q, $a(s)$, $b(t)$, $a(0)$, $b(0)$, $\nabla a(s)$ and $\nabla b(t)$ be as in [11], then*

$$\begin{aligned}\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{|a(s)||b(t)|}{qs^{p-1}+pt^{q-1}} \leq{}& \frac{1}{pq} m^{(p-1)/p} n^{(q-1)/q}\left(\sum_{s=1}^{m}(m-s+1)\left|\nabla a(s)\right|^p\right)^{1/p} \\ &\times\left(\sum_{t=1}^{n}(n-t+1)\left|\nabla b(t)\right|^q\right)^{1/q}.\end{aligned} \tag{1.2}$$

The integral analogue of inequality (1.1) is as follows [10, p.254].

Theorem B *Let p, q, p', q' and λ be as in Theorem A. If $f \in L^p(0,\infty)$ and $g \in L^q(0,\infty)$, then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{(x+y)^\lambda}\,dx\,dy \le K\left(\int_0^\infty f^p(x)\,dx\right)^{1/p}\left(\int_0^\infty g^q(y)\,dy\right)^{1/q}, \tag{1.3}$$

where $K = K(p,q)$ depends on p and q only.

In [11], Pachpatte also established a similar version of inequality (1.3) as follows.

Theorem B′ *Let p, q, $f(s)$, $g(t)$, $f(0)$, $g(0)$, $f'(s)$ and $g'(t)$ be as in [11], then*

$$\begin{aligned}&\int_0^x \int_0^y \frac{|f(s)||g(t)|}{qs^{p-1}+pt^{q-1}}\,dt\,ds\\&\quad\le \frac{1}{pq}x^{(p-1)/p}y^{(q-1)/q}\left(\int_0^x (x-s)\left|f'(s)\right|^p ds\right)^{1/p}\left(\int_0^y (y-t)\left|g'(t)\right|^q dt\right)^{1/q}.\end{aligned} \tag{1.4}$$

In the present paper we establish some new inequalities similar to Theorems A, A′, B and B′. Our results provide some new estimates to these types of inequalities.

2 Statement of results

Our main results are given in the following theorems.

Theorem 2.1 *Let $p_i > 1$ be constants and $\frac{1}{p_i} + \frac{1}{q_i} = 1$. Let $a_i(s_{1i},\dots,s_{ni})$ be real-valued functions defined for $s_{ji} = 1,2,\dots,m_{ji}$, where m_{ji} $(i,j = 1,2,\dots,n)$ are natural numbers. For convenience, we write $a_i(0,\dots,0) = 0$ and $a_i(0,s_{2i},\dots,s_{ni}) = a_i(s_{1i},0,s_{3i},\dots,s_{ni}) = \cdots = a_i(s_{1i},\dots,s_{n-1,i},0) = 0$. Define the operators ∇_i by $\nabla_i a_i(s_{1i},\dots,s_{ni}) = a_i(s_{1i},\dots,s_{ni}) - a_i(s_{1i},\dots,s_{i-1,i},s_{ii}-1,s_{i+1,i},\dots,s_{ni})$ for any function $a_i(s_{1i},\dots,s_{ni})$. Then*

$$\begin{aligned}&\sum_{s_{11}=1}^{m_{11}}\cdots\sum_{s_{n1}=1}^{m_{n1}}\sum_{s_{12}=1}^{m_{12}}\cdots\sum_{s_{n2}=1}^{m_{n2}}\cdots\sum_{s_{1n}=1}^{m_{1n}}\cdots\sum_{s_{nn}=1}^{m_{nn}}\frac{\prod_{i=1}^n |a_i(s_{1i},\dots,s_{ni})|}{(\sum_{i=1}^n (s_{1i}\cdots s_{ni})/q_i)^{\sum_{i=1}^n 1/q_i}}\\&\quad\le M\prod_{i=1}^n\left(\sum_{s_{ni}=1}^{m_{ni}}\cdots\sum_{s_{1i}=1}^{m_{1i}}\prod_{j=1}^n (m_{ji}-s_{ji}+1)\left|\nabla_n\cdots\nabla_1 a_i(s_{1i},\dots,s_{ni})\right|^{p_i}\right)^{1/p_i},\end{aligned} \tag{2.1}$$

where

$$M = M(m_{1i},\dots,m_{ni}) = \left(n - \sum_{i=1}^n 1/p_i\right)^{\sum_{i=1}^n 1/p_i - n}\prod_{i=1}^n (m_{1i}\cdots m_{ni})^{1/q_i}.$$

Remark 2.1 Let $a_i(s_{1i},\dots,s_{ni})$ change to $a_i(s_i)$ in Theorem 2.1 and in view of $a_i(0) = 0$ and $\nabla a_i(s_i) = a_i(s_i) - a_i(s_i - 1)$ for any function $a_i(s_i)$, $i = 1,2,\dots,n$, then

$$\sum_{s_1=1}^{m_1}\sum_{s_2=1}^{m_2}\cdots\sum_{s_n=1}^{m_n}\frac{\prod_{i=1}^n |a_i(s_i)|}{(\sum_{i=1}^n s_i/q_i)^{\sum_{i=1}^n 1/q_i}} \le \bar{M}\prod_{i=1}^n\left(\sum_{s_i=1}^{m_i}(m_i - s_i + 1)\left|\nabla a_i(s_i)\right|^{p_i}\right)^{1/p_i}, \tag{2.2}$$

where

$$\bar{M} = \bar{M}(m_1,\dots,m_n) = \left(n - \sum_{i=1}^n \frac{1}{p_i}\right)^{\sum_{i=1}^n 1/p_i - n}\cdot\prod_{i=1}^n m_i^{1/q_i}.$$

Remark 2.2 Taking for $n = 2$ in Remark 2.1. If $p_1, p_2 > 1$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$ and $0 < \lambda = 2 - \frac{1}{p_1} - \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} \leq 1$, then inequality (2.2) reduces to

$$\begin{aligned} &\sum_{s_1=1}^{m_1} \sum_{s_2=1}^{m_2} \frac{|a_1(s_1)||a_2(s_2)|}{(q_2 s_1 + q_1 s_2)^{\lambda}} \\ &\quad \leq \frac{1}{(\lambda q_1 q_2)^{\lambda}} m_1^{1/q_1} m_2^{1/q_2} \left(\sum_{s_1=1}^{m_1} (m_1 - s_1 + 1) \left| \nabla a_1(s_1) \right|^{p_1} \right)^{1/p_1} \\ &\qquad \times \left(\sum_{s_2=1}^{m_2} (m_2 - s_2 + 1) \left| \nabla a_2(s_2) \right|^{p_2} \right)^{1/p_2}, \end{aligned} \tag{2.3}$$

which is an interesting variation of inequality (1.1).

On the other hand, if $\lambda = 1$, then $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = 1$ and so $p_1 = q_2$, $p_2 = q_1$. In this case inequality (2.3) reduces to

$$\begin{aligned} &\sum_{s_1=1}^{m_1} \sum_{s_2=1}^{m_2} \frac{|a_1(s_1)||a_2(s_2)|}{p_1 s_1 + q_1 s_2} \\ &\quad \leq \frac{1}{p_1 q_1} m_1^{(p_1-1)/p_1} m_2^{(q_1-1)/q_1} \left(\sum_{s_1=1}^{m_1} (m_1 - s_1 + 1) \left| \nabla a_1(s_1) \right|^{p_1} \right)^{1/p_1} \\ &\qquad \times \left(\sum_{s_2=1}^{m_2} (m_2 - s_2 + 1) \left| \nabla a_2(s_2) \right|^{q_1} \right)^{1/q_1}. \end{aligned}$$

This is just a similar version of inequality (1.2) in Theorem A′.

Theorem 2.2 *Let $p_i > 1$ be constants and $\frac{1}{p_i} + \frac{1}{q_i} = 1$. Let $f_i(\tau_{1i}, \ldots, \tau_{ni})$ be real-valued nth differentiable functions defined on $[0, x_{1i}) \times \cdots \times [0, x_{ni})$, where $0 \leq x_{ji} \leq t_{ji}$, $t_{ji} \in (0, \infty)$ and $i, j = 1, 2, \ldots, n$. Suppose*

$$f_i(x_{1i}, \ldots, x_{ni}) = \int_0^{x_{1i}} \cdots \int_0^{x_{ni}} \frac{\partial^n}{\partial \tau_{1i} \cdots \partial \tau_{ni}} f_i(\tau_{1i}, \ldots, \tau_{ni}) \, d\tau_{1i} \cdots d\tau_{ni},$$

then

$$\begin{aligned} &\int_0^{t_{11}} \cdots \int_0^{t_{n1}} \int_0^{t_{12}} \cdots \int_0^{t_{n2}} \cdots \int_0^{t_{1n}} \cdots \int_0^{t_{nn}} \\ &\quad \frac{\prod_{i=1}^{n} (\int_0^{x_{1i}} \cdots \int_0^{x_{ni}} |\frac{\partial^n}{\partial \tau_{1i} \cdots \partial \tau_{ni}} f_i(\tau_{1i}, \ldots, \tau_{ni})|^{p_i} \, d\tau_{1i} \cdots d\tau_{ni})^{1/p_i}}{(\sum_{i=1}^{n} (x_{1i} \cdots x_{ni})/q_i)^{\sum_{i=1}^{n} 1/q_i}} \\ &\quad dx_{11} \cdots dx_{n1} \, dx_{12} \cdots dx_{n2} \cdots dx_{1n} \cdots dx_{nn} \\ &\leq N \prod_{i=1}^{n} \left(\int_0^{t_{1i}} \cdots \int_0^{t_{ni}} \prod_{j=1}^{n} (t_{ji} - x_{ji}) \right. \\ &\quad \left. \times \left| \frac{\partial^n}{\partial x_{1i} \cdots \partial x_{ni}} f_i(x_{1i}, \ldots, x_{ni}) \right|^{p_i} dx_{1i} \cdots dx_{ni} \right)^{1/p_i}, \end{aligned} \tag{2.4}$$

where

$$N = N(t_{1i}, \dots, t_{ni}) = \left(n - \sum_{i=1}^{n} \frac{1}{p_i}\right)^{\sum_{i=1}^{n} 1/p_i - n} \cdot \prod_{i=1}^{n} (t_{1i} \cdots t_{ni})^{1/q_i}.$$

Remark 2.3 Let $f_i(x_{1i}, \dots, x_{ni})$ change to $f_i(s_i)$ in Theorem 2.2 and in view of $f_i(0) = 0$, $i = 1, 2, \dots, n$, then

$$\begin{aligned} &\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} |f(s_i)|}{(\sum_{i=1}^{n} s_i/q_i)^{\sum_{i=1}^{n} 1/q_i}} \, ds_n \cdots ds_1 \\ &\quad \le \bar{N} \prod_{i=1}^{n} \left(\int_0^{x_i} (x_i - s_i) \left| f_i'(s_i) \right|^{p_i} ds_i \right)^{1/p_i}, \end{aligned} \tag{2.5}$$

where

$$\bar{N} = \bar{N}(x_1, \dots, x_n) = \left(n - \sum_{i=1}^{n} \frac{1}{p_i}\right)^{\sum_{i=1}^{n} 1/p_i - n} \cdot \prod_{i=1}^{n} x_i^{1/q_i}.$$

Remark 2.4 Taking for $n = 2$ in Remark 2.3, if $p_1, p_2 > 1$ are such that $\frac{1}{p_1} + \frac{1}{p_2} \ge 1$ and $0 < \lambda = 2 - \frac{1}{p_1} - \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} \le 1$, inequality (2.5) reduces to

$$\begin{aligned} &\int_0^{x_1} \int_0^{x_2} \frac{|f_1(s_1)||f_2(s_2)|}{(q_2 s_1 + q_1 s_2)^{\lambda}} \, ds_2 \, ds_1 \\ &\quad \le \frac{1}{(\lambda q_1 q_2)^{\lambda}} x_1^{1/q_1} x_2^{1/q_2} \left(\int_0^{x_1} (x_1 - s_1) \left| f_1'(s_1) \right|^{p_1} ds_1 \right)^{1/p_1} \\ &\qquad \times \left(\int_0^{x_2} (x_2 - s_2) \left| f_2'(s_2) \right|^{p_2} ds_2 \right)^{1/p_2}, \end{aligned} \tag{2.6}$$

which is an interesting variation of inequality (1.3).

On the other hand, if $\lambda = 1$, then $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = 1$ and so $p_1 = q_2$, $p_2 = q_1$. In this case inequality (2.6) reduces to

$$\begin{aligned} &\int_0^{x_1} \int_0^{x_2} \frac{|f_1(s_1)||f_2(s_2)|}{p_1 s_1 + q_1 s_2} \\ &\quad \le \frac{h_1 h_2}{p_1 q_1} x_1^{(p_1-1)/p_1} x_2^{(q_1-1)/q_1} \left(\int_0^{x_1} (x_1 - s_1) \left| f_1'(s_1) \right|^{p_1} ds_1 \right)^{1/p_1} \\ &\qquad \times \left(\int_0^{x_2} (x_2 - s_2) \left| f_2'(s_2) \right|^{q_1} ds_2 \right)^{1/q_1}. \end{aligned}$$

This is just a similar version of inequality (1.4) in Theorem B′.

3 Proofs of results

Proof of Theorem 2.1 From the hypotheses $a_i(0,\dots,0) = a_i(0,s_{2i},\dots,s_{ni}) = a_i(s_{1i},0,s_{3i},\dots,s_{ni}) = \cdots = a_i(s_{1i},\dots,s_{n-1,i},0) = 0$, we have

$$\bigl|a_i(s_{1i},\dots,s_{ni})\bigr| \le \sum_{\tau_{ni}=1}^{s_{ni}} \cdots \sum_{\tau_{1i}=1}^{s_{1i}} \bigl|\nabla_n \cdots \nabla_1 a_i(\tau_{1i},\dots,\tau_{ni})\bigr|. \tag{3.1}$$

From the hypotheses of Theorem 2.1 and in view of Hölder's inequality (see [10]) and inequality for mean [10], we obtain

$$\begin{aligned}
\prod_{i=1}^{n}\bigl|a_i(s_{1i},\dots,s_{ni})\bigr| &\le \prod_{i=1}^{n}\sum_{\tau_{ni}=1}^{s_{ni}} \cdots \sum_{\tau_{1i}=1}^{s_{1i}} \bigl|\nabla_n \cdots \nabla_1 a_i(\tau_{1i},\dots,\tau_{ni})\bigr| \\
&\le \prod_{i=1}^{n}(s_{1i}\cdots s_{ni})^{1/q_i}\left(\sum_{\tau_{ni}=1}^{s_{ni}} \cdots \sum_{\tau_{1i}=1}^{s_{1i}} \bigl|\nabla_n \cdots \nabla_1 a_i(\tau_{1i},\dots,\tau_{ni})\bigr|^{p_i}\right)^{1/p_i} \\
&\le \frac{(\sum_{i=1}^{n}(s_{1i}\cdots s_{ni})/q_i)^{\sum_{i=1}^{n}1/q_i}}{(n-\sum_{i=1}^{n}1/p_i)^{n-\sum_{i=1}^{n}1/p_i}} \\
&\quad \times \prod_{i=1}^{n}\left(\sum_{\tau_{ni}=1}^{s_{ni}} \cdots \sum_{\tau_{1i}=1}^{s_{1i}} \bigl|\nabla_n \cdots \nabla_1 a_i(\tau_{1i},\dots,\tau_{ni})\bigr|^{p_i}\right)^{1/p_i}.
\end{aligned} \tag{3.2}$$

Dividing both sides of (3.2) by $(\sum_{i=1}^{n}(s_{1i}\cdots s_{ni})/q_i)^{\sum_{i=1}^{n}1/q_i}$ and then taking sums over s_{ji} from 1 to m_{ji} $(i,j = 1,2,\dots,n)$, respectively and then using again Hölder's inequality, we obtain

$$\begin{aligned}
&\sum_{s_{11}=1}^{m_{11}} \cdots \sum_{s_{n1}=1}^{m_{n1}} \sum_{s_{12}=1}^{m_{12}} \cdots \sum_{s_{n2}=1}^{m_{n2}} \cdots \sum_{s_{1n}=1}^{m_{1n}} \cdots \sum_{s_{nn}=1}^{m_{nn}} \frac{\prod_{i=1}^{n}|\nabla_n \cdots \nabla_1 a_i(s_{1i},\dots,s_{ni})|}{(\sum_{i=1}^{n}(s_{1i}\cdots s_{ni})/q_i)^{\sum_{i=1}^{n}1/q_i}} \\
&\quad\le \left(n-\sum_{i=1}^{n}1/p_i\right)^{\sum_{i=1}^{n}1/p_i-n} \\
&\qquad \times \prod_{i=1}^{n}\left(\sum_{s_{ni}=1}^{m_{ni}} \cdots \sum_{s_{1i}=1}^{m_{1i}}\left(\sum_{\tau_{ni}=1}^{s_{ni}} \cdots \sum_{\tau_{1i}=1}^{s_{1i}} \bigl|\nabla_n \cdots \nabla_1 a_i(\tau_{1i},\dots,\tau_{ni})\bigr|^{p_i}\right)^{1/p_i}\right) \\
&\quad\le \left(n-\sum_{i=1}^{n}1/p_i\right)^{\sum_{i=1}^{n}1/p_i-n} \\
&\qquad \times \prod_{i=1}^{n}(m_{1i}\cdots m_{ni})^{1/q_i}\left(\sum_{s_{ni}=1}^{m_{ni}} \cdots \sum_{s_{1i}=1}^{m_{1i}}\left(\sum_{\tau_{ni}=1}^{s_{ni}} \cdots \sum_{\tau_{1i}=1}^{s_{1i}} \bigl|\nabla_n \cdots \nabla_1 a_i(\tau_{1i},\dots,\tau_{ni})\bigr|^{p_i}\right)\right)^{1/p_i} \\
&\quad= M\prod_{i=1}^{n}\left(\sum_{\tau_{ni}=1}^{m_{ni}} \cdots \sum_{\tau_{1i}=1}^{m_{1i}}\prod_{j=1}^{n}(m_{ji}-\tau_{ji}+1)\bigl|\nabla_n \cdots \nabla_1 a_i(\tau_{1i},\dots,\tau_{ni})\bigr|^{p_i}\right)^{1/p_i} \\
&\quad= M\prod_{i=1}^{n}\left(\sum_{s_{ni}=1}^{m_{ni}} \cdots \sum_{s_{1i}=1}^{m_{1i}}\prod_{j=1}^{n}(m_{ji}-s_{ji}+1)\bigl|\nabla_n \cdots \nabla_1 a_i(s_{1i},\dots,s_{ni})\bigr|^{p_i}\right)^{1/p_i}.
\end{aligned}$$

This concludes the proof. □

Proof of Theorem 2.2 From the hypotheses of Theorem 2.2, we have

$$\left|f_i(x_{1i},\ldots,x_{ni})\right| \le \int_0^{x_{1i}} \cdots \int_0^{x_{ni}} \left|\frac{\partial^n}{\partial\tau_{1i}\cdots\partial\tau_{ni}} f_i(\tau_{1i},\ldots,\tau_{ni})\right| d\tau_{1i}\cdots d\tau_{ni}. \tag{3.3}$$

On the other hand, by using Hölder's integral inequality (see [10]) and the following inequality for mean [10],

$$\left(\prod_{i=1}^n \lambda_i^{1/q_i}\right)^{1/\sum_{i=1}^n 1/q_i} \le \frac{1}{\sum_{i=1}^n 1/q_i}\sum_{i=1}^n \lambda_i/q_i, \quad \lambda_i > 0,$$

we obtain

$$\begin{aligned}
&\prod_{i=1}^n \left|f_i(x_{1i},\ldots,x_{ni})\right| \\
&\quad\le \prod_{i=1}^n \int_0^{x_{1i}} \cdots \int_0^{x_{ni}} \left|\frac{\partial^n}{\partial\tau_{1i}\cdots\partial\tau_{ni}} f_i(\tau_{1i},\ldots,\tau_{ni})\right| d\tau_{1i}\cdots d\tau_{ni} \\
&\quad\le \prod_{i=1}^n (x_{1i}\cdots x_{ni})^{1/q_i} \\
&\qquad\times \left(\int_0^{x_{1i}} \cdots \int_0^{x_{ni}} \left|\frac{\partial^n}{\partial\tau_{1i}\cdots\partial\tau_{ni}} f_i(\tau_{1i},\ldots,\tau_{ni})\right|^{p_i} d\tau_{1i}\cdots d\tau_{ni}\right)^{1/p_i} \\
&\quad\le \frac{(\sum_{i=1}^n (x_{1i}\cdots x_{ni})/q_i)^{\sum_{i=1}^n 1/q_i}}{(n-\sum_{i=1}^n 1/p_i)^{n-\sum_{i=1}^n 1/p_i}} \\
&\qquad\times \prod_{i=1}^n \left(\int_0^{x_{1i}} \cdots \int_0^{x_{ni}} \left|\frac{\partial^n}{\partial\tau_{1i}\cdots\partial\tau_{ni}} f_i(\tau_{1i},\ldots,\tau_{ni})\right|^{p_i} d\tau_{1i}\cdots d\tau_{ni}\right)^{1/p_i}.
\end{aligned} \tag{3.4}$$

Dividing both sides of (3.4) by $(\sum_{i=1}^n (x_{1i}\cdots x_{ni})/q_i)^{\sum_{i=1}^n 1/q_i}$ and then integrating the result inequality over x_{ji} from 1 to t_{ji} $(i,j = 1,2,\ldots,n)$, respectively and then using again Hölder's integral inequality, we obtain

$$\begin{aligned}
&\int_0^{t_{11}} \cdots \int_0^{t_{n1}} \int_0^{t_{12}} \cdots \int_0^{t_{n2}} \cdots \int_0^{t_{1n}} \cdots \int_0^{t_{nn}} \\
&\qquad \frac{\prod_{i=1}^n (\int_0^{x_{1i}} \cdots \int_0^{x_{ni}} \left|\frac{\partial^n}{\partial\tau_{1i}\cdots\partial\tau_{ni}} f_i(\tau_{1i},\ldots,\tau_{ni})\right|^{p_i} d\tau_{1i}\cdots d\tau_{ni})^{1/p_i}}{(\sum_{i=1}^n (x_{1i}\cdots x_{ni})/q_i)^{\sum_{i=1}^n 1/q_i}} \\
&\qquad dx_{11}\cdots dx_{n1}\, dx_{12}\cdots dx_{n2}\cdots dx_{1n}\cdots dx_{nn} \\
&\quad\le \left(n-\sum_{i=1}^n 1/p_i\right)^{\sum_{i=1}^n 1/p_i - n} \\
&\qquad\times \prod_{i=1}^n \int_0^{t_{1i}} \cdots \int_0^{t_{ni}} \left(\int_0^{x_{1i}} \cdots \int_0^{x_{ni}}\right. \\
&\qquad \left.\left|\frac{\partial^n}{\partial\tau_{1i}\cdots\partial\tau_{ni}} f_i(\tau_{1i},\ldots,\tau_{ni})\right|^{p_i} d\tau_{1i}\cdots d\tau_{ni}\right)^{1/p_i} dx_{1i}\cdots dx_{ni}
\end{aligned}$$

$$\begin{aligned}
&\le \left(n - \sum_{i=1}^{n} 1/p_i\right)^{\sum_{i=1}^{n} 1/p_i - n} \prod_{i=1}^{n} (t_{1i}\cdots t_{ni})^{1/q_i} \\
&\quad \times \left(\int_0^{t_{1i}} \cdots \int_0^{t_{ni}} \left(\int_0^{x_{1i}} \cdots \int_0^{x_{ni}} \right.\right. \\
&\quad \left.\left. \left|\frac{\partial^n}{\partial\tau_{1i}\cdots\partial\tau_{ni}} f_i(\tau_{1i},\ldots,\tau_{ni})\right|^{p_i} d\tau_{1i}\cdots d\tau_{ni}\right) dx_{1i}\cdots dx_{ni}\right)^{1/p_i} \\
&= N \prod_{i=1}^{n} \left(\int_0^{t_{1i}} \cdots \int_0^{t_{ni}} \prod_{j=1}^{n} (t_{ji} - x_{ji}) \left|\frac{\partial^n}{\partial x_{1i}\cdots\partial x_{ni}} f_i(x_{1i},\ldots,x_{ni})\right|^{p_i} dx_{1i}\cdots dx_{ni}\right)^{1/p_i}.
\end{aligned}$$

This concludes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
C-JZ and W-SC jointly contributed to the main results Theorems 2.1 and 2.2. All authors read and approved the final manuscript.

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