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Generalization and reverses of the left Fejér inequality for convex functions

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Abstract

In this paper we establish a generalization of the left Fejér inequality for general Lebesgue integral on measurable spaces as well as various upper bounds for the difference

$$\frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx - h\left(\frac{a+b}{2}\right),$$

where $h:[a,b]\to\mathbb{R}$ is a convex function and $g:[a,b]\to[0,\infty)$ is an integrable weight. Applications for discrete means and Hermite-Hadamard type inequalities are also provided.

Keywords: Convex functions, integral inequalities, Jensen's type inequalities, Fejér type inequalities, Lebesgue integral, Hermite-Hadamard type inequalities, special means.
2010 MSC: 26D15, 26D20.

1. Introduction

The *Hermite-Hadamard* integral inequality for convex functions $f:[a,b]\to\mathbb{R}$,

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\,dx\leqslant\frac{f(a)+f(b)}{2}\tag{HH}$$

is well-known in the literature and has many applications for special means, in information theory and in probability theory and statistics.

For related results, see for instance the research papers [1, 4–11], the monograph online [2] and the references therein.

In 1906, Fejér [3], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 1.1 (Fejér's Inequality)**.** *Consider the integral* $\int_a^b h(x)\,g(x)\,dx$*, where* h *is a convex function in the interval* (a,b) *and* g *is a positive function in the same interval such that*

$$g(a+t)=g(b-t),\ \ 0\leqslant t\leqslant\frac{1}{2}(b-a),$$

i.e., $y=g(x)$ *is a symmetric curve with respect to the straight line which contains the point* $\left(\frac{1}{2}(a+b),0\right)$ *and is*

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normal to the x*-axis. Under those conditions the following inequalities are valid:*

$$h\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leqslant \int_a^b h(x)\,g(x)\,dx \leqslant \frac{h(a)+h(b)}{2}\int_a^b g(x)\,dx. \tag{1.1}$$

If h *is concave on* (a,b)*, then the inequalities reverse in* (1.1)*.*

Clearly, for $g(x)\equiv 1$ on $[a,b]$ we get (HH).

Motivated by the above result, we establish in this paper a generalization of the left Fejér inequality (1.1) for general Lebesgue integral on measurable spaces as well as various *upper bounds* for the difference

$$\frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx - h\left(\frac{a+b}{2}\right),$$

where g and h is as above.

Applications for discrete means and Hermite-Hadamard type inequalities are also provided.

2. General results

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $\Phi : I \to \mathbb{R}$ is a convex function on I. Then Φ is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x,y\in\mathring{I}$ and $x<y$, then $\Phi'_-(x)\leqslant\Phi'_+(x)\leqslant\Phi'_-(y)\leqslant\Phi'_+(y)$ which shows that both Φ'_- and Φ'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $\Phi : I\to\mathbb{R}$, the subdifferential of Φ denoted by $\partial\Phi$ is the set of all functions $\varphi : I\to[-\infty,\infty]$ such that $\varphi\left(\mathring{I}\right)\subset\mathbb{R}$ and

$$\Phi(x)\geqslant\Phi(a)+(x-a)\,\varphi(a)\ \text{ for any } x,a\in I. \tag{2.1}$$

It is also well-known that if Φ is convex on I, then $\partial\Phi$ is nonempty, Φ'_-, $\Phi'_+\in\partial\Phi$ and if $\varphi\in\partial\Phi$, then

$$\Phi'_-(x)\leqslant\varphi(x)\leqslant\Phi'_+(x)\ \text{ for any } x\in\mathring{I}.$$

In particular, φ is a nondecreasing function.

If Φ is differentiable and convex on $\mathring{I}$, then $\partial\Phi=\{\Phi'\}$.

Let $(\Omega,\mathcal{A},\mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geqslant 0$ for μ-a.e. (almost every) $x\in\Omega$, consider the Lebesgue space

$$L_w(\Omega,\mu) := \{f:\Omega\to\mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)|\,w(x)\,d\mu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$.

In what follows we assume that $w\geqslant 0$ μ-a.e. on Ω with $\int_\Omega w d\mu = 1$.

The following result holds.

Theorem 2.1. *Let* $\Phi : [m,M]\subset\mathbb{R}\to\mathbb{R}$ *be a convex function on* $[m,M]$, $\varphi\in\partial\Phi$ *and* $f:\Omega\to\mathbb{R}$ *satisfying the condition*

$$-\infty<m\leqslant f\leqslant M<\infty$$

μ*-a.e. on* Ω *and so that* $\Phi\circ f$, $\varphi\circ f$, $(\varphi\circ f)\,f$, $f\in L_w(\Omega,\mu)$*. Then we have the inequalities*

$$0\leqslant\int_\Omega(\Phi\circ f)\,w d\mu-\Phi\left(\frac{m+M}{2}\right)-\varphi\left(\frac{m+M}{2}\right)\int_\Omega\left(f-\frac{m+M}{2}\right)w d\mu$$

$$
\begin{aligned}
&\leqslant \int_\Omega \left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right)\left(f - \frac{m+M}{2}\right) w d\mu \\
&\leqslant \operatorname*{essup}_{\Omega} \left|f - \frac{m+M}{2}\right| \left|\int_\Omega \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right| w d\mu \right. \\
&\leqslant \begin{cases} \frac{1}{2}(M-m)\int_\Omega \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right| w d\mu \\ \operatorname{essup}_\Omega \left|f - \frac{m+M}{2}\right| \left(\Phi'_-(M) - \Phi'_+(m)\right) \end{cases} \\
&\leqslant \frac{1}{2}(M-m)\left(\Phi'_-(M) - \Phi'_+(m)\right).
\end{aligned} \tag{2.2}
$$

Proof. By the gradient inequality (2.1) we have

$$
\Phi(t) - \Phi\left(\frac{m+M}{2}\right) \geqslant \left(t - \frac{m+M}{2}\right)\varphi\left(\frac{m+M}{2}\right)
$$

for any $\varphi \in \partial\Phi$ and for any $t \in [m, M]$. This inequality implies that

$$
\Phi(f(x)) \geqslant \Phi\left(\frac{m+M}{2}\right) + \left(f(x) - \frac{m+M}{2}\right)\varphi\left(\frac{m+M}{2}\right) \tag{2.3}
$$

for any $x \in \Omega$. If we multiply (2.3) by $w \geqslant 0$ μ-a.e and integrate on Ω, we get the first inequality in (2.2).

By the gradient inequality (2.1) we also have

$$
\varphi(t)\left(t - \frac{m+M}{2}\right) \geqslant \Phi(t) - \Phi\left(\frac{m+M}{2}\right)
$$

for any $\varphi \in \partial\Phi$ and for any $t \in (m, M)$.

This inequality is equivalent to

$$
\left(\varphi(t) - \varphi\left(\frac{m+M}{2}\right)\right)\left(t - \frac{m+M}{2}\right) \geqslant \Phi(t) - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right)\left(t - \frac{m+M}{2}\right)
$$

for any $\varphi \in \partial\Phi$ and for any $t \in (m, M)$. This inequality implies that

$$
\left(\varphi(f(x)) - \varphi\left(\frac{m+M}{2}\right)\right)\left(f(x) - \frac{m+M}{2}\right) \geqslant \Phi(f(x)) - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right)\left(f(x) - \frac{m+M}{2}\right) \tag{2.4}
$$

for μ-a.e $x \in \Omega$. If we multiply (2.4) by $w \geqslant 0$ μ-a.e and integrate on Ω, we get

$$
\begin{aligned}
&\int_\Omega \left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right)\left(f - \frac{m+M}{2}\right) w d\mu \\
&\quad \geqslant \int_\Omega (\Phi \circ f) w d\mu - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right)\int_\Omega \left(f - \frac{m+M}{2}\right) w d\mu,
\end{aligned}
$$

which proves the second inequality in (2.2). Now, since φ is monotonic nondecreasing on $[m, M]$, then

$$
\left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right)\left(f - \frac{m+M}{2}\right) \geqslant 0
$$

on Ω. We then have

$$
\begin{aligned}
\int_\Omega \left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right)\left(f - \frac{m+M}{2}\right) w d\mu &= \left|\int_\Omega \left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right)\left(f - \frac{m+M}{2}\right) w d\mu\right| \\
&\leqslant \int_\Omega \left|\left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right)\left(f - \frac{m+M}{2}\right)\right| w d\mu \\
&\leqslant \operatorname*{essup}_{\Omega} \left|f - \frac{m+M}{2}\right| \left|\int_\Omega \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right| w d\mu \right.
\end{aligned}
$$

and the third inequality in (2.2) is proved.

Since for any $\varphi \in \partial\Phi$ we have $\Phi'_+(m) \leqslant \varphi(t) \leqslant \Phi'_-(M)$ for any $t \in (m, M)$, then $\Phi'_+(m) \leqslant \varphi\left(\frac{m+M}{2}\right) \leqslant \Phi'_-(M)$ and

$$\left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right| \leqslant \Phi'_-(M) - \Phi'_+(m) \tag{2.5}$$

μ-a.e. on Ω. If we multiply (2.5) by $w \geqslant 0$ μ-a.e and integrate on Ω, we get

$$\int_\Omega \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right| w d\mu \leqslant \Phi'_-(M) - \Phi'_+(m)$$

and the last part in (2.2) is proved. □

We have the following result.

Corollary 2.2. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[m, M]$, $\varphi \in \partial\Phi$ and $f : \Omega \to \mathbb{R}$ satisfying the condition (2.2) μ-a.e. on Ω and so that $\Phi \circ f$, $\varphi \circ f$, $(\varphi \circ f) f$, $f \in L_w(\Omega, \mu)$. If*

$$\int_\Omega \left(f - \frac{m+M}{2}\right) w d\mu = 0, \tag{2.6}$$

then we have the inequalities

$$\begin{aligned}
0 &\leqslant \int_\Omega (\Phi \circ f) w d\mu - \Phi\left(\frac{m+M}{2}\right) \\
&\leqslant \int_\Omega (\varphi \circ f - \lambda)\left(f - \frac{m+M}{2}\right) w d\mu \\
&\leqslant \operatorname*{essup}_\Omega \left|f - \frac{m+M}{2}\right| \int_\Omega |\varphi \circ f - \lambda| w d\mu \\
&\leqslant \frac{1}{2}(M - m) \int_\Omega |\varphi \circ f - \lambda| w d\mu
\end{aligned} \tag{2.7}$$

for any $\lambda \in \mathbb{R}$.

It follows by Theorem 2.1 on observing that, if (2.6) holds true, then

$$\begin{aligned}
\int_\Omega \left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right)\left(f - \frac{m+M}{2}\right) w d\mu &= \int_\Omega (\varphi \circ f)\left(f - \frac{m+M}{2}\right) w d\mu \\
&= \int_\Omega (\varphi \circ f - \lambda)\left(f - \frac{m+M}{2}\right) w d\mu
\end{aligned}$$

for any $\lambda \in \mathbb{R}$.

Remark 2.3. From the inequality (2.7) we have

$$\begin{aligned}
0 &\leqslant \int_\Omega (\Phi \circ f) w d\mu - \Phi\left(\frac{m+M}{2}\right) \\
&\leqslant \int_\Omega (\varphi \circ f)\left(f - \frac{m+M}{2}\right) w d\mu \\
&\leqslant \operatorname*{essup}_\Omega \left|f - \frac{m+M}{2}\right| \int_\Omega |\varphi \circ f| w d\mu \\
&\leqslant \frac{1}{2}(M - m) \int_\Omega |\varphi \circ f| w d\mu.
\end{aligned}$$

Since for any $\varphi \in \partial\Phi$ we have $\Phi'_+(m) \leqslant \varphi(t) \leqslant \Phi'_-(M)$ for any $t \in (m, M)$, then from (2.7) we have

$$
\begin{aligned}
0 &\leqslant \int_\Omega (\Phi \circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) \\
&\leqslant \int_\Omega \left(\varphi \circ f - \frac{\Phi'_-(M) + \Phi'_+(m)}{2}\right)\left(f - \frac{m+M}{2}\right) w d\mu \\
&\leqslant \operatorname*{essup}_\Omega \left|f - \frac{m+M}{2}\right| \int_\Omega \left|\varphi \circ f - \frac{\Phi'_-(M) + \Phi'_+(m)}{2}\right| w d\mu \\
&\leqslant \frac{1}{2}\begin{cases} (M-m)\int_\Omega \left|\varphi \circ f - \frac{\Phi'_-(M)+\Phi'_+(m)}{2}\right| w d\mu \\ \\ \operatorname{essup}_\Omega \left|f - \frac{m+M}{2}\right| \left(\Phi'_-(M) - \Phi'_+(m)\right) \end{cases} \\
&\leqslant \frac{1}{4}(M-m)\left(\Phi'_-(M) - \Phi'_+(m)\right).
\end{aligned} \tag{2.8}
$$

We have the following result.

Theorem 2.4. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[m, M]$, $\varphi \in \partial\Phi$, and $f : \Omega \to \mathbb{R}$ satisfying the condition (2.2) μ-a.e. on Ω and so that $\Phi \circ f$, $(\varphi \circ f)^q$, $f^p \in L_w(\Omega, \mu)$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we have the inequalities:*

$$
\begin{aligned}
0 &\leqslant \int_\Omega (\Phi \circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right)\int_\Omega \left(f - \frac{m+M}{2}\right) w d\mu \\
&\leqslant \int_\Omega \left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right)\left(f - \frac{m+M}{2}\right) w d\mu \\
&\leqslant \left(\int_\Omega \left|f - \frac{m+M}{2}\right|^p w d\mu\right)^{1/p} \left(\int_\Omega \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right|^q w d\mu\right)^{1/q} \\
&\leqslant \begin{cases} \frac{1}{2}(M-m)\left(\int_\Omega \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right|^q w d\mu\right)^{1/q} \\ \\ \left(\Phi'_-(M) - \Phi'_+(m)\right)\left(\int_\Omega \left|f - \frac{m+M}{2}\right|^p w d\mu\right)^{1/p} \end{cases} \\
&\leqslant \frac{1}{2}(M-m)\left(\Phi'_-(M) - \Phi'_+(m)\right).
\end{aligned} \tag{2.9}
$$

Proof. By Hölder's integral inequality we have for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$
\begin{aligned}
&\int_\Omega \left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right)\left(f - \frac{m+M}{2}\right) w d\mu \\
&\leqslant \int_\Omega \left|\left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right)\left(f - \frac{m+M}{2}\right)\right| w d\mu \\
&\leqslant \left(\int_\Omega \left|f - \frac{m+M}{2}\right|^p w d\mu\right)^{1/p} \left(\int_\Omega \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right|^q w d\mu\right)^{1/q},
\end{aligned}
$$

which proves the third inequality in (2.9). Also, we have

$$
\left(\int_\Omega \left|f - \frac{m+M}{2}\right|^p w d\mu\right)^{1/p} \leqslant \left(\left|\frac{M-m}{2}\right|^p \int_\Omega w d\mu\right)^{1/p} = \frac{1}{2}(M-m)
$$

and

$$
\left(\int_\Omega \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right|^q w d\mu\right)^{1/q} \leqslant \left(\left(\Phi'_-(M) - \Phi'_+(m)\right)^q \int_\Omega w d\mu\right)^{1/q}
$$

that proves the last part of (2.9). □

Remark 2.5. In a similar way, the inequality (2.9) can be extended for $p = 1$ and $q = \infty$ to obtain the result:

$$\begin{aligned}
0 &\leqslant \int_{\Omega} (\Phi \circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right) \int_{\Omega} \left(f - \frac{m+M}{2}\right) w d\mu \\
&\leqslant \int_{\Omega} \left(\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right)\left(f - \frac{m+M}{2}\right) w d\mu \\
&\leqslant \operatorname*{essup}_{\Omega} \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right| \int_{\Omega} \left|f - \frac{m+M}{2}\right| w d\mu \\
&\leqslant \begin{cases} \frac{1}{2}(M-m)\operatorname{essup}_{\Omega} \left|\varphi \circ f - \varphi\left(\frac{m+M}{2}\right)\right| \\[2ex] \left(\Phi'_-(M) - \Phi'_+(m)\right) \int_{\Omega} \left|f - \frac{m+M}{2}\right| w d\mu \end{cases} \\
&\leqslant \frac{1}{2}(M-m)\left(\Phi'_-(M) - \Phi'_+(m)\right),
\end{aligned} \tag{2.10}$$

provided that $\varphi \circ f$ is essentially bounded on Ω.

Corollary 2.6. *Under the assumptions of Theorem 2.4 and if the condition (2.6) holds, then we have the simpler inequalities*

$$\begin{aligned}
0 &\leqslant \int_{\Omega} (\Phi \circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) \leqslant \int_{\Omega} (\varphi \circ f - \lambda)\left(f - \frac{m+M}{2}\right) w d\mu \\
&\leqslant \left(\int_{\Omega} \left|f - \frac{m+M}{2}\right|^p w d\mu\right)^{1/p} \left(\int_{\Omega} |\varphi \circ f - \lambda|^q\, w d\mu\right)^{1/q} \\
&\leqslant \frac{1}{2}(M-m)\left(\int_{\Omega} |\varphi \circ f - \lambda|^q\, w d\mu\right)^{1/q}
\end{aligned} \tag{2.11}$$

for any $\lambda \in \mathbb{R}$. In particular we have

$$\begin{aligned}
0 &\leqslant \int_{\Omega} (\Phi \circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) \leqslant \int_{\Omega} (\varphi \circ f)\left(f - \frac{m+M}{2}\right) w d\mu \\
&\leqslant \left(\int_{\Omega} \left|f - \frac{m+M}{2}\right|^p w d\mu\right)^{1/p} \left(\int_{\Omega} |\varphi \circ f|^q\, w d\mu\right)^{1/q} \\
&\leqslant \frac{1}{2}(M-m)\left(\int_{\Omega} |\varphi \circ f|^q\, w d\mu\right)^{1/q}
\end{aligned}$$

and

$$\begin{aligned}
0 &\leqslant \int_{\Omega} (\Phi \circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) \\
&\leqslant \int_{\Omega} \left(\varphi \circ f - \frac{\Phi'_-(M) + \Phi'_+(m)}{2}\right)\left(f - \frac{m+M}{2}\right) w d\mu \\
&\leqslant \left(\int_{\Omega} \left|f - \frac{m+M}{2}\right|^p w d\mu\right)^{1/p} \times \left(\int_{\Omega} \left|\varphi \circ f - \frac{\Phi'_-(M) + \Phi'_+(m)}{2}\right|^q w d\mu\right)^{1/q} \\
&\leqslant \frac{1}{2}\begin{cases} (M-m)\left(\int_{\Omega} \left|\varphi \circ f - \frac{\Phi'_-(M)+\Phi'_+(m)}{2}\right|^q w d\mu\right)^{1/q} \\[2ex] \left(\int_{\Omega} \left|f - \frac{m+M}{2}\right|^p w d\mu\right)^{1/p} \left(\Phi'_-(M) - \Phi'_+(m)\right) \end{cases} \\
&\leqslant \frac{1}{4}(M-m)\left(\Phi'_-(M) - \Phi'_+(m)\right).
\end{aligned} \tag{2.12}$$

We observe that, if under the assumptions of Remark 2.5, we choose f and w such that the condition (2.6) holds, then we have

$$\begin{aligned} 0 &\leqslant \int_\Omega (\Phi\circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) \\ &\leqslant \int_\Omega (\varphi\circ f-\lambda)\left(f-\frac{m+M}{2}\right) w d\mu \\ &\leqslant \operatorname*{essup}_\Omega |\varphi\circ f-\lambda| \int_\Omega \left|f-\frac{m+M}{2}\right| w d\mu \leqslant \frac{1}{2}(M-m)\operatorname*{essup}_\Omega |\varphi\circ f-\lambda| \end{aligned} \tag{2.13}$$

for any $\lambda\in\mathbb{R}$. In particular, we have

$$\begin{aligned} 0 &\leqslant \int_\Omega (\Phi\circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) \leqslant \int_\Omega (\varphi\circ f)\left(f-\frac{m+M}{2}\right) w d\mu \\ &\leqslant \operatorname*{essup}_\Omega |\varphi\circ f| \int_\Omega \left|f-\frac{m+M}{2}\right| w d\mu \leqslant \frac{1}{2}(M-m)\operatorname*{essup}_\Omega |\varphi\circ f| \end{aligned}$$

and

$$\begin{aligned} 0 &\leqslant \int_\Omega (\Phi\circ f)\, w d\mu - \Phi\left(\frac{m+M}{2}\right) \\ &\leqslant \int_\Omega \left(\varphi\circ f-\frac{\Phi'_-(M)+\Phi'_+(m)}{2}\right)\left(f-\frac{m+M}{2}\right) w d\mu \\ &\leqslant \operatorname*{essup}_\Omega \left|\varphi\circ f-\frac{\Phi'_-(M)+\Phi'_+(m)}{2}\right| \int_\Omega \left|f-\frac{m+M}{2}\right| w d\mu \\ &\leqslant \frac{1}{2}\begin{cases} (\Phi'_-(M)-\Phi'_+(m))\int_\Omega \left|f-\frac{m+M}{2}\right| w d\mu \\ (M-m)\operatorname{essup}_\Omega \left|\varphi\circ f-\frac{\Phi'_-(M)+\Phi'_+(m)}{2}\right| \end{cases} \\ &\leqslant \frac{1}{4}(M-m)\left(\Phi'_-(M)-\Phi'_+(m)\right). \end{aligned} \tag{2.14}$$

We observe that, if Φ is differentiable on (m,M) then we can replace φ by Φ' in all inequalities above. We omit the details.

3. Applications for discrete inequalities

Let $\mathbf{x}=(x_1,\dots,x_n)$ be an n-tuple with $x_i\in\mathbb{R}$, $i\in\{1,\dots,n\}$ and $\mathbf{p}=(p_1,\dots,p_n)$ a probability distribution, i.e., $p_i\geqslant 0$, $i\in\{1,\dots,n\}$ with $\sum_{i=1}^n p_i=1$.

If $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ is a convex function on $[m,M]$ and $\varphi\in\partial\Phi$, then for any $\mathbf{x}=(x_1,\dots,x_n)$ with $x_i\in(m,M)\subset\mathbb{R}$, $i\in\{1,\dots,n\}$ and any probability distribution $\mathbf{p}=(p_1,\dots,p_n)$ we have from (2.2), (2.9), and (2.10) for the discrete measure that

$$\begin{aligned} 0 &\leqslant \sum_{i=1}^n p_i\Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right)\sum_{i=1}^n p_i\left(x_i-\frac{m+M}{2}\right) \\ &\leqslant \sum_{i=1}^n p_i\left(\varphi(x_i)-\varphi\left(\frac{m+M}{2}\right)\right)\left(x_i-\frac{m+M}{2}\right) \\ &\leqslant \max_{i\in\{1,\dots,n\}}\left|x_i-\frac{m+M}{2}\right| \sum_{i=1}^n p_i\left|\varphi(x_i)-\varphi\left(\frac{m+M}{2}\right)\right| \\ &\leqslant \begin{cases} \frac{1}{2}(M-m)\sum_{i=1}^n p_i\left|\varphi(x_i)-\varphi\left(\frac{m+M}{2}\right)\right| \\ \max_{i\in\{1,\dots,n\}}\left|x_i-\frac{m+M}{2}\right|\left(\Phi'_-(M)-\Phi'_+(m)\right) \end{cases} \\ &\leqslant \frac{1}{2}(M-m)\left(\Phi'_-(M)-\Phi'_+(m)\right), \end{aligned} \tag{3.1}$$

$$
\begin{aligned}
0 &\leqslant \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right)\sum_{i=1}^{n} p_i\left(x_i - \frac{m+M}{2}\right) \\
&\leqslant \sum_{i=1}^{n} p_i\left(\varphi(x_i) - \varphi\left(\frac{m+M}{2}\right)\right)\left(x_i - \frac{m+M}{2}\right) \\
&\leqslant \left(\sum_{i=1}^{n} p_i\left|x_i - \frac{m+M}{2}\right|^p\right)^{1/p}\left(\sum_{i=1}^{n} p_i\left|\varphi(x_i) - \varphi\left(\frac{m+M}{2}\right)\right|^q\right)^{1/q} \\
&\leqslant \frac{1}{2}\times\begin{cases} (M-m)\left(\sum_{i=1}^{n} p_i\left|\varphi(x_i)-\varphi\left(\frac{m+M}{2}\right)\right|^q\right)^{1/q} \\ \left(\Phi'_-(M)-\Phi'_+(m)\right)\left(\sum_{i=1}^{n} p_i\left|x_i - \frac{m+M}{2}\right|^p\right)^{1/p} \end{cases} \\
&\leqslant \frac{1}{2}(M-m)\left(\Phi'_-(M)-\Phi'_+(m)\right),
\end{aligned} \tag{3.2}
$$

and

$$
\begin{aligned}
0 &\leqslant \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right)\sum_{i=1}^{n} p_i\left(x_i - \frac{m+M}{2}\right) \\
&\leqslant \sum_{i=1}^{n} p_i\left(\varphi(x_i) - \varphi\left(\frac{m+M}{2}\right)\right)\left(x_i - \frac{m+M}{2}\right) \\
&\leqslant \max_{i\in\{1,\dots,n\}}\left|\varphi(x_i) - \varphi\left(\frac{m+M}{2}\right)\right|\sum_{i=1}^{n} p_i\left|x_i - \frac{m+M}{2}\right| \\
&\leqslant \frac{1}{2}\times\begin{cases} (M-m)\max_{i\in\{1,\dots,n\}}\left|\varphi(x_i)-\varphi\left(\frac{m+M}{2}\right)\right| \\ \left(\Phi'_-(M)-\Phi'_+(m)\right)\sum_{i=1}^{n} p_i\left|x_i - \frac{m+M}{2}\right| \end{cases} \\
&\leqslant \frac{1}{2}(M-m)\left(\Phi'_-(M)-\Phi'_+(m)\right).
\end{aligned} \tag{3.3}
$$

If $\mathbf{x} = (x_1, \dots, x_n)$ with $x_i \in (m, M) \subset \mathbb{R}$, $i \in \{1, \dots, n\}$ and the probability distribution $\mathbf{p} = (p_1, \dots, p_n)$ is such that

$$\sum_{i=1}^{n} p_i x_i = \frac{m+M}{2}, \tag{3.4}$$

then from (2.7), (2.11), and (2.13) we have

$$
\begin{aligned}
0 &\leqslant \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) \leqslant \sum_{i=1}^{n} p_i(\varphi(x_i) - \lambda)\left(x_i - \frac{m+M}{2}\right) \\
&\leqslant \max_{i\in\{1,\dots,n\}}\left|x_i - \frac{m+M}{2}\right|\sum_{i=1}^{n} p_i|\varphi(x_i) - \lambda| \\
&\leqslant \frac{1}{2}(M-m)\sum_{i=1}^{n} p_i|\varphi(x_i) - \lambda|,
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
0 &\leqslant \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) \leqslant \sum_{i=1}^{n} p_i(\varphi(x_i) - \lambda)\left(x_i - \frac{m+M}{2}\right) \\
&\leqslant \left(\sum_{i=1}^{n} p_i\left|x_i - \frac{m+M}{2}\right|^p\right)^{1/p}\left(\sum_{i=1}^{n} p_i|\varphi(x_i) - \lambda|^q\right)^{1/q} \\
&\leqslant \frac{1}{2}(M-m)\left(\sum_{i=1}^{n} p_i|\varphi(x_i) - \lambda|^q\right)^{1/q},
\end{aligned} \tag{3.6}
$$

and

$$\begin{aligned} 0 &\leqslant \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) \leqslant \sum_{i=1}^{n} p_i (\varphi(x_i) - \lambda)\left(x_i - \frac{m+M}{2}\right) \\ &\leqslant \max_{i\in\{1,\dots,n\}} |\varphi(x_i) - \lambda| \sum_{i=1}^{n} p_i \left|x_i - \frac{m+M}{2}\right| \\ &\leqslant \frac{1}{2}(M-m) \max_{i\in\{1,\dots,n\}} |\varphi(x_i) - \lambda| \end{aligned} \tag{3.7}$$

for any $\lambda \in \mathbb{R}$. If we take $\lambda = \frac{\Phi'_-(M)+\Phi'_+(m)}{2}$ in (3.5)-(3.7), then we get

$$\begin{aligned} 0 &\leqslant \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) \\ &\leqslant \sum_{i=1}^{n} p_i \left(\varphi(x_i) - \frac{\Phi'_-(M)+\Phi'_+(m)}{2}\right)\left(x_i - \frac{m+M}{2}\right) \\ &\leqslant \max_{i\in\{1,\dots,n\}} \left|x_i - \frac{m+M}{2}\right| \sum_{i=1}^{n} p_i \left|\varphi(x_i) - \frac{\Phi'_-(M)+\Phi'_+(m)}{2}\right| \\ &\leqslant \frac{1}{2}\begin{cases} (M-m)\sum_{i=1}^{n} p_i \left|\varphi(x_i) - \frac{\Phi'_-(M)+\Phi'_+(m)}{2}\right| \\ \left(\Phi'_-(M) - \Phi'_+(m)\right) \max_{i\in\{1,\dots,n\}} \left|x_i - \frac{m+M}{2}\right| \end{cases} \\ &\leqslant \frac{1}{4}(M-m)\left(\Phi'_-(M) - \Phi'_+(m)\right), \end{aligned}$$

$$\begin{aligned} 0 &\leqslant \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) \\ &\leqslant \sum_{i=1}^{n} p_i \left(\varphi(x_i) - \frac{\Phi'_-(M)+\Phi'_+(m)}{2}\right)\left(x_i - \frac{m+M}{2}\right) \\ &\leqslant \left(\sum_{i=1}^{n} p_i \left|x_i - \frac{m+M}{2}\right|^p\right)^{1/p} \left(\sum_{i=1}^{n} p_i \left|\varphi(x_i) - \frac{\Phi'_-(M)+\Phi'_+(m)}{2}\right|^q\right)^{1/q} \\ &\leqslant \frac{1}{2}\begin{cases} (M-m)\left(\sum_{i=1}^{n} p_i \left|\varphi(x_i) - \frac{\Phi'_-(M)+\Phi'_+(m)}{2}\right|^q\right)^{1/q} \\ \left(\sum_{i=1}^{n} p_i \left|x_i - \frac{m+M}{2}\right|^p\right)^{1/p} \left(\Phi'_-(M) - \Phi'_+(m)\right) \end{cases} \\ &\leqslant \frac{1}{4}(M-m)\left(\Phi'_-(M) - \Phi'_+(m)\right), \end{aligned}$$

$$\begin{aligned} 0 &\leqslant \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi\left(\frac{m+M}{2}\right) \\ &\leqslant \sum_{i=1}^{n} p_i \left(\varphi(x_i) - \frac{\Phi'_-(M)+\Phi'_+(m)}{2}\right)\left(x_i - \frac{m+M}{2}\right) \\ &\leqslant \max_{i\in\{1,\dots,n\}} \left|\varphi(x_i) - \frac{\Phi'_-(M)+\Phi'_+(m)}{2}\right| \sum_{i=1}^{n} p_i \left|x_i - \frac{m+M}{2}\right| \end{aligned}$$

$$\leqslant \frac{1}{2}\begin{cases} (M-m)\max_{i\in\{1,\dots,n\}}\left|\varphi(x_i)-\frac{\Phi'_-(M)+\Phi'_+(m)}{2}\right| \\ \\ \left(\Phi'_-(M)-\Phi'_+(m)\right)\sum_{i=1}^n p_i\left|x_i-\frac{m+M}{2}\right| \end{cases}$$
$$\leqslant \frac{1}{4}(M-m)\left(\Phi'_-(M)-\Phi'_+(m)\right)$$

for $\mathbf{x}=(x_1,\dots,x_n)$ with $x_i\in(m,M)\subset\mathbb{R}$, $i\in\{1,\dots,n\}$ and the probability distribution $\mathbf{p}=(p_1,\dots,p_n)$ with the property that (3.4) holds true.

Consider the function $\Phi:[m,M]\subset(0,\infty)\to(0,\infty)$ defined by $\Phi(x)=x^p$ with $p\in(-\infty,0)\cup[1,\infty)$. This function is convex and for any n-tuple $\mathbf{x}=(x_1,\dots,x_n)$ with $x_i\in(m,M)$ and any probability distribution $\mathbf{p}=(p_1,\dots,p_n)$, by (3.1) we have that

$$0\leqslant \sum_{i=1}^n p_i x_i^p - p\left(\frac{m+M}{2}\right)^{p-1}\sum_{i=1}^n p_i x_i + (p-1)\left(\frac{m+M}{2}\right)^p$$
$$\leqslant p\sum_{i=1}^n p_i\left(x_i^{p-1}-\left(\frac{m+M}{2}\right)^{p-1}\right)\left(x_i-\frac{m+M}{2}\right)$$
$$\leqslant |p|\max_{i\in\{1,\dots,n\}}\left|x_i-\frac{m+M}{2}\right|\sum_{i=1}^n p_i\left|x_i^{p-1}-\left(\frac{m+M}{2}\right)^{p-1}\right|$$
$$\leqslant \begin{cases} \frac{1}{2}|p|(M-m)\sum_{i=1}^n p_i\left|x_i^{p-1}-\left(\frac{m+M}{2}\right)^{p-1}\right| \\ \\ p\max_{i\in\{1,\dots,n\}}\left|x_i-\frac{m+M}{2}\right|\left(M^{p-1}-m^{p-1}\right) \end{cases}$$
$$\leqslant \frac{1}{2}p(M-m)\left(M^{p-1}-m^{p-1}\right).$$

Similar inequalities may be stated by utilizing the other two inequalities (3.2) and (3.3).

Consider the function $\Phi:[m,M]\subset(0,\infty)\to(0,\infty)$ defined by $\Phi(x)=-\ln x$. This function is convex and for any n-tuple $\mathbf{x}=(x_1,\dots,x_n)$ with $x_i\in(m,M)$ and any probability distribution $\mathbf{p}=(p_1,\dots,p_n)$, by (3.1) we have that

$$0\leqslant \ln\left(\frac{m+M}{2}\right)+\frac{2}{m+M}\sum_{i=1}^n p_i\left(x_i-\frac{m+M}{2}\right)-\sum_{i=1}^n p_i\ln x_i$$
$$\leqslant \frac{2}{m+M}\sum_{i=1}^n p_i\frac{\left(x_i-\frac{m+M}{2}\right)^2}{x_i}$$
$$\leqslant \frac{2}{m+M}\max_{i\in\{1,\dots,n\}}\left|x_i-\frac{m+M}{2}\right|\sum_{i=1}^n p_i\left|x_i-\frac{m+M}{2}\right|$$
$$\leqslant \begin{cases} \frac{M-m}{m+M}\sum_{i=1}^n p_i\left|x_i-\frac{m+M}{2}\right| \\ \\ \frac{M-m}{mM}\max_{i\in\{1,\dots,n\}}\left|x_i-\frac{m+M}{2}\right| \end{cases}\leqslant \frac{1}{2mM}(M-m)^2,$$

or, equivalently

$$1\leqslant \frac{\frac{m+M}{2}\exp\left[\frac{2}{m+M}\sum_{i=1}^n p_i\left(x_i-\frac{m+M}{2}\right)\right]}{\prod_{i=1}^n x_i^{p_i}}$$
$$\leqslant \exp\left(\frac{2}{m+M}\sum_{i=1}^n\frac{p_i}{x_i}\left(x_i-\frac{m+M}{2}\right)^2\right)$$
$$\leqslant \exp\left[\frac{2}{m+M}\max_{i\in\{1,\dots,n\}}\left|x_i-\frac{m+M}{2}\right|\sum_{i=1}^n p_i\left|x_i-\frac{m+M}{2}\right|\right]$$

$$\leqslant \begin{cases} \exp\left[\frac{M-m}{m+M}\sum_{i=1}^{n} p_i \left|x_i - \frac{m+M}{2}\right|\right] \\ \exp\left[\frac{M-m}{mM}\max_{i\in\{1,\dots,n\}}\left|x_i - \frac{m+M}{2}\right|\right] \end{cases} \leqslant \exp\left[\frac{1}{2mM}(M-m)^2\right].$$

4. Applications for Fejér's inequality

Let $h : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function, $\varphi \in \partial h$, and $g : [a, b] \to [0, \infty)$ a Lebesgue integrable function on $[a, b]$. Then from Theorems 2.1 and 2.4, and Remark 2.5 we have the inequalities

$$\begin{aligned}
0 &\leqslant \frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)g(x)\,dx - h\left(\frac{a+b}{2}\right) - \varphi\left(\frac{a+b}{2}\right)\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left(x-\frac{a+b}{2}\right)g(x)\,dx \\
&\leqslant \frac{1}{\int_a^b g(x)\,dx}\int_a^b\left(\varphi(x)-\varphi\left(\frac{a+b}{2}\right)\right)\left(x-\frac{a+b}{2}\right)g(x)\,dx \\
&\leqslant \frac{1}{2}(b-a)\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left|\varphi(x)-\varphi\left(\frac{a+b}{2}\right)\right|g(x)\,dx \\
&\leqslant \frac{1}{2}(b-a)\left(h'_-(b)-h'_+(a)\right),
\end{aligned}$$

$$\begin{aligned}
0 &\leqslant \frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)g(x)\,dx - h\left(\frac{a+b}{2}\right) - \varphi\left(\frac{a+b}{2}\right)\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left(x-\frac{a+b}{2}\right)g(x)\,dx \\
&\leqslant \left(\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left|x-\frac{a+b}{2}\right|^p g(x)\,dx\right)^{1/p} \times \left(\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left|\varphi(x)-\frac{a+b}{2}\right|^q g(x)\,dx\right)^{1/q} \\
&\leqslant \begin{cases} \frac{1}{2}(b-a)\left(\frac{1}{\int_a^b g(x)dx}\int_a^b\left|\varphi(x)-\frac{a+b}{2}\right|^q g(x)\,dx\right)^{1/q} \\ \left(h'_-(b)-h'_+(a)\right)\left(\frac{1}{\int_a^b g(x)dx}\int_a^b\left|x-\frac{a+b}{2}\right|^p g(x)\,dx\right)^{1/p} \end{cases} \\
&\leqslant \frac{1}{2}(b-a)\left(h'_-(b)-h'_+(a)\right),
\end{aligned}$$

$$\begin{aligned}
0 &\leqslant \frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)g(x)\,dx - h\left(\frac{a+b}{2}\right) - \varphi\left(\frac{a+b}{2}\right)\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left(x-\frac{a+b}{2}\right)g(x)\,dx \\
&\leqslant \frac{1}{\int_a^b g(x)\,dx}\int_a^b\left(\varphi(x)-\varphi\left(\frac{a+b}{2}\right)\right)\left(x-\frac{a+b}{2}\right)g(x)\,dx \\
&\leqslant \operatorname*{essup}_{x\in[a,b]}\left|\varphi(x)-\varphi\left(\frac{a+b}{2}\right)\right|\frac{1}{\int_a^b g(x)\,dx}\int_a^b\left|x-\frac{a+b}{2}\right|g(x)\,dx \\
&\leqslant \begin{cases} \frac{1}{2}(b-a)\operatorname{essup}_{x\in[a,b]}\left|\varphi(x)-\varphi\left(\frac{a+b}{2}\right)\right| \\ \left(h'_-(b)-h'_+(a)\right)\frac{1}{\int_a^b g(x)dx}\int_a^b\left|x-\frac{a+b}{2}\right|g(x)\,dx \end{cases} \\
&\leqslant \frac{1}{2}(b-a)\left(h'_-(b)-h'_+(a)\right).
\end{aligned}$$

Now, if we assume that $g : [a, b] \to \mathbb{R}$ is integrable and symmetric on the interval $[a, b]$, then

$$\int_a^b\left(x-\frac{a+b}{2}\right)g(x)\,dx = 0. \tag{4.1}$$

The converse is obviously not true.

Now, if $g : [a, b] \to [0, \infty)$ is integrable and satisfies the condition (4.1), then from (2.8), (2.12), and (2.14) we have for any convex function $h : [a, b] \subset \mathbb{R} \to \mathbb{R}$ and $\varphi \in \partial h$ that

$$
\begin{aligned}
0 &\leqslant \frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx - h\left(\frac{a+b}{2}\right) \\
&\leqslant \frac{1}{\int_a^b g(x)\,dx}\int_a^b \left(\varphi(x) - \frac{h'_-(b)+h'_+(a)}{2}\right)\left(x-\frac{a+b}{2}\right) g(x)\,dx \\
&\leqslant \frac{1}{2}(b-a)\frac{1}{\int_a^b g(x)\,dx}\int_a^b \left|\varphi(x) - \frac{h'_-(b)+h'_+(a)}{2}\right| g(x)\,dx \\
&\leqslant \frac{1}{4}(b-a)\left(h'_-(b)-h'_+(a)\right),
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
0 &\leqslant \frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx - h\left(\frac{a+b}{2}\right) \\
&\leqslant \frac{1}{\int_a^b g(x)\,dx}\int_a^b \left(\varphi(x) - \frac{h'_-(b)+h'_+(a)}{2}\right)\left(x-\frac{a+b}{2}\right) g(x)\,dx \\
&\leqslant \left(\frac{1}{\int_a^b g(x)\,dx}\int_a^b \left|x-\frac{a+b}{2}\right|^p g(x)\,dx\right)^{1/p} \\
&\quad\times \left(\frac{1}{\int_a^b g(x)\,dx}\int_a^b \left|\varphi(x) - \frac{h'_-(b)+h'_+(a)}{2}\right|^q g(x)\,dx\right)^{1/q} \\
&\leqslant \frac{1}{2}\begin{cases} (b-a)\left(\frac{1}{\int_a^b g(x)dx}\int_a^b \left|\varphi(x) - \frac{h'_-(b)+h'_+(a)}{2}\right|^q g(x)\,dx\right)^{1/q} \\ \\ \left(\Phi'_-(b)-\Phi'_+(a)\right)\left(\frac{1}{\int_a^b g(x)dx}\int_a^b \left|x-\frac{a+b}{2}\right|^p g(x)\,dx\right)^{1/p} \end{cases} \\
&\leqslant \frac{1}{4}(b-a)\left(\Phi'_-(b)-\Phi'_+(a)\right),
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
0 &\leqslant \frac{1}{\int_a^b g(x)\,dx}\int_a^b h(x)\,g(x)\,dx - h\left(\frac{a+b}{2}\right) \\
&\leqslant \frac{1}{\int_a^b g(x)\,dx}\int_a^b \left(\varphi(x) - \frac{h'_-(b)+h'_+(a)}{2}\right)\left(x-\frac{a+b}{2}\right) g(x)\,dx \\
&\leqslant \operatorname*{essup}_{x\in[a,b]}\left|\varphi(x) - \frac{h'_-(b)+h'_+(a)}{2}\right| \frac{1}{\int_a^b g(x)\,dx}\int_a^b \left|x-\frac{a+b}{2}\right| g(x)\,dx \\
&\leqslant \frac{1}{2}\begin{cases} \left(h'_-(b)-h'_+(a)\right)\frac{1}{\int_a^b g(x)dx}\int_a^b \left|x-\frac{a+b}{2}\right| g(x)\,dx \\ \\ (b-a)\operatorname{essup}_{x\in[a,b]}\left|\varphi(x) - \frac{h'_-(b)+h'_+(a)}{2}\right| \end{cases} \\
&\leqslant \frac{1}{4}(b-a)\left(h'_-(b)-h'_+(a)\right).
\end{aligned}
\tag{4.4}
$$

The above inequalities provide both a generalization and several reverses for the left Fejér inequality (1.1) as announced in the introduction.

5. Applications for Hermite-Hadamard's inequality

If we take $g(x)=1$, $x\in[a,b]$ in (4.2)-(4.4), then we get for any convex function $h:[a,b]\subset\mathbb{R}\to\mathbb{R}$ and $\varphi\in\partial h$ that

$$\begin{aligned}
0 &\leqslant \frac{1}{b-a}\int_a^b h(x)\,dx - h\left(\frac{a+b}{2}\right)\\
&\leqslant \frac{1}{b-a}\int_a^b \left(\varphi(x)-\frac{h'_-(b)+h'_+(a)}{2}\right)\left(x-\frac{a+b}{2}\right)dx\\
&\leqslant \frac{1}{2}\int_a^b \left|\varphi(x)-\frac{h'_-(b)+h'_+(a)}{2}\right|dx\\
&\leqslant \frac{1}{4}(b-a)\left(h'_-(b)-h'_+(a)\right),\\
0 &\leqslant \frac{1}{b-a}\int_a^b h(x)\,dx - h\left(\frac{a+b}{2}\right)\\
&\leqslant \frac{1}{b-a}\int_a^b \left(\varphi(x)-\frac{h'_-(b)+h'_+(a)}{2}\right)\left(x-\frac{a+b}{2}\right)dx\\
&\leqslant \frac{1}{2(p+1)^{1/p}}(b-a)^{1/p}\left(\int_a^b \left|\varphi(x)-\frac{h'_-(b)+h'_+(a)}{2}\right|^q dx\right)^{1/q}\\
&\leqslant \frac{1}{4}(b-a)\left(h'_-(b)-h'_+(a)\right),
\end{aligned}$$

and

$$\begin{aligned}
0 &\leqslant \frac{1}{b-a}\int_a^b h(x)\,dx - h\left(\frac{a+b}{2}\right)\\
&\leqslant \frac{1}{b-a}\int_a^b \left(\varphi(x)-\frac{h'_-(b)+h'_+(a)}{2}\right)\left(x-\frac{a+b}{2}\right)dx\\
&\leqslant \frac{1}{4}(b-a)\operatorname*{essup}_{x\in[a,b]}\left|\varphi(x)-\frac{h'_-(b)+h'_+(a)}{2}\right|\\
&\leqslant \frac{1}{8}(b-a)\left(h'_-(b)-h'_+(a)\right).
\end{aligned}\tag{5.1}$$

For instance, if we take $h:[a,b]\subset(0,\infty)\to(0,\infty)$, $h(x)=-\ln x$ in the inequality (5.1), then we get

$$1\leqslant\frac{A(a,b)}{I(a,b)}\leqslant\exp\left(\frac{A(a,b)}{L(a,b)}-1\right)\leqslant\exp\left(\frac{1}{8}\cdot\frac{(b-a)^2}{G^2(a,b)}\right),$$

where $A(a,b):=\frac{a+b}{2}$ is the *arithmetic mean*, $G(a,b):=\sqrt{ab}$ is the *geometric mean*,

$$L(a,b):=\begin{cases}\frac{b-a}{\ln b-\ln a}, & \text{if } b\neq a,\\ b, & \text{if } b=a\end{cases}$$

is the *logarithmic mean*, and

$$I(a,b):=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } b\neq a,\\ b, & \text{if } b=a\end{cases}$$

is the *identric mean*.

Also, if we take $h : [a, b] \subset (0, \infty) \to (0, \infty)$, $h(x) = \frac{1}{x}$ in the inequality (5.1) then we get

$$0 \leqslant \frac{A(a,b) - L(a,b)}{A(a,b)L(a,b)} \leqslant \frac{L(a,b)A(a,b) - G^2(a,b)}{G^2(a,b)L(a,b)} \leqslant \frac{1}{4}\frac{A(a,b)}{G^4(a,b)}(b-a)^2.$$

We define the p-*logarithmic mean* by

$$L_p(a,b) := \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{1/p}, & \text{if } b \neq a, \\ b, & \text{if } b = a, \end{cases}$$

where $p \in \mathbb{R} \setminus \{-1, 0\}$.

We extend this mean for $\{-1, 0\}$ by putting $L_{-1}(a,b) := L(a,b)$ and $L_0(a,b) := I(a,b)$. With this convention we have that the function $\mathbb{R} \ni p \longmapsto L_p(a,b)$ is monotonic increasing.

If we take $h : [a, b] \subset (0, \infty) \to (0, \infty)$, $h(x) = x^p$, $p \in (-\infty, 0) \cup (1, \infty)$ in the inequality (5.1), then we get

$$0 \leqslant L_p^p(a,b) - A^p(a,b) \leqslant p\left(L_p^p(a,b) - L_{p-1}^{p-1}(a,b)A(a,b)\right) \leqslant \frac{1}{8}p(p-1)(b-a)^2 L_{p-2}^{p-2}(a,b).$$

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