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Norms and essential norms of composition operators from H^∞ to general weighted Bloch spaces in the polydisk

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Abstract

Let U^n be the unit polydisk of $\mathbb{C}^n$ and ϕ a holomorphic self-map of U^n. $H^\infty(U^n)$ and $\mathcal{B}^\alpha_{\log}(U^n)$ denote the space of bounded holomorphic functions and the space of general weighted Bloch functions defined on U^n, respectively, where $\alpha>0$. This paper gives some estimates of the norm and essential norm of the composition operator C_ϕ induced by ϕ from $H^\infty(U^n)$ to $\mathcal{B}^\alpha_{\log}(U^n)$. As applications, some characterizations of the boundedness and compactness of C_ϕ from $H^\infty(U^n)$ to $\mathcal{B}^\alpha_{\log}(U^n)$ are obtained. Moreover, we also characterize the weak compactness of the composition operator C_ϕ.

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1 Introduction

Let D be a bounded homogeneous domain in $\mathbb{C}^n$, $H(D)$ the class of all holomorphic functions on D. For ϕ, a holomorphic self-map of D, the linear operator defined by

$$C_\varphi(f) = f \circ \varphi, \quad f \in H(D),$$

is called the composition operator with symbol ϕ. The study of composition operators is fundamental in the study of Banach and Hilbert spaces of holomorphic functions. We refer to the books [1] and [2] for an overview of some classical results on the theory of composition operators.

Let $K(z, z)$ be the Bergman kernel function of D, and the Bergman metric $H_z(u, u)$ in D is defined by

$$H_z(u, u) = \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \log K(z,z)}{\partial z_j \partial \overline{z_k}} u_j \overline{u_k},$$

where $z = (z_1, ..., z_n) \in D$ and $u = (u_1, \ldots, u_n) \in \mathbb{C}^n$. A function $f \in H(D)$ is said to be a Bloch function if $\beta_f = \sup_{z\in D} Q_f(z)$ is finite, where

$$Q_f(z) = \sup_{u\in\mathbb{C}^n\setminus\{0\}} \frac{|(\nabla f)(z)u|}{H_z^{1/2}(u,u)},$$

$(\nabla f)(z)u = < \nabla f(z), \bar{u} > = \sum_{k=1}^{n} \frac{\partial f}{\partial z_k}(z) u_k$. By fixing a base point $z_0 \in D$, the Bloch space $\mathcal{B}(D)$ of all Bloch functions on D is a Banach space under the norm $\| f \|_{\mathcal{B}} = |f(z_0)| + \beta_f$[3]. For convenience, we assume the bounded homogeneous domain D to contain the origin and take $z_0 = 0$. In [3], Timoney proved that the space $H^\infty(D)$ of bounded holomorphic functions on a bounded homogeneous domain D is a subspace of $\mathcal{B}(D)$ and for each $f \in H^\infty(D)$, $\| f \|_{\mathcal{B}} \le C_D \| f \|_\infty$, where $C_D$is a constant depending only on the domain D and $\| f \|_\infty = \sup_{z \in D} |f(z)|$.

Let U^nbe the unit polydisk of $\mathbb{C}^n$. Timony [3] showed that $f \in \mathcal{B}(D)$ if and only if

$$\sup_{z \in U^n} \sum_{k=1}^{n} \left| \frac{\partial f}{\partial z_k}(z) \right| (1 - |z_k|^2) < \infty,$$

and $|f(0)| + \sup_{z \in U^n} \sum_{k=1}^{n} |\frac{\partial f}{\partial z_k}(z)|(1 - |z_k|^2)$ is equivalent to the Bloch norm $\| f \|_{\mathcal{B}}$. This characterization was the starting point for introducing α-Bloch spaces. For $\alpha > 0$, the α -Bloch space $\mathcal{B}^\alpha(U^n)$ is defined as follows.

$$\mathcal{B}^\alpha(U^n) = \left\{ f \in H(U^n) : \sup_{z \in U^n} \sum_{k=1}^{n} \left| \frac{\partial f}{\partial z_k}(z) \right| (1 - |z_k|^2)^\alpha < \infty \right\}.$$

Recently, Li and Liu [4] introduced the notation of general weighted Bloch spaces (Stević called these the logarithmic Bloch-type spaces in [5]) in polydisk. For $\alpha > 0$, a function $f \in H(U^n)$ is said to belong to the general weighted Bloch space $\mathcal{B}^\alpha_{\log}(U^n)$ if

$$\sup_{z \in U^n} \sum_{k=1}^{n} \left| \frac{\partial f}{\partial z_k}(z) \right| (1 - |z_k|^2)^\alpha \log \frac{2}{1 - |z_k|^2} < \infty.$$

It is easy to show that $\mathcal{B}^\alpha_{\log}(U^n)$ is a Banach space with the norm

$$\| f \|_{\mathcal{B}^\alpha_{\log}} = |f(0)| + \sup_{z \in U^n} \sum_{k=1}^{n} \left| \frac{\partial f}{\partial z_k}(z) \right| (1 - |z_k|^2)^\alpha \log \frac{2}{1 - |z_k|^2}.$$

Composition operators on various Bloch-type spaces have been studied extensively by many authors. For the unit disk $U \subset \mathbb{C}$, Madigan and Matheson [6] proved that $C_\phi$is always bounded on $\mathcal{B}(U)$. They also gave some sufficient and necessary conditions that $C_\phi$is compact on $\mathcal{B}(U)$. Since then, there were many authors generalizing the results in [6] to the unit ball, polydisk and other classical symmetric domains, see, for example, [7-17]. At the same time, there were also many papers dealing with the composition operators between Bloch-type spaces and bounded holomorphic function spaces, refer to [18,19] and the references therein for the details. Specially, Li and Liu [4] stated and proved the corresponding boundedness and compactness characterizations for $C_\phi$from $H^\infty(U^n)$ to $\mathcal{B}^\alpha_{\log}(U^n)$. But there is a little gap in the proof [[4], line 17, p. 1637]. In this paper, we apply methods developed by Montes-Rodriguez [9] to give some estimates of the norm and essential norm of $C_\phi$from $H^\infty(U^n)$ to $\mathcal{B}^\alpha_{\log}(U^n)$. Recall that the essential norm $||T||_e$of a bounded operator T between Banach spaces X and Y is defined as the distance from T to the space of compact operators from X to Y. Notice that $||T||_e = 0$ if and only if T is compact, so that estimates on $||T||_e$lead to conditions for T to be compact. For convenience, we define $||T||_e= ||T|| = \infty$ for any unbounded linear operator T. As an application of our estimates, we obtain the main results in [4] with new proofs. In addition, we also show the equivalence of the compactness and weak compactness of $C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)$.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one occurrence to the others.

2 The norm of C_ϕ

In this section, we give the following estimate of the norm of $C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)$.

Theorem 1. *Let $\alpha > 0$ and $\phi = (\phi_1, ..., \phi_n)$ be a holomorphic self-map of the unit polydisk U^n, then*

$$\sup_{z\in U^n}\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^\alpha}{1-|(\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2} \lesssim \| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|$$
$$\lesssim 1+\sup_{z\in U^n}\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^\alpha}{1-|(\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}.$$

Here and in the sequel, the symbol $A\lesssim B$(or $B \gtrsim A$) means that $A \le CB$ for some positive constant C independent of A and B. $A \sim B$ means that $A \lesssim B$ and $B\lesssim A$.

Proof. For the lower estimate: $\| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\| \gtrsim \sup_{z\in U^n}\sum_{k,l=1}^{n}|\frac{\partial\varphi_l}{\partial z_k}(z)|\frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}$.
If $\| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\| = \infty$, then the result is trivially true. Now suppose $C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)$ is bounded. For any fixed $w \in U^n$ and $k \in \{1, ..., n\}$, take the following test function

$$f(z)=\frac{(1-|(\varphi_k(w)|^2)^\alpha}{(1-\overline{(\varphi_k(w)}z_k)^\alpha},\ z\in U^n.$$

Then, $f \in H^\infty(U^n)$ and $||f||_\infty \le 2^\alpha$. Fix any $\delta \in (0, 1)$. If $|\phi_k(w)| \ge \delta$, then

$$\begin{aligned}
\infty &> \| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\| \gtrsim \| C_\varphi f\|_{\mathcal{B}^\alpha_{\log}} \\
&\ge \sup_{z\in U^n}\sum_{j=1}^{n}\left|\frac{\partial(f\circ\varphi)}{\partial z_j}(z)\right|(1-|z_j|^2)^\alpha\log\frac{2}{1-|z_j|^2} \\
&= \sup_{z\in U^n}\sum_{j=1}^{n}\left|\frac{\partial f}{\partial w_k}(\varphi(z))\frac{\partial\varphi_k}{\partial z_j}(z)\right|(1-|z_j|^2)^\alpha\log\frac{2}{1-|z_j|^2} \\
&= \sup_{z\in U^n}\sum_{j=1}^{n}\left|\alpha\overline{\varphi_k(w)}\frac{(1-|(\varphi_k(w)^2|)^\alpha}{(1-\overline{\varphi_k(w)}\varphi_k(z))^{\alpha+1}}\frac{\partial\varphi_k}{\partial z_j}(z)\right|(1-|z_j|^2)^\alpha\log\frac{2}{1-|z_j|^2} \\
&\ge \sum_{j=1}^{n}\alpha\left|\varphi_k(w)\right|\frac{(1-|w_j|^2)^\alpha}{1-|(\varphi_k(w)|^2}\left|\frac{\partial\varphi_k}{\partial z_j}(w)\right|\log\frac{2}{1-|w_j|^2} \\
&\gtrsim \sum_{j=1}^{n}\left|\frac{\partial\varphi_k}{\partial z_j}(w)\right|\frac{(1-|w_j|^2)^\alpha}{1-|(\varphi_k(w)|^2}\log\frac{2}{1-|w_j|^2}.
\end{aligned}$$

If $|\phi_k(w)| < \delta$, then

$$\sum_{j=1}^{n}\left|\frac{\partial\varphi_k}{\partial z_j}(w)\right|\frac{(1-|w_j|^2)^\alpha}{1-|(\varphi_k(w)|^2}\log\frac{2}{1-|w_j|^2} \lesssim \sum_{j=1}^{n}\left|\frac{\partial\varphi_k}{\partial z_j}(w)\right|(1-|w_j|^2)^\alpha\log\frac{2}{1-|w_j|^2}$$
$$\le \| C_\varphi z_k\|_{\mathcal{B}^\alpha_{\log}} \le \| C_\varphi :\ H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|\| z_k\|_\infty = \| C_\varphi :\ H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\| < \infty.$$

That is, for any $w \in U^n$,

$$\sum_{j=1}^{n}\left|\frac{\partial\varphi_k}{\partial z_j}(w)\right|\frac{(1-|w_j|^2)^\alpha}{1-|(\varphi_k(w)|^2}\log\frac{2}{1-|w_j|^2}\lesssim\| C_\varphi: H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|.$$

Since $k \in \{1, ..., n\}$ is arbitrary, so

$$\sup_{w\in U^n}\sum_{k,j=1}^{n}\left|\frac{\partial\varphi_k}{\partial z_j}(w)\right|\frac{(1-|w_j|^2)^\alpha}{1-|(\varphi_k(w)|^2}\log\frac{2}{1-|w_j|^2}\lesssim\| C_\varphi: H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|.$$

For the upper estimate: $\| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|\lesssim 1+\sup_{z\in U^n}\sum_{k,l=1}^{n}|\frac{\partial\varphi_l}{\partial z_k}(z)|\frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}$. We also assume $\sup_{z\in U^n}\sum_{k,l=1}^{n}|\frac{\partial\varphi_l}{\partial z_k}(z)|\frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}<\infty$, since for the other case nothing needs to be proven. For any $f \in H^\infty(U^n)$,

$$\begin{aligned}
&\sum_{k=1}^{n}\left|\frac{\partial(f\circ\varphi)}{\partial z_k}(z)\right|(1-|z_k|^2)^\alpha\log\frac{2}{1-|z_k|^2}\\
&\le \sum_{k=1}^{n}\sum_{l=1}^{n}\left|\frac{\partial f}{\partial w_l}(\varphi(z))\frac{\partial\varphi_l}{\partial z_k}(z)\right|(1-|z_k|^2)^\alpha\log\frac{2}{1-|z_k|^2}\\
&\le \sum_{l=1}^{n}\left|\frac{\partial f}{\partial w_l}(\varphi(z))\right|(1-|\varphi_l(z)|^2)\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}\\
&\lesssim \left(\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}\right)\| f\|_{\mathcal{B}}\\
&\lesssim \left(\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}\right)\| f\|_{\infty},
\end{aligned}$$

where $\| f\|_{\mathcal{B}}\lesssim\| f\|_\infty$ is used in the last line above.
Since

$$\begin{aligned}
\| C_\varphi f\|_{\mathcal{B}^\alpha_{\log}} &= |f(\varphi(0))|+\sup_{z\in U^n}\sum_{k=1}^{n}\left|\frac{\partial(f\circ\varphi)}{\partial z_k}(z)\right|(1-|z_k|^2)^\alpha\log\frac{2}{1-|z_k|^2}\\
&\lesssim \left(1+\sup_{z\in U^n}\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}\right)\| f\|_\infty.
\end{aligned}$$

Taking the supremum over all $f \in H^\infty(U^n)$ with $\|f\|_\infty \le 1$, we have

$$\| C_\varphi: H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|\lesssim 1+\sup_{z\in U^n}\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2},$$

which completes the proof.

The following corollary is obtained immediately from Theorem 1.

Corollary 2. *Let $\phi = (\phi_1, ..., \phi_n)$ be a holomorphic self-map of U^n and $\alpha > 0$. Then, $C_\varphi: H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)$ is bounded if and only if*

$$\sup_{z\in U^n}\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}<\infty.$$

3 The essential norm of C_ϕ

This section mainly gives the following estimate of the essential norm of C_ϕ from $H^\infty(U^n)$ to $\mathcal{B}^\alpha_{\log}(U^n)$.

Theorem 3. *Let $\phi = (\phi_1, ..., \phi_n)$ be a holomorphic self-map of U^n and $\alpha > 0$, then*

$$\| C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)\|_e \sim \lim_{\delta\to 0} \sup_{dist(\varphi(z),\partial U^n)<\delta} \sum_{k,l=1}^n \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log\frac{2}{1-|z_k|^2}.$$

Proof. For the lower estimate:

$$\| C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)\|_e \gtrsim \lim_{\delta\to 0} \sup_{dist(\varphi(z),\partial U^n)<\delta} \sum_{k,l=1}^n \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log\frac{2}{1-|z_k|^2}.$$

It is trivial when $C_\phi$is unbounded. So we assume that $C_\phi$is bounded. By Corollary 2,

$$\sum_{k=1}^n \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log\frac{2}{1-|z_k|^2} \lesssim 1, \quad \forall l = 1, \ldots, n. \tag{1}$$

Take $f_m(z) = z_1^m (m \geq 2)$, then $||f_m||_\infty = 1$ and $f_m(z)$ converge to zero uniformly on any compact subset of U^n. So $\| Kf_m\|_{\mathcal{B}^\alpha_{\log}} \to 0$ for any compact operator $K : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)$. Then

$$\begin{aligned}
\| C_\varphi - K \| &\geq \limsup_{m\to\infty} \| (C_\varphi - K)f_m\|_{\mathcal{B}^\alpha_{\log}} = \limsup_{m\to\infty} \| C_\varphi f_m\|_{\mathcal{B}^\alpha_{\log}} \\
&\geq \limsup_{m\to\infty} \sup_{z\in U^n} \sum_{k=1}^n \left|\frac{\partial (f_m\circ\varphi)}{\partial z_k}(z)\right| (1-|z_k|^2)^\alpha \log\frac{2}{1-|z_k|^2} \\
&\geq \limsup_{m\to\infty} \sup_{z\in A_m} \sum_{k=1}^n \left|\frac{\partial (f_m\circ\varphi)}{\partial z_k}(z)\right| (1-|z_k|^2)^\alpha \log\frac{2}{1-|z_k|^2} \\
&= \limsup_{m\to\infty} \sup_{z\in A_m} \sum_{k=1}^n \left|m\varphi_1^{m-1}(z)\frac{\partial \varphi_1}{\partial z_k}(z)\right| (1-|z_k|^2)^\alpha \log\frac{2}{1-|z_k|^2} \\
&= \limsup_{m\to\infty} \sup_{z\in A_m} \sum_{k=1}^n \left|\frac{\partial \varphi_1}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_1(z)|^2} \log\frac{2}{1-|z_k|^2} m|(\varphi_1(z)|^{m-1}(1-|(\varphi_1(z)|^2) \\
&\geq \limsup_{m\to\infty} \sup_{z\in A_m} \sum_{k=1}^n \left|\frac{\partial \varphi_1}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_1(z)|^2} \log\frac{2}{1-|z_k|^2} \liminf_{m\to\infty} \min_{z\in A_m} m|\varphi_1(z)|^{m-1}(1-|(\varphi_1(z)|^2),
\end{aligned} \tag{2}$$

where $A_m = \{z \in U^n: r_m \leq |\phi_1(z)| \leq r_{m+1}\}$, $r_m = (\frac{m-1}{m+1})^{1/2}$. Since $y = mx^{m-1}(1 - x^2)$, $x \in [0, 1)$, is increasing on $[0, r_m]$ and decreasing on $[r_m, 1)$, $\min_{z\in A_m} m|\varphi_1(z)|^{m-1}(1-|\varphi_1(z)|^2) = (\frac{m}{m+2})^{\frac{m-1}{2}} \frac{2m}{m+2} \to \frac{2}{e} (\text{as } m \to \infty)$. From (2), we have

$$\| C_\varphi - K \| \gtrsim \limsup_{m\to\infty} \sup_{z\in A_m} \sum_{k=1}^n \left|\frac{\partial \varphi_1}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_1(z)|^2} \log\frac{2}{1-|z_k|^2}.$$

It is from (1) that

$$\lim_{\delta\to 0} \sup_{dist(\varphi(z),\partial U^n)<\delta} \sum_{k=1}^n \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log\frac{2}{1-|z_k|^2} = a_l < \infty, \quad \forall l = 1, \ldots, n.$$

Then, for any $\varepsilon > 0$, there is $\delta_0 \in (0, 1)$ such that

$$\sum_{k=1}^n \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log\frac{2}{1-|z_k|^2} > a_l - \varepsilon,$$

whenever $dist\ (\phi(\mathrm{z}), \partial U^n) < \delta_0$. Again $r_m \uparrow 1$, so for m large enough,

$$\sup_{z\in A_m}\sum_{k=1}^{n}\left|\frac{\partial\varphi_1}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^{\alpha}}{1-|(\varphi_1(z)|^2}\log\frac{2}{1-|z_k|^2} > a_1-\varepsilon.$$

So $||C_\phi - K|| \gtrsim a_1 - \varepsilon$. Since K is arbitrary,

$$\| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|_e \gtrsim a_1-\varepsilon.$$

Similarly, considering the functions $f(z) = z_l^m$, $l = 2, \ldots, n$, we also have

$$\| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|_e \gtrsim a_l-\varepsilon.$$

Thus,

$$\| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|_e \gtrsim \sum_{l=1}^{n} a_l-\varepsilon,$$
$$=\lim_{\delta\to 0}\sup_{dist(\varphi(z),\partial U^n)<\delta}\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^{\alpha}}{1-|\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}-\varepsilon.$$

Again ε is arbitrary, and we obtain the desired lower estimate.
For the upper estimate:

$$\| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|_e \lesssim \lim_{\delta\to 0}\sup_{dist(\varphi(z),\partial U^n)<\delta}\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^{\alpha}}{1-|(\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}.$$

If $\lim\limits_{\delta\to 0}\sup\limits_{dist(\varphi(z),\partial U^n)<\delta}\sum_{k,l=1}^{n}|\frac{\partial\varphi_l}{\partial z_k}(z)|\frac{(1-|z_k|^2)^{\alpha}}{1-|\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}=\infty$, then the estimate is trivial too. Now we suppose $\lim\limits_{\delta\to 0}\sup\limits_{dist(\varphi(z),\partial U^n)<\delta}\sum_{k,l=1}^{n}|\frac{\partial\varphi_l}{\partial z_k}(z)|\frac{(1-|z_k|^2)^{\alpha}}{1-|\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}=\infty$, then $C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)$ is bounded by Corollary 2. Define the operators $K_m (m \geq 2)$ as follows

$$K_m f(z) = f\left(\frac{m-1}{m}z\right).$$

It is easy to see that K_m: $H^\infty(U^n) \to H^\infty(U^n)$ is compact since K_m maps every bounded sequence in $H^\infty(U^n)$ converging to zero on compact subsets of U^n to the sequence converging to zero in norm of $H^\infty(U^n)$. In addition, $||I - K_m: H^\infty(U^n) \to H^\infty(U^n)|| \leq 2$. Therefore, $C_\varphi K_m : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)$ is compact. Then

$$\begin{aligned}\| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|_e &\leq \| C_\varphi - C_\varphi K_m\| = \| C_\varphi(I-K_m)\|\\ &= \sup_{\|f\|_\infty\leq 1}\| C_\varphi(I-K_m)f\|_{\mathcal{B}^\alpha_{\log}} = \sup_{\|f\|_\infty\leq 1}|(I-K_m)f(\varphi(0))|\\ &+\sup_{\|f\|_\infty\leq 1}\sup_{z\in U^n}\sum_{k,l=1}^{n}\left|\frac{\partial(I-K_m)f}{\partial w_l}(\varphi(z))\frac{\partial\varphi_l}{\partial z_k}(z)\right|(1-|z_k|^2)^{\alpha}\log\frac{2}{1-|z_k|^2} = I_1+I_2.\end{aligned}$$

Where $I_1 = \sup\limits_{\|f\|_\infty\leq 1}|(I-K_m)f(\varphi(0))|$ and $I_2 = \sup\limits_{\|f\|_\infty\leq 1}\sup\limits_{z\in U^n}\sum_{k,l=1}^{n}|\frac{\partial(I-K_m)f}{\partial w_l}(\varphi(z))\frac{\partial\varphi_l}{\partial z_k}(z)|(1-|z_k|^2)^{\alpha}\log\frac{2}{1-|z_k|^2}$.

Fix $\delta \in (0, 1)$ and let $G_1 = \{z \in U^n: dist(\phi(z), \partial U^n) < \delta\}$, $G_2 = U^n \backslash G_1 = \{z \in U^n: dist(\phi(z), \partial U^n) \geq \delta\}$, which is a compact subset of U^n.

$$I_2 = \sup_{\|f\|_\infty \le 1} \sup_{z \in G_1} \sum_{k,l=1}^{n} \left| \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) \frac{\partial \varphi_l}{\partial z_k}(z) \right| (1-|z_k|^2)^\alpha \log \frac{2}{1-|z_k|^2}$$
$$+ \sup_{\|f\|_\infty \le 1} \sup_{z \in G_2} \sum_{k,l=1}^{n} \left| \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) \frac{\partial \varphi_l}{\partial z_k}(z) \right| (1-|z_k|^2)^\alpha \log \frac{2}{1-|z_k|^2} = J_1 + J_2.$$

Where

$$\begin{aligned}
J_1 &= \sup_{\|f\|_\infty \le 1} \sup_{z \in G_1} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} \left| \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) \right| (1-|\varphi_l(z)|^2) \\
&\lesssim \sup_{\|f\|_\infty \le 1} \sup_{z \in G_1} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} \| (I-K_m)f \|_{\mathcal{B}} \\
&\lesssim \sup_{\|f\|_\infty \le 1} \sup_{z \in G_1} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} \| (I-K_m)f \|_{\infty} \\
&\le \sup_{z \in G_1} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} \| I-K_m \| \\
&\lesssim \sup_{z \in G_1} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^\alpha}{1-|(\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2}.
\end{aligned}$$

And

$$\begin{aligned}
J_2 &= \sup_{\|f\|\infty \le 1} \sup_{z \in G_2} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^\alpha}{1-|(\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} | \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) | (1-|(\varphi_l(z)|^2) \\
&\lesssim \sum_{l=1}^{n} \sup_{\|f\|_\infty \le 1} \sup_{z \in G_2} \left| \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) \right|.
\end{aligned}$$

It is clear that the sequence of operators $\{I - K_m\}_m$satisfies $\lim_{m \to \infty} (I-K_m)f = 0$ for each $f \in H(U^n)$, and the space $H(U^n)$ endowed with the compact open topology τ is a Fréchet space. Further, D_j: $(H(U^n), \tau) \to (H(U^n), \tau)$ defined by $D_j f = \frac{\partial f}{\partial z_j}$ is a contionuous linear operator. Therefore, by the Banach-Steinhaus theorem, the sequence $\{D_j^\circ (I - K_m)\}_m$converges to zero uniformly on compact subsets of $(H(U^n), \tau)$. Since, by Montel's normal theorem, the closed unit ball of $H^\infty(U^n)$ is a compact subset of $(H(U^n), \tau)$, we conclude that

$$\lim_{m \to \infty} \sup_{\|f\|_\infty \le 1} \sup_{z \in G_2} \left| \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) \right| = 0, \ l = 1, \ldots, n.$$

Thence, $J_2 \to 0$ (as $m \to \infty$).

Similarly, we know that $I_1 = \sup_{\|f\|_\infty \le 1} |(I-K_m)f(\varphi(0))| \to 0, \text{ (as } m \to \infty).$

Consequently,

$$\begin{aligned}
\| C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n) \|_e &\le \limsup_{m \to \infty} \| C_\varphi (I-K_m) \| \\
&\le \limsup_{m \to \infty} I_1 + \limsup_{m \to \infty} J_1 + \limsup_{m \to \infty} J_2 \\
&\lesssim \sup_{dist(\varphi(z), \partial U^n) < \delta} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2}.
\end{aligned}$$

Thus

$$\| C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n) \|_e \leq \lim_{\delta \to 0} \sup_{dist(\varphi(z), \partial U^n) < \delta} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2}.$$

The proof is complete.

As an application, we have the following corollary.

Corollary 4. *Let* $\phi = (\phi_1, ..., \phi_n)$ *be a holomorphic self-map of* U^n *and* $\alpha > 0$. *Then the following are equivalent.*

(1) $C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)$ *is compact.*

(2) $C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)$ *is weakly compact.*

(3) $\lim_{\varphi(z) \to \partial U^n} \sum_{k,l=1}^{n} |\frac{\partial \varphi_l}{\partial z_k}(z)| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} = 0.$

Proof. (1)⇒ (2) is obvious, and (3)⇒ (1) follows immediately from Theorem 3. So it suffices to prove (2)⇒ (3). Now assume that $C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)$ is weakly compact. If (3) is not true, then there is a sequence $\{z^j\} \subset U^n$ and $\varepsilon_0 > 0$ such that $w^j = \phi(z^j) \to \partial U^n$ (as $j \to \infty$) together with

$$\sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z^j) \right| \frac{(1-|z_k^j|^2)^\alpha}{1-|(\varphi_l(z^j)|^2} \log \frac{2}{1-|z_k^j|^2} \geq \varepsilon_0, \tag{3}$$

for each $j \geq 1$. Since C_ϕ is weakly compact, C_ϕ is bounded. Then, by Corollary 2,

$$\sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z^j) \right| \frac{(1-|z_k^j|^2)^\alpha}{1-|\varphi_l(z^j)|^2} \log \frac{2}{1-|z_k^j|^2} \lesssim 1.$$

Extracting a subsequence of $\{z^j\}$, if needed, we may assume that $\lim_{j \to \infty} |\varphi_l(z^j)|$ exists for every l and

$$\left| \frac{\partial \varphi_l}{\partial z_k}(z^j) \right| \frac{(1-|z_k^j|^2)^\alpha}{1-|(\varphi_l(z^j)|^2} \log \frac{2}{1-|z_k^j|^2} \to a_{lk} \in [0, \infty), \quad (\text{as } j \to \infty).$$

From (3), there are k_0 and l_0 such that $a_{l_0 k_0} > 0$, i.e.,

$$\left| \frac{\partial \varphi_{l_0}}{\partial z_{k_0}}(z^j) \right| \frac{(1-|z_{k_0}^j|^2)^\alpha}{1-|(\varphi_{l_0}(z^j)|^2} \log \frac{2}{1-|z_{k_0}^j|^2} \to a_{l_0 k_0} > 0. \tag{4}$$

If $|w_{l_0}^j| \to 1$, define $f_j(z) = \dfrac{1-|w_{l_0}^j|^2}{1-z_{l_0}\overline{w_{l_0}^j}}$. Then, the sequence $\{f_j\}_j \subset H^\infty(U^n)$ is bounded and converges to zero uniformly on any compact subset of U^n. That is, $\{f_j\}$ weakly converges to zero in $H^\infty(U^n)$. Because $H^\infty(U^n)$ has Dunford-Pettis property (See Theorem 5.3 in [20] for $H^\infty(U)$, and note the proof there works also for $H^\infty(U^n)$), the weak compactness of $C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)$ implies that $\| C_\varphi f_j \|_{\mathcal{B}^\alpha_{\log}} \to 0 (\text{as } j \to \infty)$. But this is impossible since using (4) we may estimate that for each $j \geq 1$,

$$\begin{aligned}
\| C_\varphi f_j \|_{\mathcal{B}^\alpha_{\log}} &\geq \sum_{k=1}^{n} \left| \frac{\partial (f_j \circ \varphi)}{\partial z_k}(z^j) \right| (1 - |z^j_k|^2)^\alpha \log \frac{2}{1 - |z^j_k|^2} \\
&= \sum_{k=1}^{n} \left| \frac{\partial f_j}{\partial w_{l_0}}(\varphi(z^j)) \frac{\partial \varphi_{l_0}}{\partial z_k}(z^j) \right| (1 - |z^j_k|^2)^\alpha \log \frac{2}{1 - |z^j_k|^2} \\
&= |w^j_{l_0}| \sum_{k=1}^{n} \frac{(1 - |z^j_k|^2)^\alpha}{1 - |\varphi_{l_0}(z^j)|^2} \left| \frac{\partial \varphi_{l_0}}{\partial z_k}(z^j) \right| \log \frac{2}{1 - |z^j_k|^2} \\
&\geq |w^j_{l_0}| \left| \frac{\partial \varphi_{l_0}}{\partial z_{k_0}}(z^j) \right| \frac{(1 - |z^j_{k_0}|^2)^\alpha}{1 - |\varphi_{l_0}(z^j)|^2} \log \frac{2}{1 - |z^j_{k_0}|^2} \to a_{l_0 k_0} > 0.
\end{aligned}$$

If $|w^j_{l_0}| \to \rho < 1$. Since $w^j \to \partial U^n$, there is $l_1 \in \{1, \ldots, n\} \backslash \{l_0\}$ such that $|w^j_{l_1}| \to 1$. If there exists k_1 such that

$$\left| \frac{\partial \varphi_{l_1}}{\partial z_{k_1}}(z^j) \right| \frac{(1 - |z^j_{k_1}|^2)^\alpha}{1 - |\varphi_{l_1}(z^j)|^2} \log \frac{2}{1 - |z^j_{k_1}|^2} \to a_{l_1 k_1} > 0,$$

then as in the last paragraph above we obtain the desired contradiction using the following test functions:

$$g_j(z) = \frac{1 - |w^j_{l_1}|^2}{1 - z_{l_1} \overline{w^j_{l_1}}}.$$

Thus, we may assume that

$$\left| \frac{\partial \varphi_{l_1}}{\partial z_k}(z^j) \right| \frac{(1 - |z^j_k|^2)^\alpha}{1 - |(\varphi_{l_1}(z^j)|^2} \log \frac{2}{1 - |z^j_k|^2} \to 0, \ (\text{as } j \to \infty), \tag{5}$$

for each k. We now define the test functions $h_j$as follows

$$h_j(z) = (z_{l_0} + 2) \frac{1 - |w^j_{l_1}|^2}{1 - z_{l_1} \overline{w^j_{l_1}}}.$$

Then, $||h_j||_\infty \lesssim 1$ and $h_j$converge to zero uniformly on any compact subset of U^n. But for any j large enough

$$\begin{aligned}
\| C_\varphi h_j \|_{\mathcal{B}^\alpha_{\log}} &\geq \sum_{k=1}^{n} \left| \frac{\partial (h_j \circ \varphi)}{\partial z_k}(z^j) \right| (1 - |z^j_k|^2)^\alpha \log \frac{2}{1 - |z^j_k|^2} \\
&= \sum_{k=1}^{n} \left| \frac{\partial \varphi_{l_0}}{\partial z_k}(z^j) + (w^j_{l_0} + 2) \overline{w^j_{l_1}} \frac{1}{1 - |w^j_{l_1}|^2} \frac{\partial \varphi_{l_1}}{\partial z_k}(z^j) \right| |(1 - |z^j_k|^2)^\alpha \log \frac{2}{1 - |z^j_k|^2} \\
&\geq \sum_{k=1}^{n} (1 - |z^j_k|^2)^\alpha \left| \frac{\partial \varphi_{l_0}}{\partial z_k}(z^j) \right| \log \frac{2}{1 - |z^j_k|^2} - \sum_{k=1}^{n} \left| w^j_{l_1} ||w^j_{l_0} + 2|| \frac{\partial \varphi_{l_1}}{\partial z_k}(z^j) \right| \frac{(1 - |z^j_k|^2)^\alpha}{1 - |w^j_{l_1}|^2} \log \frac{2}{1 - |z^j_k|^2} \\
&\gtrsim \sum_{k=1}^{n} (1 - |z^j_k|^2)^\alpha \left| \frac{\partial \varphi_{l_0}}{\partial z_k}(z^j) \right| \log \frac{2}{1 - |z^j_k|^2} \\
&\gtrsim (1 - |z^j_{k_0}|^2)^\alpha \left| \frac{\partial \varphi_{l_0}}{\partial z_{k_0}}(z^j) \right| \log \frac{2}{1 - |z^j_{k_0}|^2} \\
&\gtrsim \frac{(1 - |z^j_{k_0}|^2)^\alpha}{1 - |(\varphi_{l_0}(z^j)|^2} \left| \frac{\partial \varphi_{l_0}}{\partial z_{k_0}}(z^j) \right| \log \frac{2}{1 - |z^j_{k_0}|^2} \to a_{l_0 k_0} > 0,
\end{aligned}$$

the inequalities in the third and fourth lines above follow from (5), and the last line is due to $|\varphi_{l_0}(z^j)| \to \rho < 1$. This contradicts again $\| C_\varphi h_j \|_{\mathcal{B}^\alpha_{\log}} \to 0$, which completes the proof.

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Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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The authors declare that they have no competing interests.

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References

1. Cowen, C, MacCluer, B: Composition Operators on Spaces of Analytic Functions. CRC Press, Boca Raton (1995)
2. Shapiro, J: Composition Operators and Classical Function Theory. Springer, New York (1993)
3. Timoney, R: Bloch functions in several complex variables I. Bull Lond Math Soc. **12**, 241–267 (1980). doi:10.1112/blms/12.4.241
4. Li, H, Liu, P: Composition operators between H^∞ and generally weighted Bloch spaces in polydisk. Acta Math Sci. **29A**, 1634–1641 (2009)
5. Stević, S, Agarwal, R: Weighted composition operators from logarithmic Bloch-type spaces to Bloch-type spaces. J Inequal Appl **21** (2009). Article ID 964814
6. Madigan, K, Matheson, A: Compact composition operators on the Bloch space. Trans Amer Math Soc. **347**, 2679–2687 (1995). doi:10.2307/2154848
7. Stević, S: On new Bloch-type spaces. Appl Math Comput. **215**, 841–849 (2009). doi:10.1016/j.amc.2009.06.009
8. MacCluer, B, Zhao, R: Essential norms of weighted composition operators between Bloch-type spaces. Rocky Mountain J Math. **33**, 1437–1458 (2003). doi:10.1216/rmjm/1181075473
9. Montes-Rodriguez, A: The essential norm of a composition operator on Bloch spaces. Pac J Math. **188**, 339–351 (1999). doi:10.2140/pjm.1999.188.339
10. Zhao, R: Essential norms of composition operators between Bloch type spaces. Proc Am Math Soc. **138**, 2537–2546 (2010). doi:10.1090/S0002-9939-10-10285-8
11. Zhou, Z: Composition operators on the Lipschitz spaces in polydiscs. Sci China A. **46**, 33–381 (2003)
12. Zhou, Z, Shi, J: Composition operators on the Bloch spaces in polydiscs. Complex Var. **46**, 73–88 (2001)
13. Lindström, M, Makhmutov, S, Taskinen, J: The essential norm of a Bloch-to-$Q_p$composition operator. Can Math Bull. **47**, 49–59 (2004). doi:10.4153/CMB-2004-007-6
14. Montes-Rodriguez, A: The Pick-Schwarz lemma and composition operators on Bloch spaces. Rend Circ Mat Palermo. **56**(2 Suppl), 167–170 (1998)
15. Zhou, Z, Shi, J: Compactness of composition operators on the Bloch space in classical bounded symmetric domains. Mich Math J. **50**, 381–405 (2002). doi:10.1307/mmj/1028575740
16. Zhou, Z, Liu, Y: The essential norms of composition operators between generalized Bloch spaces in the polydisc and their applications. J Inequal Appl **22** (2006). Article ID 90742
17. Montes-Rodriguez, A: Weighted composition operators on weighted Banach spaces of analytic functions. J Lond Math Soc. **61**, 872–884 (2000). doi:10.1112/S0024610700008875
18. Ohno, S: Weighted composition operators between H^∞ and the Bloch space. Taiwan J Math. **5**, 555–563 (2001)
19. Stević, S: Composition operators between H^∞ and α-Bloch spaces on the polydisc. Zeitschrift für Anal und ihre Anwendungen. **25**, 457–466 (2006)
20. Bourgain, J: New Banach space properties of the disc algebra and H^∞. Acta Math. **152**, 1–48 (1984). doi:10.1007/BF02392189

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