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A general composite iterative method for generalized mixed equilibrium problems, variational inequality problems and optimization problems

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Abstract

In this article, we introduce a new general composite iterative scheme for finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of fixed points of an infinite family of nonexpansive mappings and the set of solutions of a variational inequality problem for an inverse-strongly monotone mapping in Hilbert spaces. It is shown that the sequence generated by the proposed iterative scheme converges strongly to a common element of the above three sets under suitable control conditions, which solves a certain optimization problem. The results of this article substantially improve, develop, and complement the previous well-known results in this area.

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1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot, \cdot\rangle$ and induced norm $||\cdot||$. Let C be a nonempty closed convex subset of H and $S : C \to C$ be a self-mapping on C. Let us denote by $F(S)$ the set of fixed points of S and by P_C the metric projection of H onto C.

Let $B : C \to H$ be a nonlinear mapping and $\phi : C \to \mathbb{R}$ be a function, and Θ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

Then, we consider the following generalized mixed equilibrium problem of finding $x \in C$ such that

$$\Theta(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C, \tag{1.1}$$

which was recently introduced by Peng and Yao [1]. The set of solutions of the problem (1.1) is denoted by $GMEP(\Theta, \phi, B)$. Here, some special cases of the problem (1.1) are stated as follows:

If $B = 0$, then the problem (1.1) reduces the following mixed equilibrium problem of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C, \tag{1.2}$$

which was studied by Ceng and Yao [2] (see also [3]). The set of solutions of the problem (1.2) is denoted by $MEP(\Theta,\phi)$.

If $\phi = 0$ and $B = 0$, then the problem (1.1) reduces the following equilibrium problem of finding $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of the problem (1.3) is denoted by $EP(\Theta)$.

If $\phi = 0$ and $\Theta(x, y) = 0$ for all $x, y \in C$, then the problem (1.1) reduces the following variational inequality problem of finding $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of the problem (1.4) is denoted by $VI(C, B)$.

The problem (1.1) is very general in the sense that it includes, as special cases, fixed point problems, optimization problems, variational inequality problems, minmax problems, Nash equilibrium problems in noncooperative games, and others; see [2,4-6].

Recently, in order to study the problem (1.3) coupled with the fixed point problem, many authors have introduced some iterative schemes for finding a common element of the set of the solutions of the problem (1.3) and the set of fixed points of a countable family of nonexpansive mappings; see [7-16] and the references therein.

In 2008, Su et al. [17] gave an iterative scheme for the problem (1.3), the problem (1.4) for an inverse-strongly monotone mapping, and fixed point problems of nonexpansive mappings. In 2009, Yao et al. [18] considered an iterative scheme for the problem (1.2), the problem (1.4) for a Lipschitz and relaxed-cocoercive mapping and fixed point problems of nonexpansive mappings, and in 2008, Peng and Yao [1] studied an iterative scheme for the problem (1.1), the problem (1.4) for a monotone, and Lipschitz continuous mapping and fixed point problems of nonexpansive mappings.

In particular, in 2010, Jung [9] introduced the following new composite iterative scheme for finding a common element of the set of solutions of the problem (1.3) and the set of fixed points of a nonexpansive mapping: $x_1 \in C$ and

$$\begin{cases} \Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n Ty_n, \quad n \geq 1, \end{cases} \tag{1.5}$$

where T is a nonexpansive mapping, f is a contraction with constant $k \in (0, 1)$, $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$, and $\{r_n\} \subset (0, \infty)$. He showed that the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.5) converge strongly to a point in $F(T) \cap EP(\Theta)$ under suitable conditions.

On the other hand, the following optimization problem has been studied extensively by many authors:

$$\min_{x\in\Omega} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x),$$

where $\Omega = \bigcap_{n=1}^{\infty} C_n,\ C_1,\ C_2,\ \cdots$ are infinitely many closed convex subsets of H such that $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$, $u \in H$, $\mu \geq 0$ is a real number, A is a strongly positive bounded linear operator on H (i.e., there is a constant $\bar{\gamma} > 0$ such that $\langle Ax,\ x\rangle \geq \bar{\gamma}\|x\|^2$, $\forall x \in H$) and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for all $x \in H$). For this kind of optimization problems, see, for example, Bauschke and Borwein [19], Combettes [20], Deutsch and Yamada [21], Jung [22], and Xu [23] when $\Omega = \bigcap_{i=1}^{N} C_i$; and $h(x) = \langle x, b\rangle$ for a given point b in H.

In 2009, Yao et al. [3] considered the following iterative scheme for the problem (1.2) and optimization problems:

$$\begin{cases} \Theta(y_n,\ y) + \varphi(y) - \varphi(y_n) + \dfrac{1}{r}\langle K'(y_n) - K'(x_n),\ y - y_n\rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n(I + \mu A))W_n y_n, \quad n \geq 1, \end{cases} \tag{1.6}$$

where $u \in H$; $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in (0,1), $\mu > 0$, $r > 0$, $\gamma > 0$; $K'(x)$ is the Fréchet derivative of a functional $K : H \to \mathbb{R}$ at x; and W_n is the so-called W-mapping related to a sequence $\{T_n\}$ of nonexpansive mappings. They showed that under appropriate conditions, the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.6) converge strongly to a solution of the optimization problem:

$$\min_{x \in \bigcap_{n=1}^{\infty} F(T_n) \cap MEP(\Theta,\varphi)} \frac{\mu}{2}\langle Ax,\ x\rangle + \frac{1}{2}||x - u||^2 - h(x).$$

In 2010, using the method of Yao et al. [3], Jaiboon and Kumam [24] also introduced a general iterative method for finding a common element of the set of solutions of the problem (1.2), the set of fixed points of a sequence $\{T_n\}$ of nonexpansive mappings, and the set of solutions of the problem (1.4) for a α-inverse-strongly monotone mapping. We point out that in the main results of [3,24], the condition of the sequentially continuity from the weak topology to the strong topology for the derivative K' of the function $K : C \to \mathbb{R}$ is very strong. Even if $K(x) = \frac{\|x\|^2}{2}$, then $K'(x) = x$ is not sequentially continuous from the weak topology to the strong topology.

In this article, inspired and motivated by above mentioned results, we introduce a new iterative method for finding a common element of the set of solutions of a generalized mixed equilibrium problem (1.1), the set of fixed points of a countable family of nonexpansive mappings, and the set of solutions of the variational inequality problem (1.4) for an inverse-strongly monotone mapping in a Hilbert space. We show that under suitable conditions, the sequence generated by the proposed iterative scheme converges strongly to a common element of the above three sets, which is a solution of a certain optimization problem. The results of this article can be viewed as an improvement and complement of the recent results in this direction.

2 Preliminaries and lemmas

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. In the following, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

First, we know that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0, 1)$ such that $||f(x) - f(y)|| \le k||x - y||$, $x, y \in C$. A mapping $T : C \to C$ is called *nonexpansive* if $||Tx - Ty|| \le ||x - y||$, $x, y \in C$.

In a real Hilbert space H, we have

$$\left\|\lambda x + (1-\lambda)y\right\|^2 = \lambda\|x\|^2 + (1-\lambda)\left\|y\right\|^2 - \lambda(1-\lambda)\left\|x-y\right\|^2$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$. For every point $x \in H$, there exists the unique nearest point in C, denoted by $P_C(x)$, such that

$$||x - P_C(x)|| \le ||x - y||$$

for all $y \in C$. P_C is called the *metric projection* of H onto C. It is well known that P_C is nonexpansive and P_C satisfies

$$\langle x - y,\ P_C(x) - P_C(y)\rangle \ge \left\|P_C(x) - P_C(y)\right\|^2 \tag{2.1}$$

for every $x, y \in H$. Moreover, $P_C(x)$ is characterized by the properties:

$$\left\|x-y\right\|^2 \ge \left\|x - P_C(x)\right\|^2 + \left\|y - P_C(x)\right\|^2$$

and

$$u = P_C(x) \Leftrightarrow \langle x - u,\ u - y\rangle \ge 0 \text{ for all } x \in H,\ y \in C\,.$$

In the context of the variational inequality problem for a nonlinear mapping F, this implies that

$$u \in VI(C,\ F) \Leftrightarrow u = P_C(u - \lambda Fu)\,, \quad \text{for any } \lambda > 0. \tag{2.2}$$

It is also well known that H satisfies the *Opial condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \left\|x_n - y\right\|$$

holds for every $y \in H$ with $y \neq x$.

A mapping F of C into H is called *α-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle x - y,\ Fx - Fy\rangle \ge \alpha\left\|Fx - Fy\right\|^2, \quad \forall x,\ y \in C.$$

We know that if $F = I - T$, where T is a nonexpansive mapping of C into itself and I is the identity mapping of H, then F is $\frac{1}{2}$-inverse-strongly monotone and $VI(C, F) = F(T)$. A mapping F of C into H is called *strongly monotone* if there exists a positive real number η such that

$$\langle x - y,\ Fx - Fy\rangle \ge \eta\left\|x - y\right\|^2, \quad \forall x,\ y \in C.$$

In such a case, we say F is η-strongly monotone. If F is η-strongly monotone and *κ-Lipschitz continuous*, that is, $||Fx - Fy|| \le \kappa||x - y||$ for all $x, y \in C$, then F is $\frac{\eta}{\kappa^2}$-inverse-strongly monotone. If F is an α-inverse-strongly monotone mapping of C into H, then it is obvious that F is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that for all x, $y \in C$ and $\lambda > 0$,

$$\begin{aligned}\|(I-\lambda F)x-(I-\lambda F)y\|^2 &= \|(x-y)-\lambda(Fx-Fy)\|^2\\ &= \|x-y\|^2-2\lambda\langle x-y,\ Fx-Fy\rangle+\lambda^2\|Fx-Fy\|^2\\ &\le \|x-y\|^2+\lambda(\lambda-2\alpha)\ \|Fx-Fy\|^2.\end{aligned}$$

Hence, if $\lambda \le 2\alpha$, then $I - \lambda F$ is a nonexpansive mapping of C into H. The following result for the existence of solutions of the variational inequality problem for inverse-strongly monotone mappings was given in Takahashi and Toyoda [25].

Proposition *Let C be a bounded closed convex subset of a real Hilbert space, and F be an α-inverse-strongly monotone mapping of C into H. Then, $VI(C, F)$ is nonempty.*

A set-valued mapping $Q : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Qx$ and $g \in Qy$ imply $\langle x - y, f - g \rangle \ge 0$. A monotone mapping $Q : H \to 2^H$ is *maximal* if the graph $G(Q)$ of Q is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping Q is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \ge 0$ for every $(y, g) \in G(Q)$ implies $f \in Qx$. Let F be an inverse-strongly monotone mapping of C into H, and let $N_C v$ be the *normal cone* to C at v, that is, $N_C v = \{w \in H : \langle v - u, w\rangle \ge 0$, for all $u \in C\}$, and define

$$Qv = \begin{cases} Fv + N_C v, & v \in C\\ \emptyset & v \notin C.\end{cases}$$

Then, Q is maximal monotone and $0 \in Qv$ if and only if $v \in VI(C, F)$; see [26,27].

For solving the equilibrium problem for a bifunction $\Theta : C \times C \to \mathbb{R}$, let us assume that Θ and ϕ satisfy the following conditions:

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \le 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t\downarrow 0}\Theta(tz+(1-t)x,\ y)\le\Theta(x,\ y);$$

(A4) for each $x \in C$, $y\ \alpha\ \Theta(x, y)$ is convex and lower semicontinuous;

(A5) For each $y \in C$, $x\ \alpha\ \Theta(x, y)$ is weakly upper semicontinuous;

(B1) For each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z,\ y_x)+\varphi(y_x)-\varphi(z)+\frac{1}{r}\langle y_x-z,\ z-x\rangle<0;$$

(B2) C is a bounded set;

The following lemmas were given in [1,4].

Lemma 2.1 ([4]) *Let C be a nonempty closed convex subset of H, and Θ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\Theta(z,\ y)+\frac{1}{r}\langle y-z,\ z-x\rangle\ge 0,\quad \forall y\in C.$$

Lemma 2.2 ([1]) *Let C be a nonempty closed convex subset of H. Let Θ be a bifunction form $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A5) *and $\phi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function. For $r > 0$ and $x \in H$, define a mapping $S_r : H \to C$ as follows:*

$$S_r(x) = \{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}$$

for all $z \in H$. *Assume that either* (B1) *or* (B2) *holds. Then, the following hold:*

(1) *For each* $x \in H$, $S_r(x) \neq \emptyset$;

(2) S_r *is single-valued;*

(3) S_r *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$||S_r x - S_r y||^2 \leq \langle S_r x - S_r y, x - y\rangle;$$

(4) $F(S_r) = MEP\ (\Theta, \phi)$;

(5) $MEP\ (\Theta, \phi)$ *is closed and convex.*

We also need the following lemmas for the proof of our main results.

Lemma 2.3 ([23]) *Let* $\{s_n\}$ *be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \beta_n, \quad n \geq 1,$$

where $\{\lambda_n\}$ *and* $\{\beta_n\}$ *satisfy the following conditions*:

(i) $\{\lambda_n\} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\prod_{n=1}^{\infty} (1 - \lambda_n) = 0$,

(ii) $\limsup_{n\to\infty} \frac{\beta_n}{\lambda_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\beta_n| < \infty$,

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 *In a Hilbert space, there holds the inequality*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 2.5 (Aoyama et al. [28]) *Let C be a nonempty closed convex subset of H and* $\{T_n\}$ *be a sequence of nonexpansive mappings of C into itself. Suppose that*

$$\sum_{n=1}^{\infty} \sup\{||T_{n+1}z - T_n z|| : z \in C\} < \infty.$$

Then, for each $y \in C$, $\{T_n y\}$ *converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by* $Ty = \lim_{n\to\infty} T_n y$ *for all* $y \in C$. *Then*, $\lim_{n\to\infty} \sup \{||Tz - T_n z|| : z \in C\} = 0$.

The following lemma can be found in [3](see also Lemma 2.1 in [22]).

Lemma 2.6 *Let C be a nonempty closed convex subset of a real Hilbert space H and* $g : C \to \mathbb{R} \cup \{\infty\}$ *be a proper lower semicontinuous differentiable convex function. If* x^* *is a solution to the minimization problem*

$$g(x^*) = \inf_{x\in C} g(x),$$

then

$$\langle g'(x), x - x^*\rangle \geq 0, \quad x \in C.$$

In particular, if x^* *solves the optimization problem*

$$\min_{x\in C} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}||x - u||^2 - h(x),$$

then

$$\langle u + (\gamma f - (I + \mu A))x^*, \ x - x^*\rangle \le 0, \quad x \in C,$$

where h is a potential function for γf.

3 Main results

In this section, we introduce a new composite iterative scheme for finding a common point of the set of solutions of the problem (1.1), the set of fixed points of a countable family of nonexpansive mappings, and the set of solutions of the problem (1.4) for an inverse-strongly monotone mapping.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A5) *and $\phi : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Let F, B be two α, β-inverse-strongly monotone mappings of C into H, respectively. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\Omega_1 := \bigcap_{n=1}^{\infty} F(T_n) \cap VI(C, F) \cap GMEP(\Theta, \varphi, B) \neq \emptyset$. Let $\mu > 0$ and $\gamma > 0$ be real numbers. Let f be a contraction of C into itself with constant k $\in (0, 1)$ and A be a strongly positive bounded linear operator on C with constant $\bar{\gamma} \in (0, 1)$ such that $0 < \gamma < \frac{(1+\mu)\bar{\gamma}}{k}$. Assume that either* (B1) *or* (B2) *holds. Let $u \in C$, and let $\{x_n\}$,$\{y_n\}$, and $\{u_n\}$ be sequences generated by $x_1 \in C$ and*

$$\begin{cases} \Theta(u_n, y) + \langle Bx_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ y_n = \alpha_n(u + \gamma f(x_n)) + (I - \alpha_n(I + \mu A))T_n P_C(u_n - \lambda_n F u_n), \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n T_n P_C(y_n - \lambda_n F y_n), \quad n \ge 1, \end{cases} \tag{IS}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$, $\lambda_n \in [a, b] \subset (0, 2\alpha)$ and $r_n \in [c, d] \subset (0, 2\beta)$. Let $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(C1) $\alpha_n \to 0$ $(n \to \infty)$; $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\beta_n \subset [0, a)$ *for all $n \ge 0$ and for some $a \in (0, 1)$;*

(C3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Suppose that $\sum_{n=1}^{\infty} \sup\{||T_{n+1}z - T_n z|| : z \in D\} < \infty$ for any bounded subset D of C. Let T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $q \in \Omega_1$, which is a solution of the optimization problem:

$$\min_{x \in \Omega_1} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}||x - u||^2 - h(x), \quad \text{(OP1)}$$

where h is a potential function for γf.

Proof First, from $\alpha_n \to 0$ $(n \to \infty)$ in the condition (C1), we assume, without loss of generality, that $\alpha_n \le (1 + \mu||A||)^{-1}$ and $2((1 + \mu)\bar{\gamma} - \gamma k)\alpha_n < 1$ for $n \ge 1$. We know that if A is bounded linear self-adjoint operator on H, then

$$||A|| = \sup\{|\langle Au, u\rangle| : u \in H, \ ||u|| = 1\}.$$

Observe that

$$\begin{aligned}\langle (I-\alpha_n(I+\mu A))u,\ u\rangle &= 1-\alpha_n-\alpha_n\mu\langle Au,\ u\rangle\\ &\geq 1-\alpha_n-\alpha_n\mu\|A\|\\ &\geq 0,\end{aligned}$$

which is to say $I - \alpha_n(I + \mu A)$ is positive. It follows that

$$\begin{aligned}\|I-\alpha_n(I+\mu A)\| &= \sup\{\langle (I-\alpha_n(I+\mu A))u,\ u\rangle\ :\ u\in H,\ \|u\|=1\}\\ &= \sup\{1-\alpha_n-\alpha_n\mu\langle Au,\ u\rangle\ :\ u\in H,\ \|u\|=1\}\\ &\leq 1-\alpha_n(1+\mu\bar{\gamma})\\ &< 1-\alpha_n(1+\mu)\bar{\gamma}.\end{aligned}$$

Let us divide the proof into several steps. From now on, we put $z_n = P_C(u_n - \lambda_n Fu_n)$ and $w_n = P_C(y_n - \lambda_n Fy_n)$.

Step 1: We show that $\{x_n\}$ is bounded. To this end, let $p \in \Omega_1 := \bigcap_{n=1}^{\infty} F(T_n) \cap VI(C, F) \cap GMEP(\Theta, \varphi, B)$ and $\{S_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.2. Then $p = T_n p = S_{r_n}(p - r_n Bp)$ and $p = P_C(p - \lambda_n Fp)$ from (2.2). From $z_n = P_C(u_n - \lambda_n Fu_n)$ and the fact that P_C and $I - \lambda_n F$ are nonexpansive, it follows that

$$\|z_n-p\| \leq \|(I-\lambda_nF)u_n-(I-\lambda_nF)p\| \leq \|u_n-p\|.$$

Also, by $u_n = S_{r_n}(x_n - r_n Bx_n) \in C$ and the β-inverse-strongly monotonicity of B, we have with $r_n \in (0, 2\beta)$,

$$\begin{aligned}\|u_n-p\|^2 &\leq \|x_n-r_nBx_n-(p-r_nBp)\|^2\\ &\leq \|x_n-p\|^2-2r_n\langle x_n-p,\ Bx_n-Bp\rangle+r_n^2\|Bx_n-Bp\|^2\\ &\leq \|x_n-p\|^2+r_n(r_n-2\beta)\beta\|Bx_n-Bp\|^2\\ &\leq \|x_n-p\|^2,\end{aligned}$$

that is, $\|u_n - p\| \leq \|x_n - p\|$, and so

$$\|z_n-p\| \leq \|x_n-p\|. \tag{3.1}$$

Similarly, we have

$$\|w_n-p\| \leq \|y_n-p\|. \tag{3.2}$$

Now, set $\bar{A} = (I + \mu A)$. Then, from (IS) and (3.1), we obtain

$$\begin{aligned}&\|y_n-p\|\\ &\leq (1-(1+\mu\bar{\gamma})\alpha_n)\|z_n-p\|+\alpha_n\|u\|\\ &\quad+\alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-\bar{A}p\|\\ &\leq (1-(1+\mu\bar{\gamma})\alpha_n)\|z_n-p\|+\alpha_n\|u\|+\alpha_n\gamma k\|x_n-p\|+\alpha_n\|\gamma f(p)-\bar{A}p\|\\ &\leq (1-((1+\mu)\bar{\gamma}-\gamma k)\alpha_n)\|x_n-p\|+\alpha_n(\|\gamma f(p)-\bar{A}p\|+\|u\|)\\ &= (1-((1+\mu)\bar{\gamma}-\gamma k)\alpha_n)\|x_n-p\|+((1+\mu)\bar{\gamma}-\gamma k)\alpha_n\frac{\|\gamma f(p)-\bar{A}p\|+\|u\|}{(1+\mu)\bar{\gamma}-\gamma k}.\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), it follows that

$$\begin{aligned} \|x_{n+1}-p\| &\le (1-\beta_n)\|y_n-p\| + \beta_n\|w_n-p\| \\ &\le (1-\beta_n)\|y_n-p\| + \beta_n\|y_n-p\| \\ &= \|y_n-p\| \\ &\le \max\left\{\|x_n-p\|, \frac{\|\gamma f(p)-\bar{A}p\|+\|u\|}{(1+\mu)\bar{\gamma}-\gamma k}\right\}. \end{aligned} \tag{3.4}$$

By induction, it follows from (3.4) that

$$\|x_n-p\| \le \max\left\{\|x_1-p\|, \frac{\|\gamma f(p)-\bar{A}p\|+\|u\|}{(1+\mu)\bar{\gamma}-\gamma k}\right\}, \quad n \ge 1.$$

Therefore, $\{x_n\}$ is bounded. Hence $\{u_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{f(x_n)\}$, $\{Fu_n\}$, $\{Fy_n\}$, and $\{\bar{A}T_nz_n\}$ are bounded. Moreover, since $\|T_nz_n - p\| \le \|x_n - p\|$ and $\|T_nw_n - p\| \le \|y_n - p\|$, $\{T_nz_n\}$ and $\{T_nw_n\}$ are also bounded, and since $\alpha_n \to 0$ in the condition (C1), we have

$$\|y_n - T_nz_n\| = \alpha_n\|(u+\gamma f(x_n)) - \bar{A}T_nz_n\| \to 0 \quad (\text{as } n \to \infty). \tag{3.5}$$

Step 2: We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Indeed, since $I - \lambda_nF$ and P_C are nonexpansive, we have

$$\begin{aligned} \|z_n - z_{n-1}\| &= \|P_C(u_n-\lambda_nFu_n) - P_C(u_{n-1}-\lambda_{n-1}Fu_{n-1})\| \\ &\le \|(I-\lambda_nF)u_n - (I-\lambda_nF)u_{n-1} + (\lambda_n-\lambda_{n-1})Fu_{n-1}\| \\ &\le \|u_n-u_{n-1}\| + |\lambda_n-\lambda_{n-1}|\,\|Fu_{n-1}\|. \end{aligned} \tag{3.6}$$

Similarly, we get

$$\|w_n - w_{n-1}\| \le \|y_n-y_{n-1}\| + |\lambda_n-\lambda_{n-1}|\,\|Fy_{n-1}\|. \tag{3.7}$$

On the other hand, from $u_{n-1} = S_{r_{n-1}}(x_{n-1} - r_nBx_{n-1})$ and $u_n = S_{r_n}(x_n - r_nBx_n)$, it follows that

$$\begin{aligned} &\Theta(u_{n-1}, y) + \langle Bx_{n-1}, y-u_{n-1}\rangle \\ &\quad + \varphi(y) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}}\langle y-u_{n-1}, u_{n-1}-x_{n-1}\rangle \ge 0, \quad \forall y \in C, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\Theta(u_n, y) + \langle Bx_n, y-u_n\rangle \\ &\quad + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge 0, \quad \forall y \in C. \end{aligned} \tag{3.9}$$

Substituting $y = u_n$ into (3.8) and $y = u_{n-1}$ into (3.9), we obtain

$$\Theta(u_{n-1}, u_n) + \langle Bx_{n-1}, u_n-u_{n-1}\rangle + \varphi(u_n) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}}\langle u_n-u_{n-1}, u_{n-1}-x_{n-1}\rangle \ge 0$$

and

$$\Theta(u_n, u_{n-1}) + \langle Bx_n, u_{n-1}-u_n\rangle + \varphi(u_{n-1}) - \varphi(u_n) + \frac{1}{r_n}\langle u_{n-1}-u_n, u_n-x_n\rangle \ge 0.$$

From (A2), we have

$$\langle u_n - u_{n-1},\ Bx_{n-1} - Bx_n + \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n}\rangle \geq 0,$$

and then

$$\langle u_n - u_{n-1},\ r_{n-1}(Bx_{n-1} - Bx_n) + u_{n-1} - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n)\rangle \geq 0.$$

Hence, it follows that

$$\begin{aligned}\langle u_n - u_{n-1},\ &(I - r_{n-l}B)x_n - (I - r_{n-1}B)x_{n-1} \\ &+ u_{n-1} - u_n + u_n - x_n - \frac{r_{n-1}}{r_n}(u_n - x_n)\rangle \geq 0.\end{aligned} \tag{3.10}$$

Without loss of generality, let us assume that there exists a real number c such that $r_n > c > 0$ for all $n \geq 1$. Then, by (3.10) and the fact that $(I - r_{n-1}B)$ is nonexpansive, we have

$$\begin{aligned}&||u_n - u_{n-1}||^2 \\ \leq\ & \langle u_n - u_{n-1},\ (I - r_{n-1}B)x_n - (I - r_{n-1}B)x_{n-1} + (1 - \frac{r_{n-1}}{r_n})(u_n - x_n)\rangle \\ \leq\ & ||u_n - u_{n-1}|| \left\{||(I - r_{n-1}B)x_n - (I - r_{n-1}B)x_{n-1}|| + \left|1 - \frac{r_{n-1}}{r_n}\right| ||u_n - x_n||\right\} \\ \leq\ & ||u_n - u_{n-1}|| \left\{||x_n - x_{n-1}|| + \left|1 - \frac{r_{n-1}}{r_n}\right| ||u_n - x_n||\right\},\end{aligned}$$

which implies that

$$\begin{aligned}||u_n - u_{n-1}|| &\leq ||x_n - x_{n-1}|| + \frac{1}{r_n}|r_n - r_{n-1}|\ ||u_n - x_n|| \\ &\leq ||x_n - x_{n-1}|| + \frac{M_1}{c}|r_n - r_{n-1}|,\end{aligned} \tag{3.11}$$

where $M_1 = \sup\{||u_n - x_n|| : n \geq 1\}$. Substituting (3.11) into (3.6), we have

$$||z_n - z_{n-1}|| \leq ||x_n - x_{n-1}|| + \frac{M_1}{c}|r_n - r_{n-1}| + |\lambda_n - \lambda_{n-1}|\ ||Fu_{n-1}||. \tag{3.12}$$

Simple calculations show that

$$\begin{aligned}y_n - y_{n-1} = &(\alpha_n - \alpha_{n-1})(u + \gamma f(x_{n-1}) - \bar{A}T_{n-1}z_{n-1}) + \alpha_n\gamma(f(x_n) - f(x_{n-1})) \\ &+ (I - \alpha_n\bar{A})(T_nz_n - T_nz_{n-1}) + (I - \alpha_n\bar{A})(T_nz_{n-1} - T_{n-1}z_{n-1}).\end{aligned}$$

Hence, by (3.11) and (3.12), we obtain

$$
\begin{aligned}
\|y_n - y_{n-1}\| \le\ & |\alpha_n - \alpha_{n-1}|(\|u\| + \gamma\|f(x_{n-1})\| + \|\bar{A}\|\ \|T_{n-1}z_{n-1}\|) \\
& + \alpha_n\gamma k\|x_n - x_{n-1}\| + (1-(1+\mu)\bar{\gamma}\alpha_n)\|z_n - z_{n-1}\| \\
& + (1-(1+\mu)\bar{\gamma}\alpha_n)\|T_n z_{n-1} - T_{n-1}z_{n-1}\| \\
\le\ & |\alpha_n - \alpha_{n-1}|(\|u\| + \gamma\|f(x_{n-1})\| + \|\bar{A}\|\ \|T_{n-1}z_{n-1}\|) \\
& + \alpha_n\gamma k\|x_n - x_{n-1}\| + (1-(1+\mu)\bar{\gamma}\alpha_n)\|x_n - x_{n-1}\| \\
& + \frac{M_1}{c}|r_n - r_{n-1}| + |\lambda_n - \lambda_{n-1}|\ \|Fu_{n-1}\| + \sup_{z\in D_1}\|T_n z - T_{n-1}z\|,
\end{aligned}
\tag{3.13}
$$

where D_1 is a bounded subset of C containing $\{z_n\}$. Also observe that

$$
\begin{aligned}
x_{n+1} - x_n =\ & (1-\beta_n)(y_n - y_{n-1}) + (\beta_n - \beta_{n-1})(T_{n-1}w_{n-1} - y_{n-1}) \\
& + \beta_n(T_n w_n - T_n w_{n-1}) + \beta_n(T_n w_{n-1} - T_{n-1}w_{n-1}).
\end{aligned}
\tag{3.14}
$$

By (3.7), (3.13) and (3.14), we have

$$
\begin{aligned}
& \|x_{n+1} - x_n\| \\
\le\ & (1-\beta_n)\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|(\|T_{n-1}x_{n-1}\| + \|y_{n-1}\|) \\
& + \beta_n\|w_n - w_{n-1}\| + \beta_n\|T_n w_{n-1} - T_{n-1}w_{n-1}\| \\
\le\ & (1-\beta_n)\|y_n - y_{n-1}\| + \beta_n\|y_n - y_{n-1}\| + \beta_n|\lambda_n - \lambda_{n-1}|\ \|Fy_{n-1}\| \\
& + |\beta_n - \beta_{n-1}|(\|T_{n-1}w_{n-1}\| + \|y_{n-1}\|) + \beta_n\|T_n w_{n-1} - T_{n-1}w_{n-1}\| \\
\le\ & \|y_n - y_{n-1}\| + |\lambda_n - \lambda_{n-1}|\ \|Fy_{n-1}\| \\
& + |\beta_n - \beta_{n-1}|(\|T_{n-1}w_{n-1}\| + \|y_{n-1}\|) + \|T_n w_{n-1} - T_{n-1}w_{n-1}\| \\
\le\ & (1-((1+\mu)\bar{\gamma} - \gamma k)\alpha_n)\|x_n - x_{n-1}\| \\
& + |(\alpha_n - \alpha_{n-1}|(\|u\| + \gamma\|f(x_{n-1})\| + \|\bar{A}\|\ \|T_{n-1}z_{n-1}\|) \\
& + |\lambda_n - \lambda_{n-1}|(\|Fy_{n-1}\| + \|Fu_{n-1}\|) + \frac{M_1}{c}|r_n - r_{n-1}| \\
& + |\beta_n - \beta_{n-1}|(\|T_{n-1}w_{n-1}\| + \|y_{n-1}\|) + 2\sup_{z\in D_2}\{\|T_n z - T_{n-1}z\| \\
\le\ & (1-((1+\mu)\bar{\gamma} - \gamma k)\alpha_n)\|x_n - x_{n-1}\| + \frac{M_1}{c}|r_n - r_{n-1}| \\
& + M_2|\alpha_n - \alpha_{n-1}| + M_3|\lambda_n - \lambda_{n-1}| + M_4|\beta_n - \beta_{n-1}| \\
& + 2\sup_{z\in D_2}\|T_n z - T_{n-1}z\|,
\end{aligned}
\tag{3.15}
$$

where D_2 is a bounded subset of C containing $\{z_n\}$ and $\{w_n\}$, $M_2 = \sup\{\|u\| + \gamma\|f(x_n)\| + \|\bar{A}\|\ \|T_n z_n\| : n \ge 1\}$, $M_3 = \sup\{\|Fy_n\| + \|Fu_n\| : n \ge 1\}$, and $M_4 = \sup\{\|T_n w_n\| + \|y_n\| : n \ge 1\}$. From the conditions (C1) and (C3) and the condition $\sum_{n=1}^{\infty}\sup\{\|T_{n+1}z - T_z\| \ : z \in D\} < \infty$ for any bounded subset D of C, it is easy to see that

$$
\lim_{n\to\infty}((1+\mu)\bar{\gamma} - \gamma k)\alpha_n = 0, \quad \sum_{n=1}^{\infty}((1+\mu)\bar{\gamma} - \gamma k)\alpha_n = \infty,
$$

and

$$\sum_{n=2}^{\infty}\left(\frac{M_1}{c}|r_n - r_{n-1}| + M_2|\alpha_n - \alpha_{n-1}| + M_3|\lambda_n - \lambda_{n-1}| + M_4|\beta_n - \beta_{n-1}| + 2\sup_{z\in D_2}||T_n z - T_{n-1}z||\right) < \infty.$$

Applying Lemma 2.3 to (3.15), we obtain

$$\lim_{n\to\infty}||x_{n+1} - x_n|| = 0.$$

Moreover, from (3.11), it follows that

$$\lim_{n\to\infty}||u_{n+1} - u_n|| = 0.$$

From (3.12) and (3.13), we also have

$$\lim_{n\to\infty}||z_{n+1} - z_n|| = 0 \text{ and } \lim_{n\to\infty}||y_{n+1} - y_n|| = 0.$$

Step 3: We show that $\lim_{n-\infty}||x_n - u_n|| = 0$. To this end, let $p \in \Omega_1$. Since S_{r_n} is firmly nonexpansive and $u_n = S_{r_n}(x_n - r_n Bx_n)$, we have

$$\begin{aligned}
||u_n - p||^2 &\le \langle S_{r_n}(x_n - r_n Bx_n) - S_{r_n}(p - r_n Bp),\ x_n - r_n Bx_n - (p - r_n Bp)\rangle \\
&= \frac{1}{2}\{||u_n - p||^2 + ||x_n - r_n Bx_n - (p - r_n Bp)||^2\} \\
&\quad - \frac{1}{2}\{||x_n - r_n Bx_n - (p - r_n Bp) - (u_n - p)||^2\} \\
&\le \frac{1}{2}\{||u_n - p||^2 + ||x_n - p||^2 - ||x_n - u_n - r_n(Bx_n - Bp)||^2\} \\
&\le \frac{1}{2}\{||u_n - p||^2 + ||x_n - p||^2 - ||x_n - u_n||^2 \\
&\quad + 2r_n\langle Bx_n - Bp,\ x_n - u_n\rangle - r_n^2||Bx_n - Bp||^2\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&||u_n - p||^2 \\
&\le ||x_n - p||^2 - ||x_n - u_n||^2 + 2r_n\langle Bx_n - Bp,\ x_n - u_n\rangle - r_n^2||Bx_n - Bp||^2 \\
&\le ||x_n - p||^2 - ||x_n - u_n||^2 + 2r_n\langle Bx_n - Bp,\ x_n - u_n\rangle \\
&\le ||x_n - p||^2 - ||x_n - u_n||^2 + 2r_n||Bx_n - Bp||\ ||x_n - u_n||.
\end{aligned} \tag{3.16}$$

On the other hand, since $z_n = P_C(u_n - \lambda_n F u_n)$, we get

$$\begin{aligned}
||z_n - p||^2 &\le ||(I - \lambda_n F)u_n - (I - \lambda_n F)p||^2 \\
&\le ||u_n - p||^2.
\end{aligned} \tag{3.17}$$

From (3.16), (3.17), and the convexity of $|| \ ||^2$, we obtain

$$
\begin{aligned}
||y_n - p||^2 &\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + (1 - \alpha_n)||T_n z_n - p||^2 \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + (1 - \alpha_n)||z_n - p||^2 \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + (1 - \alpha_n)||u_n - p||^2 \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 \\
&\quad + (1 - \alpha_n)\{||x_n - p||^2 - ||x_n - u_n||^2 + 2r_n||x_n - u_n|| \, ||Bx_n - Bp||\} \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + ||x_n - p||^2 \\
&\quad - (1 - \alpha_n)||x_n - u_n||^2 + 2(1 - \alpha_n)r_n||x_n - u_n|| \, ||Bx_n - Bp||.
\end{aligned} \tag{3.18}
$$

On the another hand, we note that

$$
\begin{aligned}
||u_n - p||^2 &\le ||(x_n - r_n Bx_n) - (p - r_n Bp)||^2 \\
&\le ||x_n - p||^2 - 2r_n\langle x_n - p, \ Bx_n - Bp\rangle + r_n^2||Bx_n - Bp||^2 \\
&\le ||x_n - p||^2 - 2r_n\beta||Bx_n - Bp||^2 + r_n^2||Bx_n - Bp||^2.
\end{aligned} \tag{3.19}
$$

Using the convexity of $|| \ ||^2$, (3.2), (3.18), and (3.19), we have

$$
\begin{aligned}
||x_{n+1} - p||^2 &\le ||y_n - p||^2 \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + (1 - \alpha_n)||u_n - p||^2 \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 \\
&\quad + (1 - \alpha_n)\{||x_n - p||^2 - 2r_n\beta||Bx_n - Bp||^2 + r_n^2||Bx_n - Bp||^2\} \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + ||x_n - p||^2 \\
&\quad + (1 - \alpha_n)r_n(r_n - 2\beta)||Bx_n - Bp||^2.
\end{aligned} \tag{3.20}
$$

Hence, we have

$$
\begin{aligned}
&(1 - \alpha_n)c(2\beta - d)||Bx_n - Bp||^2 \\
&\le (1 - \alpha_n)r_n(2\beta - r_n)||Bx_n - Bp||^2 \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + (||x_n - p||^2 - ||x_{n+1} - p||^2) \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + ||x_n - x_{n+1}||(||x_n - p|| + ||x_{n+1} - p||).
\end{aligned}
$$

From the condition (C1), $\{r_n\} \subset [c, d] \subset (0, 2\beta)$ and Step 2, it follows that

$$
\lim_{n\to\infty} ||Bx_n - Bp||^2 = 0. \tag{3.21}
$$

Also, by (3.16) and (3.20), we have

$$
\begin{aligned}
||x_{n+1} - p||^2 &\le ||y_n - p||^2 \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + (1 - \alpha_n)||u_n - p||^2 \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 \\
&\quad + (1 - \alpha_n)\{||x_n - p||^2 - ||x_n - u_n||^2 + 2r_n||x_n - u_n|| \, ||Bx_n - Bp||\} \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + ||x_n - p||^2 \\
&\quad - (1 - \alpha_n)||x_n - u_n||^2 + 2r_n(1 - \alpha_n)||x_n - u_n|| \, ||Bx_n - Bp||,
\end{aligned}
$$

and so

$$\begin{aligned}(1-\alpha_n)\|x_n-u_n\|^2 &\le \alpha_n\|u+\gamma f(x_n)+(I-\bar{A})T_nz_n-p\|^2+\|x_n-p\|^2\\ &\quad -\|x_{n+1}-p\|^2+2r_n(1-\alpha_n)\|x_n-u_n\|\,\|Bx_n-Bp\|\\ &\le \alpha_nM_5+\|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)\\ &\quad +2r_n(1-\alpha_n)\|x_n-u_n\|\,\|Bx_n-Bp\|,\end{aligned}$$

where $M_5 = \sup\{\|u+\gamma f(x_n)+(I-\bar{A})T_nz_n-p\|^2 : n \ge 1\}$. Since $\|x_{n+1} - x_n\| \to 0$, $\alpha_n \to 0$, and $\|Bx_n - Bp\| \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{3.22}$$

Moreover, since $\liminf_{n\to\infty} r_n > 0$, we also have

$$\lim_{n\to\infty}\frac{\|x_n-u_n\|}{r_n} = \lim_{n\to\infty}\frac{1}{r_n}\|x_n-u_n\| = 0. \tag{3.23}$$

Step 4: We show that $\lim_{n\to\infty} \|x_n - T_nz_n\| = 0$ and $\lim_{n-\infty} \|x_n - y_n\| = 0$. Indeed, since $z_n = P_C(u_n - \lambda_nFu_n)$ and $w_n = P_C(y_n - \lambda_nFy_n)$, we obtain with the condition (C2)

$$\begin{aligned}\|x_{n+1}-y_n\| &\le \beta_n(\|T_nz_n-T_nw_n\|+\|y_n-T_nz_n\|)\\ &\le a(\|z_n-w_n\|+\|y_n-T_nz_n\|)\\ &\le a(\|u_n-y_n\|+\|y_n-T_nz_n\|)\\ &\le a(\|u_n-x_n\|+\|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|+\|y_n-T_nz_n\|),\end{aligned}$$

which implies that

$$\|x_{n+1}-y_n\| \le \frac{a}{1-a}(\|u_n-x_n\|+\|x_n-x_{n+1}\|+\|y_n-T_nz_n\|).$$

Thus, from (3.5), Step 2, and Step 3, we have

$$\|x_{n+1}-y_n\| \to 0 \ \text{ as } \ n\to\infty.$$

Also we have

$$\|x_n-y_n\| \le \|x_n-x_{n+1}\|+\|x_{n+1}-y_n\| \to 0 \text{ as } n\to\infty.$$

Since $\lim_{n\to\infty} \|y_n - T_nz_n\| = 0$ by (3.5) in Step 1, we obtain

$$\|x_n-T_nz_n\| \le \|x_n-y_n\|+\|y_n-T_nz_n\| \to 0 \ \text{ as } \ n\to\infty.$$

Step 5: We show that $\lim_{n\to\infty} \|T_nz_n - z_n\| = 0$. Let $p \in \Omega_1$. Using the convexity of $\|\ \|^2$, we compute

$$\begin{aligned}
||y_n - p||^2 &\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + (1-\alpha_n)||T_n z_n - p||^2 \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + (1-\alpha_n)||z_n - p||^2 \\
&= \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 \\
&\quad + (1-\alpha_n)||P_C(u_n - \lambda_n F u_n) - P_C(p - \lambda_n F p)||^2 \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 \\
&\quad + (1-\alpha_n)||(u_n - p) - \lambda_n(F u_n - F p)||^2 \\
&= \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 \\
&\quad + (1-\alpha_n)\{||u_n - p||^2 - 2\lambda_n\langle u_n - p,\ F u_n - F p\rangle + \lambda_n^2||F u_n - F p||^2\} \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 \\
&\quad + (1-\alpha_n)\{||u_n - p||^2 - 2\lambda_n\alpha||F u_n - F p||^2 + \lambda_n^2||F u_n - F p||^2\} \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 \\
&\quad + ||x_n - p||^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\alpha)\,||F u_n - F p||^2.
\end{aligned} \tag{3.24}$$

Using (3.24), we obtain

$$\begin{aligned}
||x_{n+1} - p||^2 &\le ||y_n - p||^2 \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 \\
&\quad + ||x_n - p||^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\alpha)\,||F u_n - F p||^2.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&(1-\alpha_n)a(2\alpha - b)||F u_n - F p||^2 \\
&\le (1-\alpha_n)\lambda_n(2\alpha - \lambda_n)||F u_n - F p||^2 \\
&\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 \\
&\quad + (||x_n - p|| - ||x_{n+1} - p||)(||x_n - p|| + ||x_{n+1} - p||) \\
&\le \alpha_n M_5 + ||x_n - x_{n+1}||(||x_n - p|| + ||x_{n+1} - p||).
\end{aligned}$$

where $M_5 = \sup\{||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 : n \ge 1\}$. From the condition (C1), $\lambda_n \in [a, b] \subset (0, 2\alpha)$, and Step 2, it follows that

$$\lim_{n\to\infty} ||F u_n - F p|| = 0. \tag{3.25}$$

On the other hand, using $z_n = P_C (u_n - \lambda_n F u_n)$ and (2.1), we observe that

$$\begin{aligned}
||z_n - p||^2 &\le \langle (u_n - \lambda_n F u_n) - (p - \lambda_n F p),\ z_n - p\rangle \\
&\le \frac{1}{2}\{||u_n - p||^2 + ||z_n - p||^2 - ||(u_n - z_n) - \lambda_n(F u_n - F p)||^2\} \\
&\le \frac{1}{2}\{||x_n - p||^2 + ||z_n - p||^2 - ||u_n - z_n||^2 \\
&\quad + 2\lambda_n\langle u_n - z_n,\ F u_n - F p\rangle - \lambda_n^2||F u_n - F p||^2\},
\end{aligned}$$

and so

$$\begin{aligned}
||z_n - p||^2 &\le ||x_n - p||^2 - ||u_n - z_n||^2 \\
&\quad + 2\lambda_n\langle u_n - z_n,\ F u_n - F p\rangle - \lambda_n^2||F u_n - F p||^2
\end{aligned} \tag{3.26}$$

Thus, from (3.26), we have

$$\begin{aligned}||y_n - p||^2 &\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + (1-\alpha_n)||T_n z_n - p||^2 \\ &\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + (1-\alpha_n)||z_n - p||^2 \\ &\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 \\ &\quad + (1-\alpha_n)\{||x_n - p||^2 - ||u_n - z_n||^2 \\ &\qquad + 2\lambda_n\langle u_n - z_n,\ Fu_n - Fp\rangle - \lambda_n^2||Fu_n - Fp||^2\} \\ &\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + ||x_n - p||^2 \\ &\quad - (1-\alpha_n)||u_n - z_n||^2 + 2\lambda_n(1-\alpha_n)\langle u_n - z_n,\ Fu_n - Fp\rangle \\ &\quad - \lambda_n^2||Fu_n - Fp||^2.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}||x_{n+1} - p||^2 &\le ||y_n - p||^2 \\ &\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + ||x_n - p||^2 \\ &\quad - (1-\alpha_n)||u_n - z_n||^2 + 2\lambda_n(1-\alpha_n)\langle u_n - z_n,\ Fu_n - Fp\rangle \\ &\quad - \lambda_n^2||Fu_n - Fp||^2,\end{aligned} \tag{3.27}$$

which implies that

$$\begin{aligned}&(1-\alpha_n)||u_n - z_n||^2 \\ &\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + ||x_n - p||^2 - ||x_{n+1} - p||^2 \\ &\quad +2\lambda_n(1-\alpha_n)||u_n - z_n||\ ||Fu_n - Fp|| \\ &\le \alpha_n||u + \gamma f(x_n) + (I - \bar{A})T_n z_n - p||^2 + ||x_{n+1} - x_n||(||x_n - p|| + ||x_{n+1} - p||) \\ &\quad +2\lambda_n(1-\alpha_n)||u_n - z_n||\ ||Fu_n - Fp||.\end{aligned}$$

From $||x_{n+1} - x_n|| \to 0$, $\alpha_n \to 0$ and $||Fu_n - Fp|| \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} ||u_n - z_n|| = 0. \tag{3.28}$$

Since $||T_n z_n - z_n|| \le ||T_n z_n - x_n|| + ||x_n - u_n|| + ||u_n - z_n||$, from Step 3, Step 4, and (3.28), we conclude that

$$\lim_{n\to\infty} ||T_n z_n - z_n|| = 0. \tag{3.29}$$

We notice that by the assumption on T, (3.29) and Lemma 2.5,

$$\begin{aligned}\lim_{n\to\infty} ||Tz_n - z_n|| &\le \lim_{n\to\infty}(||Tz_n - T_n z_n|| + ||T_n z_n - z_n||) \\ &\le \lim_{n\to\infty} \sup\{||Ty - T_n y|| : y \in C\} + \lim_{n\to\infty} ||T_n z_n - z_n|| = 0.\end{aligned} \tag{3.30}$$

Step 6: We show that

$$\limsup_{n\to\infty}\langle u + (\gamma f - (I + \mu A))q,\ y_n - q\rangle = \limsup_{n\to\infty}\langle u + (\gamma f - \bar{A})q,\ y_n - q\rangle \le 0,$$

where q is a solution of the optimization problem (OP1). To this end, first we prove that

$$\limsup_{n\to\infty}\langle u + (\gamma f - \bar{A})q,\ Tz_n - q\rangle \le 0.$$

Since $\{z_n\}$ is bounded, we can choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty} \langle u + (\gamma f - \bar{A})q,\ Tz_n - q\rangle = \lim_{i\to\infty} \langle u + (\gamma f - \bar{A})q,\ Tz_{n_i} - q\rangle. \tag{3.31}$$

Without loss of generality, we may assume that $\{z_{n_i}\}$ converges weakly to $z \in C$. From $||Tz_n - z_n|| \to 0$ in (3.30), it follows that $Tz_{n_i} \rightharpoonup z$.

Now, we will show that $z \in \Omega_1$. First, we show that$z \in \bigcap_{n=1}^{\infty} F(T_n) = F(T)$. Assume that $z \notin F(T)$. Since $z_{n_i} \rightharpoonup z$ and $Tz \neq z$, by the Opial condition, we obtain

$$\begin{aligned}
\liminf_{i\to\infty} ||z_{n_i} - z|| &< \liminf_{i\to\infty} ||z_{n_i} - Tz|| \\
&\le \liminf_{i\to\infty} (||z_{n_i} - Tz_{n_i}|| + ||Tz_{n_i} - Tz||) \\
&\le \liminf_{i\to\infty} ||z_{n_i} - z||,
\end{aligned}$$

which is a contradiction. Thus we have $z \in \bigcap_{n=1}^{\infty} F(T_n) = F(T)$.

Next, we prove that $z \in VI(C, F)$. Let

$$Qv = \begin{cases} Fv + N_C v, & v \in C \\ \emptyset, & v \notin C, \end{cases}$$

where $N_C v$ is normal cone to C at v. We already know that, in this case, the mapping Q is maximal monotone, and $0 \in Qv$ if and only if $v \in VI(C, F)$. Let $(v, w) \in G(Q)$. Since $w - Fv \in N_C v$ and $z_n \in C$, we have

$$\langle v - z_n,\ w - Fv\rangle \ge 0.$$

On the other hand, from $z_n = P_C(u_n - \lambda_n Fu_n)$, we have

$$\langle v - z_n,\ z_n - (u_n - \lambda_n Fu_n)\rangle \ge 0,$$

that is,

$$\langle v - z_n,\ \frac{z_n - u_n}{\lambda_n} + Fu_n\rangle \ge 0.$$

Thus, we obtain

$$\begin{aligned}
\langle v - z_{n_i}, w\rangle &\ge \langle v - z_{n_i}, Fv\rangle \\
&\ge \langle v - z_{n_i}, Fv\rangle - \langle v - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} + Fu_{n_i}\rangle \\
&= \langle v - z_{n_i}, Fv - Fz_{n_i}\rangle + \langle v - z_{n_i}, Fz_{n_i} - Fu_{n_i}\rangle \\
&\quad - \langle v - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\
&\ge \langle v - z_{n_i}, Fz_{n_i} - Fu_{n_i}\rangle - \langle v - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle.
\end{aligned} \tag{3.32}$$

Since $||z_n - u_n|| \to 0$ in (3.28) and F is α-inverse-strongly monotone, it follows from (3.32) that

$$\langle v - z, w\rangle \ge 0, \quad \text{as } i \to \infty.$$

Since Q is maximal monotone, we have $z \in Q^{-1}0$ and hence $z \in VI(C, F)$.

Finally, we show that $z \in GMEP(\Theta, \phi, B)$. By $u_n = S_{r_n}(x_n - r_nBx_n)$; we know that

$$\Theta(u_n, y) + \langle Bx_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

It follows from (A2) that

$$\langle Bx_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \Theta(y,\ u_n), \quad \forall y \in C.$$

Hence

$$\langle Bx_{n_i}, y - u_{n_i}\rangle + (\varphi(y) - (\varphi(u_{n_i}) + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \geq \Theta(y,\ u_{n_i}), \quad \forall y \in C \tag{3.33}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1 - t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $\Theta(y_t, z) \leq 0$. Hence, from (3.33), we have

$$\begin{aligned}
\langle y_t - u_{n_i}, By_t\rangle &\geq \langle y_t - u_{n_i}, By_t\rangle - \varphi(y_t) + \varphi(u_{n_i}) - \langle y_t - u_{n_i}, Bx_{n_i}\rangle \\
&\quad - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + \Theta(y_t, u_{n_i}) \\
&= \langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle + \langle y_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle \\
&\quad - \varphi(y_t) + \varphi(u_{n_i}) - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + \Theta(y_t, u_{n_i})
\end{aligned}$$

Since $||u_{n_i} - x_{n_i}|| \to 0$ from Step 3, we have $||Bu_{n_i} - Bx_{n_i}|| \to 0$ and $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$. Also by $||u_n - z_n|| \to 0$ in (3.28), we have $u_{n_i} \rightharpoonup z$. Moreover, from the inverse-strongly monotonicity of B, we have $\langle y_t - u_{n_i},\ By_t - Bu_{n_i}\rangle \geq 0$. Hence, from (A4) and the weak lower semicontinuity of ϕ, if follows that

$$\langle y_t - z, By_t\rangle \geq -\varphi(y_t) + \varphi(z) + \Theta(y_t, z) \ \text{ as } i \to \infty. \tag{3.34}$$

From (A1), (A4), and (3.34), we also obtain

$$\begin{aligned}
0 &= \Theta(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&\leq t\Theta(y_t, y) + (1 - t)\Theta(y_t, z) + t\varphi(y_t) + (1 - t)\varphi(z) - \varphi(y_t) \\
&\leq t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1 - t)\langle y_t - z, By_t\rangle \\
&= t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1 - t)t\langle y - z, By_t\rangle,
\end{aligned}$$

and hence

$$0 \leq \Theta(y_t, y) + \varphi(y) - \varphi(y_t) + (1 - t)\langle y - z, By_t\rangle. \tag{3.35}$$

Letting $t \to 0$ in (3.35), we have for each $y \in C$

$$\Theta(z, y) + \langle Bz, y - z\rangle + \varphi(y) - \varphi(z) \geq 0.$$

This implies that $z \in GMEP(\Theta, \phi, B)$. Therefore $z \in \Omega_1$.

Now, from Lemma 2.6 and (3.31), we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\langle u + (\gamma f - \bar{A})q, Tz_n - q\rangle &= \lim_{i\to\infty}\langle u + (\gamma f - \bar{A})q, Tz_{n_i} - q\rangle \\
&= \langle u + (\gamma f - \bar{A})q, z - q\rangle \\
&\leq 0.
\end{aligned} \tag{3.36}$$

Since $\lim_{n\to\infty} ||x_n - Tz_n|| \le ||x_n - T_n z_n|| + ||T_n z_n - Tz_n|| \to 0$, from Step 4 and Lemma 2.5, and from (3.36), we conclude that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle u + (\gamma f - \bar{A})q, y_n - q\rangle \\
&\le \limsup_{n\to\infty} \langle u + (\gamma f - \bar{A})q, y_n - Tz_n\rangle + \limsup_{n\to\infty} \langle u + (\gamma f - \bar{A})q, Tz_n - q\rangle \\
&\le \limsup_{n\to\infty} ||u + (\gamma f - \bar{A})q||\, ||y_n - Tz_n|| + \limsup_{n\to\infty} \langle u + (\gamma f - \bar{A})q, Tz_n - q\rangle \\
&\le 0.
\end{aligned}$$

Step 7: We show that $\lim_{n\to\infty} ||x_n - q|| = 0$ and $\lim_{n\to\infty} ||u_n - q|| = 0$, where q is a solution of the optimization problem (OP1). Indeed, from (IS) and Lemma 2.4, we have

$$\begin{aligned}
&||x_{n+1} - q||^2 \le ||y_n - q||^2 \\
&\le ||(I - \alpha_n \bar{A})(T_n z_n - q)||^2 + 2\alpha_n \langle u + \gamma f(x_n) - \bar{A}q, y_n - q\rangle \\
&\le (1 - (1+\mu)\bar{\gamma}\alpha_n)^2 ||z_n - q||^2 + 2\alpha_n \gamma \langle f(x_n) - f(q), y_n - q\rangle \\
&\quad + 2\alpha_n \langle u + \gamma f(q) - \bar{A}q, y_n - q\rangle) \\
&\le (1 - (1+\mu)\bar{\gamma}\alpha_n)^2 ||x_n - q||^2 + 2\alpha_n \gamma k ||x_n - q||\, ||y_n - q|| \\
&\quad + 2\alpha_n \langle u + (\gamma f - \bar{A})q, y_n - q\rangle \\
&\le (1 - (1+\mu)\bar{\gamma}\alpha_n)^2 ||x_n - q||^2 + 2\alpha_n \gamma k ||x_n - q|| (||y_n - x_n|| + ||x_n - q||) \\
&\quad + 2\alpha_n \langle u + (\gamma f - \bar{A})q,\ y_n - q\rangle \\
&= (1 - 2((1+\mu)\bar{\gamma} - \gamma k)\alpha_n) ||x_n - q||^2 \\
&\quad + \alpha_n^2 ((1+\mu)\bar{\gamma})^2 ||x_n - q||^2 + 2\alpha_n \gamma k ||x_n - q||\, ||y_n - x_n|| \\
&\qquad + 2\alpha_n \langle u + (\gamma f - \bar{A})q, y_n - q\rangle,
\end{aligned}$$

that is,

$$\begin{aligned}
||x_{n+1} - q||^2 &\le (1 - 2((1+\mu)\bar{\gamma} - \gamma k)\alpha_n) ||x_n - q||^2 + \alpha_n^2 ((1+\mu)\bar{\gamma})^2 M_6 \\
&\quad + 2\alpha_n \gamma k ||y_n - x_n|| M_6 + 2\alpha_n \langle u + (\gamma f - \bar{A})q,\ y_n - q\rangle \\
&= (1 - \overline{\alpha_n}) ||x_n - q||^2 + \overline{\beta_n},
\end{aligned}$$

where $M_6 = \sup\{||x_n - q|| : n \ge 1\}$, $\overline{\alpha_n} = 2((1+\mu)\bar{\gamma} - \gamma k)\alpha_n$ and

$$\overline{\beta_n} = \alpha_n [\alpha_n ((1+\mu)\bar{\gamma})^2 M_6^2 + 2\gamma k ||y_n - x_n|| M_6 + 2\langle u + (\gamma f - \bar{A})q, y_n - q\rangle].$$

From the condition (C1), $||y_n - x_n|| \to 0$ in Step 4, and Step 6, it is easily seen that $\sum_{n=1}^{\infty} \overline{\alpha_n} = \infty$, $\sum_{n=1}^{\infty} \overline{\alpha_n} = \infty$, and $\limsup_{n\to\infty} \frac{\overline{\beta_n}}{\overline{\alpha_n}} \le 0$. Hence, by Lemma 2.3, we conclude $x_n \to q$ as $n \to \infty$. Moreover, from Step 3, we obtain that $u_n \to q$ as $n \to \infty$. This completes the proof. □

Corollary 3.2 *Let* H, C, Θ, ϕ, F, A, f, μ, γ, $\bar{\gamma}$ *and* k *be as in Theorem 3.1. Let* $\{T_n\}$ *be a sequence of nonexpansive mappings of* C *into itself such that* $\Omega_2 := \bigcap_{n=1}^{\infty} F(T_n) \cap VI(C, F) \cap MEP(\Theta, \varphi) \neq \emptyset$. *Assume that either* (B1) *or* (B2) *holds. Let* $u \in C$ *and let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_1 \in C$ *and*

$$\begin{cases} \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ y_n = \alpha_n (u + \gamma f(x_n)) + (I - \alpha_n (I + \mu A)) T_n P_C (u_n - \lambda_n F u_n), \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n T_n P_C (y_n - \lambda_n F y_n), \quad n \ge 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$, $\lambda_n \in [a, b] \subset (0, 2\alpha)$, *and* $r_n \in [c, d] \subset (0, 2\beta)$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$, *and* $\{r_n\}$ *satisfy the conditions* (C1), (C2), and (C3) *in Theorem 3.1. Suppose that* $\sum_{n=1}^{\infty} \sup\{||T_{n+1}z - T_n z|| : z \in D\} < \infty$*for any bounded subset D of C. Let T be a mapping of C into itself defined by* $Tz = \lim_{n\to\infty} T_n z$ *for all* $z \in C$ *and suppose that* $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. *Then* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $q \in \Omega_2$, *which is a solution of the optimization problem:*

$$\min_{x\in\Omega_2} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}||x-u||^2 - h(x), \quad \text{(OP2)}$$

where h is a potential function for γf.

Proof Putting $B = 0$ in Theorem 3.1, we obtain the required result. □

As direct consequences of Theorem 3.1, we can also obtain the following new strong convergence theorems for the problem (1.1) and fixed point problem.

Corollary 3.3 *Let* H, C, Θ, ϕ, B, A, f, μ, γ, $\bar{\gamma}$, *and k be as in Theorem 3.1. Let T be a nonexpansive mapping of C into itself such that* $\Omega_3 := F(T) \cap GMEP(\Theta, \phi, B) \neq \emptyset$. *Assume that either* (B1) *or* (B2) *holds. Let* $u \in C$ *and let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_1 \in C$ *and*

$$\begin{cases} \Theta(u_n, y) + \langle Bx_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + (I - \alpha_n(I + \mu A))Tu_n, & n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$, *and* $r_n \in [c, d] \subset (0; 2\beta)$. *Let* $\{\alpha_n\}$ *and* $\{r_n\}$ *satisfy the following conditions:*

(i) $\alpha_n \to 0$ $(n \to \infty)$; $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$

Then $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $q \in \Omega_3$, *which is a solution of the optimization problem:*

$$\min_{x\in\Omega_3} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}||x-u||^2 - h(x), \quad \text{(OP3)}$$

where h is a potential function for γf.

Proof Putting $F = 0$, $\beta_n = 0$, and $T_n = T$ for $n \geq 1$ in Theorem 3.1, we obtain the required result. □

A mapping $S : C \to C$ is called strictly pseudocontractive if there exists r with $0 \leq r < 1$ such that

$$||Sx - Sy||^2 \leq ||x-y||^2 + r||(I-S)x - (I-S)y||^2 \text{ for all } x, y \in C.$$

Such a mapping S is called r-strictly pseudocontractive. Putting $F = I - S$, we know that

$$\langle x - y, Fx - Fy\rangle \geq \frac{1-r}{2}||Fx - Fy||^2 \text{ for all } x, y \in C;$$

see, for instance, [12,29]. Hence, we have the following result.

Corollary 3.4 *Let H, C,* Θ, ϕ, *B, A, f,* μ, γ, $\bar{\gamma}$, *and k be as in Theorem 3.1. Let S be a r-strictly pseudocontractive mappings of C into itself. Let T be a nonexpansive mapping of C into itself such that* $\Omega_4 := F(T) \cap F(S) \cap GMEP(\Theta,\phi, B) \neq \emptyset$. *Assume that either*

(B1) *or* (B2) *holds. Let* $u \in C$ *and let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_1 \in C$ *and*

$$\begin{cases} \Theta(u_n, y) + \langle Bx_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + (I - \alpha_n(I + \mu A))T(u_n - \lambda_n(u_n - Su_n)), \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$,$\{\lambda_n\} \subset [0, 1]$, $\lambda_n \in [a, b] \subset (0, 1 - r)$, *and* $r_n \in [c, d] \subset (0, 2\beta)$. *Let* $\{\alpha_n\}$, $\{\lambda_n\}$, *and* $\{r_n\}$ *satisfy the following conditions:*

(i) $\alpha_n \to 0$ $(n \to \infty)$; $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$; $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $q \in \Omega_4$, *which is a solution of the optimization problem*

$$\min_{x\in\Omega_4} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}||x - u||^2 - h(x), \qquad \text{(OP4)}$$

where h *is a potential function for* γf.

Proof Putting $F = I - S$, $\beta_n = 0$, and $T_n = T$ for $n \geq 1$, from Theorem 3.1, we obtain the required result. □

Remark 3.1

(1) Theorem 3.1, Corollary 3.2, Corollary 3.3, and Corollary 3.4 improve and develop the corresponding results, which were obtained recently by many authors in various references, for example, see [2,3,7-9,11-18,24,30]. In particular, we mention that the iterative scheme (3.1) in Theorem 3.1 of [24] is not well defined without the assumption $C \pm C \subset C$ on a nonempty closed convex subset C of H.

(2) The condition $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ imposed on $\{\alpha_n\}$ can be replaced by the perturbed control condition $|\alpha_{n+1} - \alpha_n| < o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=1}^{\infty} \sigma_n < \infty$ or the condition $\alpha_n \in (0, 1]$, $n \geq 1$, and $\lim_{n\to\infty} \alpha_n/\alpha_{n+1} = 1$.

(3) Some special cases of the generalized mixed equilibrium problem (1.1) are known as follows:

(i) If $\phi = 0$, then the problem (1.1) reduced the following generalized equilibrium problem (*GEP*) of finding $x \in C$ such that

$$\Theta(x, y) + \langle Bx, y - x\rangle \geq 0, \quad \forall y \in C, \tag{3.31a}$$

which was studied by Takahashi and Takahashi [30].

(ii) If $\Theta(x, y) = 0$ for all $x, y \in C$, then the problem (1.1) reduces the following generalized variational inequality problem (*GVI*) of finding $x \in C$ such that

$$\langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{3.32a}$$

(iii) If $B = 0$ and $\Theta(x, y) = 0$ for all $x, y \in C$, then the problem (1.1) reduces the following minimization problem of finding $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{3.33a}$$

Applying Theorem 3.1 together with the one in the following assumptions instead of (B1), we can also establish the new corresponding results for the above mentioned problems:

(B2) C is a bounded set;

(B3) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \backslash D_x$,

$$\varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(B4) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \backslash D_x$,

$$\Theta(z, y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0.$$

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Competing interests
The author declares that they have no competing interests.

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