

Full Length Research Paper

On best 1:1 codes for generalized quantitative-qualitative measure of inaccuracy

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In coding theory, generally, we come across the problem of efficient coding of messages to be sent over a noiseless channel and attempt to maximize the number of messages that can be sent through a channel in a given time. Therefore, we find the minimum value of a mean codeword length subject to a given constraint on codeword lengths. As the codeword lengths are integers, the minimum value lies between two bounds, so, a noiseless coding theorem seeks to find these bounds for a given mean and a given constraint. In this paper, we are modifying the noiseless coding theorem as obtained by Bilal and Pirzada (2006) for binary, as well as D size 1:1 codes. The coding theorem as obtained by us is less constrained.

Key words: Quantitative-qualitative measure, inaccuracy, noiseless coding theorem.

INTRODUCTION

For uniquely decipherable codes, Shannon (1948) found the lower bounds for the arithmetic mean by using his entropy. Longo (1976) obtained the lower bound for useful mean codeword length in terms of quantitative-qualitative measure of entropy, introduced by Belis and Guiasu (1968). Guiasu and Picard (1971) proved a noiseless coding theorem and obtained the lower bounds for similar useful mean codeword length. Gurdial and Pessoa (1977) extended this by finding the lower bounds for useful mean codeword length of order α and type β . Further, it is important to note that for other standard means like the geometric mean, the harmonic mean and the power mean, the lower bounds can be found in principle, but except for the arithmetic mean, no closed expression for the lower bounds can be obtained.

For a pair of discrete probability distribution $P = \{(p_1, p_2, \dots, p_n), p_i > 0, \sum_{i=1}^n p_i = 1\}$ together with a utility distribution $U = (u_1, u_2, \dots, u_n)$, where $u_i > 0$ is the utility of the i^{th} event, where probability of occurrence is $p_i > 0$. Belis and Guiasu (1968) proposed a measure of

information, called the useful information by Longo (1976), as:

$$H(P; U) = - \sum_{i=1}^N u_i p_i \log p_i \quad (1)$$

If we consider P as a probability of a set of N events on the basis of an experiment whose prior probability distribution is $Q = \{(q_1, q_2, \dots, q_n), q_i > 0, \sum_{i=1}^N q_i = 1\}$, then Kullback – Leibler's measure (1981) of inaccuracy that P provides about Q is given by:

$$H(P / Q) = \sum_{i=1}^N p_i \log \left(\frac{p_i}{q_i} \right) \quad (2)$$

Taneja and Tuteja (1984), considering u_i independent of p_i and directly proportional to its importance, characterized axiomatically the quantitative-qualitative measure of relative information on:

$$H(P / Q; U) = \sum_{i=1}^N u_i p_i \log \left(\frac{p_i}{q_i} \right) \quad (3)$$

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The measure (Equation 3) was characterized and generalized for complete probability distribution by various authors. Bhaker and Hooda (1993) considered P and Q as the posterior and prior generalized probability distributions, respectively of an experiment having utility distribution U and characterized the following measures of useful relative information:

$$H(P/Q;U) = \frac{\sum_{i=1}^N u_i p_i \log\left(\frac{p_i}{q_i}\right)}{\sum_{i=1}^N u_i p_i} \quad (4)$$

and

$$H_\alpha(P/Q;U) = \frac{1}{1-\alpha} \frac{\sum_{i=1}^N u_i p_i^\alpha q_i^{1-\alpha}}{\sum_{i=1}^N u_i p_i}, \alpha \neq 1 \quad (5)$$

It is interesting to note that the measure (Equation 5) tends to measure (Equation 4) when $\alpha \rightarrow 1$ and further to Kullback-Leibler's (1981) measure in case the utilities are ignored, that is, $u_i = 1$ for each i .

In the literature of Aczel and Daroezy (1975) and Guiasu and Picard (1971), there exists a number of parametric generalized measures of information. Longo (1976), Gurdial and Pessoa (1977), Autar and Khan (1989) and Khan Bilal and Pirzada studied the generalized coding theorem by considering different generalized measures proposed by various authors, for example, Belis and Guiasu (1968) and Guiasu and Picard (1971) under the condition of unique decipherability (Feinstein, 1958).

Bilal and Pirzada (2006) considered a new function:

$$H_\alpha^\beta(P/Q;U) = \frac{1}{1-\alpha} \log_D \left\{ \frac{\sum_{i=1}^N u_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^N u_i p_i} \right\},$$

where

$$\alpha \neq 1, \beta > 0, \alpha + \beta > 1, p_i > 0, u_i > 0,$$

$$\sum_{i=1}^N p_i = 1 \text{ and } \sum_{i=1}^N a_i = 1 \quad (6)$$

and corresponding by introducing a parametric mean

length tLu_α^β called the generalized useful mean length of codewords weighted with the function of utilities and power probabilities as:

$$tLu_\alpha^\beta = \frac{1}{t} \log_D \left\{ \sum_{i=1}^N p_i^\beta \left(\frac{u_i}{\sum_{i=1}^N u_i p_i} \right)^{t+1} D^{l_i} \right\}, t \neq 0 \quad (7)$$

They obtained the noiseless coding theorem:

$$H_\alpha^\beta(P/Q;U) \leq tLu_\alpha^\beta < H_\alpha^\beta(P/Q;U) + 1 \quad (8)$$

under the modify Kraft's inequality $\sum_{i=1}^N p_i^{\beta-1} D^{-l_i} q_i \leq 1$.

Our motivation for studying this paper is to modify the aforementioned coding theorem for 1:1 codes of D size and binary codes, which is less constraint.

MAIN RESULTS ON CODING THEOREMS

Let X be a random variable taking on a finite number of values x_1, x_2, \dots, x_N with probabilities (p_1, p_2, \dots, p_N) and utilities $(\mu_1, \mu_2, \dots, \mu_N)$. Let $l_i, i = 1, 2, \dots, N$ be the lengths of the code words in the best 1:1 binary code (0, 1, 00, 10, 01, 11, 000, ...) for encoding the random variable X, l_i is the length of the code word assigned to the output X_i . It is clear that $l_1 \leq l_2 \leq l_3 \leq \dots$ and in general, $l_i =$

$\left\lceil \log_2 \frac{i+2}{2} \right\rceil$, where $\lceil S \rceil$ denotes the smallest integer greater than or equal to S.

Thus the average 'useful' codeword length for the best 1:1 binary code can be defined as:

$$tL_{1:1} \mu_\alpha^\beta = \frac{1}{t} \log_2 \left\{ \sum_{i=1}^N p_i^\beta \left(\frac{u_i}{\sum_{i=1}^N u_i p_i} \right)^{t+1} 2^{\left\lceil \log_2 \left(\frac{i+2}{2} \right) \right\rceil} \right\} \quad (9)$$

We will now prove the following theorem.

Theorem

For $H_\alpha^\beta(P/Q;U), tLu_\alpha^\beta$ and $tL_{1:1} \mu_\alpha^\beta$ as defined in

Equations (6), (7) and (9), prove the following inequalities:

$$tL_{1:1}\mu_\alpha^\beta \geq H_\alpha^\beta(P/Q;U) - \log_2$$

$$\left[\sum_{i=1}^N p_i^{\beta-1} q_i \left(\frac{i+2}{2} \right)^{-1} \right]. \quad (10)$$

OR

$$tL_{1:1}\mu_\alpha^\beta \geq t\mu_\alpha^\beta - \log_2 \left[\sum_{i=1}^N p_i^{\beta-1} q_i \left(\frac{1}{i+2} \right) \right] - 2. \quad (11)$$

Proof

From Equation (9), we have:

$$tL_{1:1}\mu_\alpha^\beta \geq \frac{1}{t} \log_2 \left[\sum_{i=1}^N p_i^\beta \left(\frac{\mu_i}{\sum_{i=1}^N \mu_i p_i} \right)^{t+1} \left(\frac{i+2}{2} \right)^t \right]. \quad (12)$$

Now

$$\begin{aligned} H_\alpha^\beta(P/Q;U) - tL_{1:1}\mu_\alpha^\beta &\leq \frac{1}{1-\alpha} \log_2 \left[\frac{\sum_{i=1}^N \mu_i^{\alpha\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^N \mu_i p_i} \right] \\ &\quad - \frac{1}{t} \log_2 \left[\sum_{i=1}^N p_i^\beta \left(\frac{\mu_i}{\sum_{i=1}^N \mu_i p_i} \right)^{t+1} \left(\frac{i+2}{2} \right)^t \right] \\ &\leq \frac{1+t}{t} \log_2 \left[\frac{\sum_{i=1}^N \mu_i^{\alpha\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^N \mu_i p_i} \right] - \frac{1}{t} \log_2 \left[\sum_{i=1}^N p_i^\beta \left(\frac{\mu_i}{\sum_{i=1}^N \mu_i p_i} \right)^{t+1} \left(\frac{i+2}{2} \right)^t \right] \\ &\leq \log_2 \left[\frac{\sum_{i=1}^N \mu_i^{\alpha\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^N \mu_i p_i} \right]^{\frac{1+t}{t}} + \log_2 \left[\sum_{i=1}^N p_i^\beta \left(\frac{\mu_i}{\sum_{i=1}^N \mu_i p_i} \right)^{t+1} \left(\frac{i+2}{2} \right)^t \right]^{\frac{1}{t}} \end{aligned}$$

$$\leq \log_2 \left[\sum_{i=1}^N \mu_i p_i^{\alpha\beta-1} q_i^{1-\alpha} \right]^{\frac{1+t}{t}} \cdot \left[\sum_{i=1}^N p_i^\beta \mu_i^{t+1} \left(\frac{i+2}{2} \right)^t \right]^{\frac{1}{t}}$$

After applying holders inequality, we have:

$$\leq \log_2 \left[\sum_{i=1}^N p_i^{\beta-1} q_i \left(\frac{i+2}{2} \right)^{-1} \right], \quad B > 0 \quad (13)$$

which gives Equation (10). Using the result of Equation (8):

$$tLu_{i\alpha}^\beta < 1 + H_\alpha^\beta(P/Q;U) \quad \text{Or}$$

$$tLu_{i\alpha}^\beta - tL_{1:1} < 1 + H_\alpha^\beta(P/Q;U) - tL_{1:1}\mu_\alpha^\beta$$

$$\leq 1 + \log_2 \left[\sum_{i=1}^N p_i^\beta q_i \left(\frac{2}{i+2} \right) \right], \quad B > 0$$

$$\leq 2 + \log_2 \left[\sum_{i=1}^N \left(p_i^\beta q_i \left(\frac{1}{i+2} \right) \right) \right], \quad B > 0 \quad (14)$$

which proves Equation (11).

Particular cases

- 1) If $\beta = 1$, $t \rightarrow 0$ and $\mu_i = 1 \quad \forall i$, then result reduces as proved by Shannon (1948).
- 2) If $\beta = 1$, $t \rightarrow 0$ then result reduces as proved by Guiasu and Picard (1971).

In general, when the code alphabet is of order D, then the set of available codeword is $\{0, 1, 2, \dots, (D-1); 00, 01, \dots, (D-1)(D-1); \dots\}$ and by inspection, we have $l_1 = 1, l_2 = 1, \dots, l_D = 1, l_{D+1} = 2, \dots; l_{D(D+1)} = 2$ etc., and thus $l_i =$

$$\left\lceil \log_D \left(\frac{(D-1)}{D} i + 1 \right) \right\rceil,$$

where $\lceil S \rceil$ denotes the smallest integer greater than or equal to a.

Thus the average 'useful' codeword length for the best 1:1 'D' size code can be defined as:

$$tL_{1:1}\mu_\alpha^\beta = \frac{1}{t} \log_D \left[\sum_{i=1}^N p_i^\beta \left(\frac{\mu_i}{\sum_{i=1}^N \mu_i p_i} \right)^{t+1} \right] D \left\lceil \log_D \left(\frac{(D-1)}{D} i + 1 \right) \right\rceil \quad t \neq 0 \quad (15)$$

We prove the following theorem.

Theorem

For $H_\alpha^\beta(P/Q;U)$, $tL\mu_\alpha^\beta$ and $tL_{1:1}\mu_\alpha^\beta$ as defined in Equations (6), (7) and (15), prove the following inequalities:

$$tL_{1:1}\mu_\alpha^\beta \geq H_\alpha^\beta(P/Q;U) - \log_D \left[\sum_{i=1}^N p_i^{\beta-1} q_i \left(\frac{D}{(D-1)i+D} \right) \right] \quad (16)$$

OR

$$tL_{1:1}\mu_\alpha^\beta \geq tL\mu_\alpha^\beta - \log_D \left[\sum_{i=1}^N p_i^{\beta-1} q_i \left(\frac{1}{(D-1)i+D} \right) \right] - 2. \quad (17)$$

Proof

From Equation (15), we have:

$$tL_{1:1}\mu_\alpha^\beta \geq \frac{1}{t} \log_D \left[\sum_{i=1}^N p_i^\beta \left(\frac{\mu_i}{\sum_{i=1}^N \mu_i p_i} \right)^{t+1} \left(\frac{(D-1)i+D}{D} \right)^t \right]. \quad (18)$$

Now

$$H_\alpha^\beta(P/Q;U) - tL_{1:1}\mu_\alpha^\beta \leq \frac{1}{1-\alpha} \log_D \left[\sum_{i=1}^N \left(\frac{\mu_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^N \mu_i p_i} \right) \right]$$

$$- \frac{1}{t} \log_D \left[\sum_{i=1}^N p_i^\beta \left(\frac{\mu_i}{\sum_{i=1}^N \mu_i p_i} \right)^{t+1} \left(\frac{(D-1)i+D}{D} \right)^t \right].$$

$$\leq \frac{1+t}{t} \log_D \left[\sum_{i=1}^N \frac{\mu_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^N \mu_i p_i} \right] - \frac{1}{t} \log_D \left[\sum_{i=1}^N p_i^\beta \left(\frac{\mu_i}{\sum_{i=1}^N \mu_i p_i} \right)^{t+1} \left(\frac{(D-1)i+D}{D} \right)^t \right]$$

$$\leq \log_D \left[\sum_{i=1}^N \frac{\mu_i p_i^{\alpha+\beta-1} q_i^{1-\alpha}}{\sum_{i=1}^N \mu_i p_i} \right]^{\frac{1+t}{t}} \cdot \left[\sum_{i=1}^N p_i^\beta \left(\frac{\mu_i}{\sum_{i=1}^N \mu_i p_i} \right)^{t+1} \left(\frac{(D-1)i+D}{D} \right)^t \right]^{\frac{1}{t}}$$

$$\leq \log_D \left[\sum_{i=1}^N \mu_i p_i^{\alpha+\beta-1} q_i^{1-\alpha} \right]^{\frac{1+t}{t}} \cdot \left[\sum_{i=1}^N p_i^\beta \mu_i^{t+1} \left(\frac{(D-1)i+D}{D} \right)^t \right]^{\frac{1}{t}}.$$

After applying holder's inequality, we have:

$$\leq \log_D \left[\sum_{i=1}^N p_i^{\beta-1} q_i \left(\frac{(D-1)i+D}{D} \right)^{-1} \right], \quad \beta > 0 \quad (19)$$

which gives Equation (16).

Using the result, proved by Bilal and Pirzada (2006):

$$tL\mu_\alpha^\beta < 1 + H_\alpha^\beta(P/Q;U)$$

Now

$$\begin{aligned} tL\mu_\alpha^\beta - tL_{1:1}\mu_\alpha^\beta &< 1 + H_\alpha^\beta(P/Q;U) - tL_{1:1}\mu_\alpha^\beta \\ &\leq 1 + \log_D \left[\sum_{i=1}^N p_i^{\beta-1} q_i \left(\frac{(D-1)i+D}{D} \right)^{-1} \right], \quad \beta > 0 \\ &\leq 2 + \log_D \left[\sum_{i=1}^N p_i^{\beta-1} q_i \left(\frac{1}{(D-1)i+D} \right) \right]. \end{aligned} \quad (20)$$

which proves Equation (17).

Particular cases

For $D = 2$, all the results of the aforementioned theorem reduces that of the theorem proved earlier on.

DISCUSSION AND CONCLUSION

Error-correcting codes play an important role in many areas of science and engineering. Error-correcting codes constitute one of the key ingredients in achieving the high degree of reliability required in modern data transmission and storage systems. We come across the problem of efficient coding of messages to be sent over a noiseless channel and attempt to maximize the number of messages that can be sent through a channel in a given time. Therefore, we find the minimum value of a mean codeword length subject to a given constraint on code-word lengths. We have modified the noiseless coding theorem as obtained by Bilal and Pirzada (2006) for binary as well as D size 1:1 codes. The coding theorem as obtained by us is less constrained.

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