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Full Length Research Paper

Weakened Mannheim curves

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In the discussion of Bertrand and Mannheim curves, it is always assumed (explicitly or implicitly) that the curvature is nowhere zero. In this paper, we adapt this requirement on the Mannheim curves and investigate into the properties of two types of similar curves (the Frenet-Mannheim curves and the weakened Mannheim curves) under weakened conditions.

Key words: Mannheim curves, Frenet-Mannheim curves, weakened-Mannheim curves.

INTRODUCTION

In the study of the fundamental theory and the characterizations of the space curves, the corresponding relations between the curves are very interesting and important problem. The well-known Bertrand curve is characterized as a kind of such corresponding relation between the two curves. For the Bertrand curve α, it shares the normal lines with another curve β, called Bertrand mate or Bertrand partner curve of α (Liu and Wang , 2008). The properties of two types of similar curves (the Frenet-Bertrand curves and the Weakened Bertrand curves) were investigated under weakened conditions by Lai (1967). In recent works, the Mannheim curves in both Euclidean and Minkowski 3-space was studied and they obtained the necessary and sufficient conditions between the curvature and the torsion for a curve to be the Mannheim partner curves by Liu and Wang (2008). Meanwhile, the detailed discussion concerned with the Mannheim curves can be found in literatures (Liu and Wang, 2008; Orbay and Kasap, 2009). In this paper, our main purpose is to carry out some results which were given in Lai (1967) to the Frenet-Mannheim curves and the Weakened Mannheim curves and we assume that the angle between tangent vectors T_β and T_α is a constant such that $\langle T_\alpha, T_\beta \rangle = \cos\theta \neq 0$.

PRELIMINARIES

In this paper, we weaken the conditions. We shall adopt the definition of a C^∞ Frenet curve as a C^∞ regular curve $\alpha(s), s \in I$, for which there exists a C^∞ family of the Frenet frames, that is, right-handed orthonormal frames $\{T_\alpha(s), N_\alpha(s), B_\alpha(s)\}$ where $T_\alpha(s) = \alpha'(s)$, satisfies the Frenet equations

$$\begin{bmatrix} T'_\alpha(s) \\ N'_\alpha(s) \\ B'_\alpha(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\alpha & 0 \\ -\kappa_\alpha & 0 & \tau_\alpha \\ 0 & -\tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} T_\alpha(s) \\ N_\alpha(s) \\ B_\alpha(s) \end{bmatrix} \tag{1}$$

for some C^∞ scalar functions $\kappa_\alpha(s), \tau_\alpha(s)$ which are called the curvature and torsion of $\alpha(s)$.

Definition 1

Let E^3 be 3-dimensional Euclidean space with the standard inner product $\langle , \rangle$. If there exists a corresponding relationship between the space curves α and β such that, at the corresponding points of the curves, the principal normal lines of β coincides with the binormal lines of α, then β is called a Mannheim curve, and α a

Mannheim partner curve of β. The pair $\{\alpha, \beta\}$ is said to be a Mannheim pair (Liu and Wang, 2008).

Definition 2

A Mannheim curve $\beta(s^*), s^* \in I$ is a C^∞ regular curve with non-zero curvature for which there exists another (different) C^∞ regular curve $\alpha(s)$ where $\alpha(s)$ is of class C^∞ and $\alpha'(s) \neq 0$ (s being the arc length of $\alpha(s)$ only), also with non-zero curvature, in bijection with it in such a manner that the principal normal to $\beta(s^*)$ and the binormal to $\alpha(s)$ at each pair of corresponding points coincide with the line joining the corresponding points. The curve $\alpha(s)$ is called a Mannheim conjugate of $\beta(s^*)$.

Definition 3

A Frenet-Mannheim curve $\beta(s^*)$ (briefly called a FM curve) is a C^∞ Frenet curve for which there exists another C^∞ Frenet curve $\alpha(s)$, where $\alpha(s)$ is of class C^∞ and $\alpha'(s) \neq 0$, in bijection with it so that, by suitable choice of the Frenet frames, the principal normal vector $N_\beta(s^*)$ and binormal vector $B_\alpha(s)$ at corresponding points on $\beta(s^*), \alpha(s)$, both lie on the line joining the corresponding points. The curve $\alpha(s)$ is called a FM conjugate of $\beta(s^*)$.

Definition 4

A weakened Mannheim curve $\beta(s^*), s^* \in I^*$ (briefly called a WM curve) is a C^∞ regular curve for which there exists another C^∞ regular curve $\alpha(s), s \in I$, where s is the arc-length of $\alpha(s)$, and a homeomorphism $\sigma : I \to I^*$ such that (i) here exists two (disjoint) closed subsets Z, N of I with void interiors such that $\sigma \in C^\infty$ on $L \setminus \mathrm{N}$, $\left(\frac{ds^*}{ds}\right) = 0$ on Z, $\sigma^{-1} \in C^\infty$ on $\sigma(L \setminus \mathrm{Z})$ and $\left(\frac{ds}{ds^*}\right) = 0$ on $\sigma(N)$. (ii) The line joining corresponding points s, s^* of $\alpha(s)$ and $\beta(s^*)$ is orthogonal to $\alpha(s)$ and $\beta(s^*)$ at the points s, s^* respectively, and is along the principal normal to $\beta(s^*)$ or $\alpha(s)$ at the points s, s^* whenever it is well defined. The curve $\alpha(s)$ is called a WM conjugate of $\beta(s^*)$. Thus, for a WM curve, not only we drop the requirement of $\alpha(s)$ being a Frenet curve, but also allow $\left(\frac{ds^*}{ds}\right)$ to be zero on a subset with void interior $\left(\frac{ds^*}{ds}\right) = 0$ on an interval would destroy the injectivity of the mapping σ. Since $\left(\frac{ds^*}{ds}\right) = 0$ implies that $\left(\frac{ds}{ds^*}\right)$ does not exist, the apparently artificial requirements in (i) are in fact quite natural. It is clear that a Mannheim curve is necessarily a FM curve, and a FM curve is necessarily a WM curve. It will be proved in Theorem 3 that under certain conditions a WM curve is also a FM curve.

FRENET-MANNHEIM CURVE

Here, we study the structure and characterization of FM curves. We begin with a lemma, the method used in which is classical.

Lemma 1

Let $\beta(s^*), s^* \in I^*$ be a FM curve and $\alpha(s)$ a FM conjugate of $\beta(s^*)$. Let

$$\beta(s^*) = \alpha(s) + \lambda(s) B_\alpha(s) \tag{2}$$

Then the distance $|\lambda|$ between corresponding points of $\alpha(s), \beta(s^*)$ is a constant, and there is a constant angle θ such that $\langle T_\alpha, T_\beta \rangle = \cos\theta$ and

$(i) \sin\theta = -2\lambda\tau_\alpha \cos\theta$

$(ii) (1 + \lambda\varepsilon\kappa_\beta)\sin\theta = \lambda\tau_\beta \cos\theta$

$(iii)\cos^2\theta = (1+\lambda\varepsilon\kappa_\beta)$

$(iv)\sin^2\theta = \lambda^2\tau_\alpha\tau_\beta.$

Proof

From (1), it follows that $\lambda(s) = \langle \beta(s^*) - \alpha(s), B_\alpha(s)\rangle$ is of class C^∞. Differentiation of (2) with respect to s gives

$$T_\beta \frac{ds^*}{ds} = T_\alpha + \lambda' B_\alpha - \lambda\tau_\alpha N_\alpha. \qquad (3)$$

Since by hypothesis, we have $B_\alpha = \varepsilon N_\beta$ with $\varepsilon = \pm 1$, the scalar multiplication of (3) by B_α gives $\lambda' = 0 \Rightarrow \lambda = \text{constant}.$ Therefore

$$T_\beta \frac{ds^*}{ds} = T_\alpha - \lambda\tau_\alpha N_\alpha. \qquad (4)$$

But by definition of FM curve, we have $\frac{ds^*}{ds} \neq 0$, so that T_β is C^∞ function of s. Hence $\langle T_\alpha, T_\beta\rangle' = \kappa_\alpha\langle N_\alpha, T_\beta\rangle + \frac{ds^*}{ds}\kappa_\beta\langle T_\alpha, N_\beta\rangle = 0.$ Consequently, $\langle T_\alpha, T_\beta\rangle$ is a constant, and there exists a constant angle θ such that

$$T_\beta = T_\alpha\cos\theta + N_\alpha\sin\theta. \qquad (5)$$

Taking the vector product of (4) and (5), we obtain $\sin\theta = -2\lambda\tau_\alpha\cos\theta$ which is (i). Now write $\alpha(s) = \beta(s^*) - \varepsilon\lambda(s)N_\beta(s).$ Therefore

$$T_\alpha = \frac{ds^*}{ds}\left[(1+\lambda\varepsilon\kappa_\beta)T_\beta - \lambda\varepsilon\tau_\beta B_\beta\right]. \qquad (6)$$

On the other hand, Equation (3 and 4) gives

$$B_\beta = T_\beta \wedge N_\beta = -\varepsilon N_\alpha\cos\theta + \varepsilon T_\alpha\sin\theta.$$

Using (5) again, we get

$$T_\alpha = T_\beta\cos\theta - \varepsilon B_\beta\sin\theta. \qquad (7)$$

Taking the vector product of (5) to (7), we obtain $(1+\lambda\varepsilon\kappa_\beta)\sin\theta = \lambda\tau_\beta\cos\theta,$ which is (ii). On the other hand, the comparison of (4) and (5) gives

$$\frac{ds^*}{ds}\cos\theta = 1, \qquad (8)$$

$$\frac{ds^*}{ds}\sin\theta = \lambda\tau_\alpha. \qquad (9)$$

Similarly (6), (7) give

$$\frac{ds^*}{ds}(1+\lambda\varepsilon\kappa_\beta) = \cos\theta, \qquad (10)$$

$$\frac{ds^*}{ds}(\lambda\tau_\beta) = \sin\theta. \qquad (11)$$

The properties (iii) and (iv) then easily follow from (7) to (11).

Theorem 1

Let $\beta(s^*), s^* \in I^*$ be a C^∞ Frenet curve with τ_β nowhere zero and satisfying the equation for constants λ with $\lambda \neq 0.$ Then $\beta(s^*)$ is a non-planar FM curve

$$(1+\lambda\varepsilon\kappa_\beta)\sin\theta = \lambda\tau_\beta\cos\theta. \qquad (12)$$

Proof

Define the curve $\beta(s^*)$ with the position vector $\beta(s^*) = \alpha(s) + \lambda(s)B_\alpha(s).$ Then, denoting differentiation with respect to s by a dash, we have $\beta'(s^*) = T_\alpha - \lambda\tau_\alpha N_\alpha.$ Since $\tau_\alpha \neq 0$, it follows that $\beta(s^*)$ is a C^∞ regular curve. Then $T_\beta\frac{ds^*}{ds} = T_\alpha - \lambda\tau_\alpha N_\alpha.$ Hence $\frac{ds^*}{ds} = \sqrt{1-\lambda^2\tau_\alpha^2}.$ and, using (12), $T_\beta = T_\alpha\cos\theta + N_\alpha\sin\theta,$ notices that from (12), we have $\sin\theta \neq 0.$ Therefore $\frac{T_\beta}{ds^*}\frac{ds^*}{ds} = -\kappa_\alpha T_\alpha\sin\theta + \kappa_\alpha N_\alpha\cos\theta + \tau_\alpha B_\alpha\sin\theta.$ Now define $N_\beta = \varepsilon B_\alpha,$ $\kappa_\beta = \frac{\varepsilon}{\frac{ds^*}{ds}}\tau_\alpha\sin\theta.$ These are C^∞ functions of s (and hence of s^*), and $\frac{T_\beta}{ds^*} = \kappa_\beta N_\beta.$ Further define $B_\beta = T_\beta \wedge B_\alpha$ and $\tau_\beta = -\left\langle \frac{B_\beta}{ds^*}, N_\beta \right\rangle.$ These are also C^∞ functions on I^*. It is then easy to verify that with the frame $\{T_\beta, N_\beta, B_\beta\}$ and the functions κ_β, τ_β, the curve $\beta(s^*)$ becomes a C^∞ Frenet curve. But B_α and N_β lie on the line joining corresponding points of $\alpha(s)$ and $\beta(s^*)$. Thus, $\beta(s^*)$ is a FM curve and $\alpha(s)$ is a FM conjugate of $\beta(s^*)$.

Lemma 2

A necessary and sufficient condition for a C^∞ regular curve β to be a FM curve with a FM conjugate. Then β should be either a line or a non-planar circular helix.

Proof

$\Rightarrow$: Let β has a FM conjugate α which is a line. Then $\kappa_\alpha = 0.$ Using Lemma 1, (iii) and (i), (ii), we have

$$\cos^2\theta = (1 + \lambda\varepsilon\kappa_\beta), \tag{13}$$

and then

$$\cos^2\theta \sin\theta = \lambda\tau_\beta \cos\theta, \tag{14}$$

$$\sin\theta = -2\lambda\tau_\alpha \cos\theta. \tag{15}$$

From (15), it follows that $\cos\theta \neq 0.$ Hence (14) is equivalent to

$$\lambda\tau_\beta = \cos\theta \sin\theta. \tag{16}$$

Case 1

$\sin\theta = 0.$ Then $\cos\theta = \pm 1$, so that (13) implies that $\kappa_\beta = 0$ and β is a line. We note also that (16) implies that $\tau_\beta = 0.$

Case 2

$\sin\theta \neq 0.$ Then $\cos\theta \neq \pm 1$ and (13), (16) imply that κ_β, τ_β are non-zero constants, and β is a non-planar circular helix.

$\Leftarrow$: If β is a non-planar circular helix $\beta = (a\cos t, a\sin t, bt)$ where $t = \frac{s}{\sqrt{a^2 + b^2}}$, we may take $N_\beta = (-\cos t, \sin t, 0).$ Now put $\lambda = a$, then the curve β with $\beta = \alpha + \lambda B_\alpha$ will be a line along the $z-$axis, and can be made into a FM conjugate of β if N_β is defined to be equal to B_α.

Theorem 2

Let $\beta(s^*)$ be a plane C^∞ Frenet curve with zero torsion and whose curvature is either bounded below or bounded above. Then β is a FM curve and has FM conjugates which are the plane curves.

Proof

Let β be a curve satisfying the conditions of the hypothesis. Then there are non-zero numbers λ such that $\kappa_\beta < -\left(\frac{1}{\lambda}\right)$ on I or $\kappa_\beta > -\left(\frac{1}{\lambda}\right)$ on I. For any such λ, consider the plane curve α with position vector $\alpha = \beta - \lambda N_\beta$. Then $T_\alpha = (1 + \lambda\kappa_\beta)T_\beta$. Since $(1 + \lambda\kappa_\beta) \neq 0$, α is a C^∞ regular curve, and $T_\alpha = T_\beta$. Then it is a straightforward matter to verify that α is a FM conjugate of β.

WEAKENED MANNHEIM CURVES

Definition 5

Let D be a subset of a topological space X. A function on X into a set Y is said to be $D-$piecewise constant if it is a constant on each component of D.

Lemma 3

Let X be a proper interval on the real line and D an open subset of X. Then a necessary and sufficient condition for every continuous, $D-$piecewise constant real function on X to be constant is that $X \setminus \mathrm{D}$ should have empty dense-in-itself kernel.

We notice, however, that if D is dense in X, any C^1 and $D-$piecewise constant real function on X must be constant, even if D has non-empty dense-in-itself kernel.

Theorem 3

A WM curve for which N and Z have empty dense-in-itself kernels is a FM curve.

Proof

Let $\beta(s^*), s^* \in I^*$ be a WM curve and $\alpha(s), s \in I$ a WM conjugate of $\beta(s^*)$. It follows from the definition that $\alpha(s)$ and $\beta(s^*)$ are curves, each of which has a C^∞ family of tangent vectors $T_\alpha(s), T_\beta(s^*)$. Let

$$\beta(s) = \beta(\sigma(s)) = \alpha(s) + \lambda(s) B_\alpha(s), \tag{17}$$

where $B_\alpha(s)$ is some unit vector function and $\lambda(s) \geq 0$ is some scalar function. Let $D = I \setminus \mathrm{N}$, $D^* = I^* \setminus \sigma(\mathrm{Z})$. Then $s^*(s) \in C^\infty$ on D^*.

Step 1

To prove $\lambda = \text{constant}$. Since $\lambda = \|\beta(s) - \alpha(s)\|$, it is continuous on I and is of class C^∞ on every interval of D on which it is nowhere zero. Let $P = \{s \in I : \lambda(s) \neq 0\}$ and X any component of P. Then P, and hence also X, is open in I. Let L be any component interval of $X \cap D$. Then on L, $\lambda(s)$ and $B_\alpha(s)$ are of class C^∞, and from (17) we have $\beta'(s) = \alpha'(s) + \lambda'(s) B_\alpha(s) + \lambda(s) B'_\alpha(s)$. Now by definition of a WM curve we have $\langle \alpha'(s), B_\alpha(s) \rangle = 0 = \langle \beta'(s^*), B_\alpha(s) \rangle$.

Hence, using the identity $\langle B'_\alpha(s), B_\alpha(s) \rangle = 0$, we have $\lambda'(s) \langle B_\alpha(s), B_\alpha(s) \rangle = 0$. Therefore $\lambda = \text{constant}$ on L. hence λ is a constant on each interval of the set $X \cap D$. But by hypothesis $X \setminus \mathrm{D}$ has empty dense-in-itself kernel. It follows from Lemma 3.2 that λ a is constant (and non-zero) on X. Since λ is continuous on I, X must be closed in I. But X is also open in I. Therefore by connectedness we must have $X = I$, that is, λ is a constant on I.

Step 2

To prove the existence of two frames $\{T_\alpha(s), N_\alpha(s), B_\alpha(s)\}, \{T_\beta(s^*), N_\beta(s^*), B_\beta(s^*)\}$ which are Frenet frames for $\alpha(s), \beta(s^*)$ on D, D^*, respectively. Since λ is a non-zero constant, it follows from (17) that $B_\alpha(s)$ is continuous on I and C^∞ on D and is always orthogonal to $T_\alpha(s)$. Now defining $B_\alpha(s) = T_\alpha(s) \wedge N_\alpha(s)$. Then $\{T_\alpha(s), N_\alpha(s), B_\alpha(s)\}$ forms a right-handed orthonormal frame for $\alpha(s)$ which is continuous on I and C^∞ on D.

Now from the definition of WM curve, we see that there exists a scalar function $\kappa_\beta(s^*)$ such that $T'_\beta(s^*) = \kappa_\beta(s^*) N_\beta(s^*)$ on I^*. Hence $\kappa_\beta(s^*) = \langle T'_\beta(s^*), N_\beta(s^*) \rangle$ is continuous on I^* and C^∞ on D^*. Thus the first Frenet formula holds on D^*. Then it is straightforward to show that there exists a C^∞ function $\tau_\alpha(s)$ on D such that the Frenet formulas hold. Thus $\{T_\alpha(s), N_\alpha(s), B_\alpha(s)\}$ is a Frenet frame for $\alpha(s)$ on D.

Similarly, there exists a right-handed orthonormal frame $\{T_\beta(s^*), N_\beta(s^*), B_\beta(s^*)\}$ for $\beta(s^*)$ which is continuous on I^* and is a Frenet frame for $\beta(s^*)$ on D^*. Moreover, we can choose $B_\alpha(s) = N_\beta(\sigma(s))$.

Step 3

To prove that $N \neq \phi, Z \neq \phi.$ We first notice that on D we have $\langle T_\beta, T_\alpha \rangle' = \left\langle \kappa_\beta N_\beta \frac{ds^*}{ds}, T_\alpha \right\rangle + \langle T_\beta, \kappa_\alpha N_\alpha \rangle = 0,$ so that $\langle T_\beta, T_\alpha \rangle$ is a constant on each component of D and hence on I by Lemma 2. Consequently, there exists an angle θ such that $T_\beta = T_\alpha \cos\theta + N_\alpha \sin\theta.$ Further, $B_\alpha(s) = N_\beta(\sigma(s))$ and so $B_\beta(s^*) = -T_\alpha(s)\sin\theta + N_\alpha \cos\theta.$ Thus $\{T_\beta(s^*), N_\beta(s^*), B_\alpha(s)\}$ is also of class C^∞ on $D.$ On the other hand $\{T_\beta(s^*), N_\beta(s^*), B_\beta(s^*)\}$ is of class C^∞ with respect to s^* on $D^*.$ Writing (17) in the form $\alpha = \beta - \lambda N_\beta$ and differentiating with respect to s on $D \cap \sigma^{-1}(D^*)$, we have $T_\alpha = \frac{ds^*}{ds}\left[(1+\lambda\kappa_\beta)T_\beta - \lambda\tau_\beta B_\beta\right].$ But $T_\alpha = T_\beta \cos\theta - B_\beta \sin\theta.$ Hence $\frac{ds^*}{ds}(1+\lambda\kappa_\beta) = \cos\theta$ and

$$\lambda\tau_\beta = -\sin\theta. \tag{18}$$

Since $\kappa_\beta(s^*) = \langle T_\beta', N_\beta \rangle$ is defined and continuous on I^* and $\sigma^{-1}(D^*)$ is dense, it follows by continuity that (18) holds throughout $D.$

Case 1

$\cos\theta \neq 0.$ Then (18) implies that $\frac{ds^*}{ds} \neq 0$ on $D.$ Hence $Z \neq \phi.$ Similarly $N \neq \phi.$

Case 2

$\cos\theta = 0.$ Then

$$T_\beta = \pm N_\alpha. \tag{19}$$

Differentiation of (17) with respect to s in D gives $T_\beta \frac{ds^*}{ds} = T_\alpha - \lambda\tau_\alpha N_\alpha.$ Hence, using (19), we have $\frac{ds^*}{ds} = \mp\lambda\tau_\alpha.$ Therefore $\tau_\alpha = \mp\frac{1}{\lambda}\frac{ds^*}{ds},$ and so also on I, by Lemma 2. It follows that τ_α is nowhere zero on $I.$ Consequently, $\beta(s^*) = \alpha(s) + \lambda(s)B_\alpha(s)$ is of class C^∞ on $I^*.$ Hence $N \neq \phi.$ Similarly $Z \neq \phi.$

Conclusıon

In this paper, we have given some characterizations about Mannheim curves under “weakened conditions”. Further research can be done by applying similar characterizations to the non-Euclidean geometries such as Minkowski (Lorentz) space, Galilean space and pseudo-Galilean space etc.

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