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Hardy-Littlewood maximal function on noncommutative Lorentz spaces

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Abstract
This paper is mainly devoted to the study of the Hardy-Littlewood maximal function on noncommutative Lorentz spaces and to obtaining (p,q)-(p,q)-type inequality for the Hardy-Littlewood maximal function on noncommutative Lorentz spaces.

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Keywords: noncommutative Lorentz spaces; Hardy-Littlewood maximal function; von Neumann algebra

1 Introduction

In [1] Nelson defined the measure topology of τ-measurable operators affiliated with a semi-finite von Neumann algebra. Fack and Kosaki [2] investigated generalized s-numbers of τ-measurable operators and proved dominated convergence theorems for a gage and convexity (or concavity) inequality.

As for noncommutative maximal inequalities, a version of ergodic theory was given by Junge [3] and Junge, Xu [4]. In 2007, Mei [5] presented a version of noncommutative Hardy-Littlewood maximal inequality for an operator-valued function. In this paper, we study another version of Hardy-Littlewood maximal inequality introduced by Bekjan [6]. In [6], Bekjan defined the Hardy-Littlewood maximal function for τ-measurable operators and, among other things, obtained weak $(1,1)$-type and (p,p)-type inequalities for the Hardy-Littlewood maximal function. In [6], for an operator T affiliated with a semi-finite von Neumann algebra, the Hardy-Littlewood maximal function of T is defined by

$$MT(x) = \sup_{r>0} \frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \tau\big(|T| E_{[x-r,x+r]}(|T|)\big).$$

The classical Hardy-Littlewood maximal function of a Lebesgue measurable function $f: \mathcal{R} \to \mathcal{R}$ denoted by $Mf(x)$ is defined as

$$Mf(x) = \sup_{r>0} \frac{1}{m([x-r,x+r])} \int_{[x-r,x+r]} \big|f(t)\big|\,dt,$$

where m is a Lebesgue measure on $(-\infty,+\infty)$ (*cf.* [7]). Moreover, a natural generalization of this is the case $f: \mathcal{R} \to \mathcal{R}$ and μ, a Borel measure on $(-\infty,+\infty)$, where

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu([x-r,x+r])} \int_{[x-r,x+r]} \big|f(t)\big|\,d\mu(t).$$

As discussed by Bekjan in [6], let $\mu(A) = \tau(E_A(|T|))$, where A is a Borel subset of $(-\infty, +\infty)$. Then μ is a Borel measure and

$$MT(x) = \sup_{r>0} \frac{1}{\mu([x-r, x+r])} \int_{[x-r,x+r]} t\, d\mu(t),$$

i.e., $MT(x)$ is the Hardy-Littlewood maximal function $M_\mu f(x)$ of $f : \mathcal{R} \to \mathcal{R}$ defined by

$$f(t) = \begin{cases} t, & t \in \sigma(|T|), \\ 0, & t \notin \sigma(|T|), \end{cases} \tag{1.1}$$

with respect to μ.

In view of spectral theory, $|T|$ is represented as

$$|T| = \int_{\sigma(|T|)} t\, dE_t, \tag{1.2}$$

and $MT(|T|)$ is represented as $MT(x)$. Thus, for T, $MT(|T|)$ is considered as the operator analogue of the Hardy-Littlewood maximal function in the classical case. Therefore, roughly speaking, $MT(|T|)$ stands in relation to T as $Mf(x)$ stands in relation to f in classical analysis.

In this paper, we study the Hardy-Littlewood maximal function on noncommutative Lorentz spaces. By primarily adapting the techniques in [8], we obtain the (p,q)-(p,q)-type inequality for the Hardy-Littlewood maximal function on noncommutative Lorentz spaces.

The remainder of this paper is organized as follows. Section 2 consists of some notations and preliminaries, including the noncommutative Lorentz spaces and their properties. In Section 3, we present the main result of this paper.

2 Preliminaries

Throughout the paper, let $\mathcal{M}$ be a finite von Neumann algebra acting on the Hilbert space $\mathcal{H}$ with a normal faithful tracial state τ, and C will be a numerical constant not necessarily the same in each instance. The identity in $\mathcal{M}$ is denoted by 1, and we denote by $\mathcal{M}_{\text{proj}}$ the lattice of (orthogonal) projections in $\mathcal{M}$. A linear operator $T : \text{dom}(T) \to \mathcal{H}$, with domain $\text{dom}(T) \subseteq \mathcal{H}$, is said to be affiliated with $\mathcal{M}$ if $uT = Tu$ for all unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. The closed densely defined linear operator T affiliated with $\mathcal{M}$ is called τ-measurable if for every $\epsilon > 0$ there exists an orthogonal projection $P \in \mathcal{M}_{\text{proj}}$ such that $P(H) \subseteq \text{dom}(T)$ and $\tau(1-P) < \epsilon$. The collection of all τ-measurable operators is denoted by $\widetilde{\mathcal{M}}$. With the sum and product defined as the respective closures of the algebraic sum and product, $\widetilde{\mathcal{M}}$ is an $*$-algebra. For a positive self-adjoint operator T affiliated with $\mathcal{M}$, we set

$$\tau(T) = \sup_n \tau\left(\int_0^n \lambda\, dE_\lambda\right) = \int_0^\infty \lambda\, d\tau(E_\lambda),$$

where $T = \int_0^\infty \lambda\, dE_\lambda$ is the spectral decomposition of T.

Let T be a τ-measurable operator and $t > 0$. The 'tth singular number (or generalized s-number) of T' is defined by

$$\mu_t(T) = \inf\bigl\{\|TE\| : E \in \mathcal{M}_{\text{proj}}, \tau(1-E) \le t\bigr\}.$$

By Proposition 2.2 of [2], we have

$$\mu_t(T) = \inf\bigl\{s \ge 0 : \lambda_s(T) \le t\bigr\} \quad (t > 0),$$

where $\lambda_s(T) = \tau(E_{(s,\infty)}(|T|))$ $(s \ge 0)$ and $E_{(s,\infty)}(|T|)$ is the spectral projection of $|T|$ corresponding to the interval (s,∞). The reader is referred to [2] for basic properties and detailed information on generalized s-numbers and the distribution function of τ-measurable operators.

Definition 2.1 (See, *e.g.*, [9]) Let T be a τ-measurable operator affiliated with a finite von Neumann algebra $\mathcal{M}$, and let $0 < p, q \le \infty$. Define

$$\|T\|_{L^{p,q}(\mathcal{M})} = \begin{cases} (\int_0^\infty (t^{\frac{1}{p}}\mu_t(T))^q \frac{dt}{t})^{\frac{1}{q}} & \text{if } q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}}\mu_t(T) & \text{if } q = \infty. \end{cases} \tag{2.1}$$

The set of all $T \in \widetilde{\mathcal{M}}$ with $\|T\|_{L^{p,q}(\mathcal{M})} < \infty$ is called the noncommutative Lorentz space, denoted by $L^{p,q}(\mathcal{M})$ with indices p and q.

For convenience, we need the following Hardy inequalities in [10].

Lemma 2.2 *If $q \ge 1$, $r > 0$ and $f \ge 0$, then*

$$\left(\int_0^\infty \left[\int_0^t f(y)\,dy\right]^q t^{-r-1}\,dt\right)^{\frac{1}{q}} \le \frac{q}{r}\left(\int_0^\infty \bigl[yf(y)\bigr]^q y^{-r-1}\right)^{\frac{1}{q}}$$

and

$$\left(\int_0^\infty \left[\int_t^\infty f(y)\,dy\right]^q t^{r-1}\,dt\right)^{\frac{1}{q}} \le \frac{q}{r}\left(\int_0^\infty \bigl[yf(y)\bigr]^q y^{r-1}\right)^{\frac{1}{q}}.$$

Lemma 2.3 *Let $0 < r_2 < p < \infty$ and $0 < q, s < \infty$, then*

$$L^{p,q}(\mathcal{M}) \subset L^{r_2,s}(\mathcal{M}).$$

Let $L_{\text{loc}}(\mathcal{M};\tau)$ be the set of all τ-measurable operators such that

$$\tau\bigl(|T|E_I\bigl(|T|\bigr)\bigr) < +\infty$$

for all bounded intervals $I \subset [0,+\infty)$.

Definition 2.4 (See, *e.g.*, [6]) Let $T \in L_{\mathrm{loc}}(\mathcal{M};\tau)$, the maximal function of T is defined by

$$MT(x) = \sup_{r>0} \frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \tau\bigl(|T|E_{[x-r,x+r]}\bigl(|T|\bigr)\bigr)$$

(let $\frac{0}{0} = 0$). M is called the Hardy-Littlewood maximal operator.

Remark 2.5 By the introduction of [6], we know that $MT(|T|)$ is represented as $MT(x)$. Hence, for $T \in L_{\mathrm{loc}}(\mathcal{M};\tau)$, $MT(|T|)$ is considered as the operator analogue of the Hardy-Littlewood maximal function in the classical case. Therefore, roughly speaking, $MT(|T|)$ stands in relation to T as $Mf(x)$ stands in relation to f in classical analysis. Also, in [6], it was proved that $MT(|T|)$ defined in Definition 2.4 was weak $(1,1)$-type and (p,p)-type. We refer the readers to [6] for more details and basic properties of $MT(|T|)$.

3 Main result

Lemma 3.1 *Let* $0 < q < \infty$, $1 \le p, p_0, p_1 < \infty$ *and* $p_0 \ne p_1$ *such that*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{for some } 0 < \theta < 1.$$

Assume that $\mathcal{M}$ *has no minimal projection, then there exists a constant* C *such that* $\forall T \in L^{p,q}(\mathcal{M})$ *we have*

$$\|MT\|_{p,q} \le C\|T\|_{p,q}. \tag{3.1}$$

Proof We assume that $p_0 < p_1$. Theorem 2 of [6] and Lemma 2.3 imply that

$$\|MT\|_{p_0,\infty} \le \|MT\|_{p_0} \le C\|T\|_{p_0} \le C\|T\|_{p_0,m} \tag{3.2}$$

and

$$\|MT\|_{p_1,\infty} \le \|MT\|_{p_1} \le C\|T\|_{p_1} \le C\|T\|_{p_1,m}, \tag{3.3}$$

where $m = \frac{1}{2}\min(1,q)$.

By Lemma 1.8 of [11], for all $t \in (0,1)$, we can take $P \in \mathcal{M}_{\mathrm{proj}}$ such that $P|T| = |T|P$ and $\tau(P) = t$. Set $T_1 = |T|P$, $T_2 = |T| - T_1$, it is easy to check that $T_1 \in L^{p_0,m}(\mathcal{M})$ and $T_2 \in L^{p_1,m}(\mathcal{M})$. Indeed, we see that $\mu_\nu(T_1) = \mu_\nu(|T|P) = \mu_\nu(T)\chi_{[0,t]}$ and $\mu_\nu(T_2) = \mu_\nu(|T|P^\perp) = \mu_{\nu+t}(T)$. Thus we obtain

$$\begin{aligned}
\|T_1\|_{p_0,m}^m &= \int_0^\infty \nu^{\frac{m}{p_0}-1}\mu_\nu\bigl(|T|P\bigr)^m\,d\nu = \int_0^t \nu^{\frac{m}{p_0}-1}\mu_\nu(T)^m\,d\nu \\
&= \int_0^t \bigl(\nu^{\frac{1}{p}}\mu_\nu(T)\bigr)^m \nu^{\frac{m}{p_0}-\frac{m}{p}-1}\,d\nu \\
&\le \|T\|_{p,\infty}^m \int_0^t \nu^{\frac{m}{p_0}-\frac{m}{p}-1}\,d\nu \\
&\le \left(\frac{q}{p}\right)^{\frac{m}{q}} \|T\|_{p,q}^m \frac{1}{\frac{m}{p_0}-\frac{m}{p}} t^{\frac{m}{p_0}-\frac{m}{p}} < \infty
\end{aligned}$$

and

$$\begin{aligned}
\|T_2\|_{p_1,m}^m &= \int_0^\infty v^{\frac{m}{p_1}-1}\mu_v\bigl(|T|P^\perp\bigr)^m\,dv = \int_0^\infty v^{\frac{m}{p_1}-1}\mu_{v+t}(T)^m\,dv \\
&\le \int_0^t v^{\frac{m}{p_1}-1}\mu_t(T)^m\,dv + \int_t^\infty v^{\frac{m}{p_1}-1}\mu_v(T)^m\,dv \\
&\le \frac{p_1}{m}t^{\frac{m}{p_1}}\mu_t(T)^m + \sup_{v>t}\bigl(v^{\frac{1}{p}}\mu_v(T)\bigr)^m \int_t^\infty v^{\frac{m}{p_1}-\frac{m}{p}-1}\,dv \\
&= \frac{p_1}{m}t^{\frac{m}{p_1}}\mu_t(T)^m + \|T\|_{p,\infty}^m \frac{1}{\frac{m}{p}-\frac{m}{p_1}}t^{\frac{m}{p_1}-\frac{m}{p}} \\
&\le \frac{p_1}{m}t^{\frac{m}{p_1}-\frac{m}{p}}\Bigl(\sup_{t>0}t^{\frac{1}{p}}\mu_t(T)\Bigr)^m + \Bigl(\frac{q}{p}\Bigr)^{\frac{m}{q}}\frac{1}{\frac{m}{p}-\frac{m}{p_1}}t^{\frac{m}{p_1}-\frac{m}{p}}\|T\|_{p,q}^m \\
&= \Biggl[\Bigl(\frac{q}{p}\Bigr)^{\frac{m}{q}}t^{\frac{m}{p_1}-\frac{m}{p}}\Biggl(\frac{p_1}{m}+\frac{1}{\frac{m}{p}-\frac{m}{p_1}}\Biggr)\Biggr]\|T\|_{p,q}^m < \infty.
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\bigl(|T|E_{[x-r,x+r]}\bigl(|T|\bigr)\bigr) \\
&\quad\le \frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\bigl(|T|PE_{[x-r,x+r]}\bigl(|T|\bigr)\bigr) \\
&\qquad + \frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\bigl(|T|P^\perp E_{[x-r,x+r]}\bigl(|T|\bigr)\bigr) \\
&\quad= \frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\bigl(|T_1|E_{[x-r,x+r]}\bigl(|T_1|\bigr)\bigr) \\
&\qquad + \frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\bigl(|T_2|E_{[x-r,x+r]}\bigl(|T_2|\bigr)\bigr) \\
&\quad\le \frac{1}{\tau(E_{[x-r,x+r]}(|T_1|))}\tau\bigl(|T_1|E_{[x-r,x+r]}\bigl(|T_1|\bigr)\bigr) \\
&\qquad + \frac{1}{\tau(E_{[x-r,x+r]}(|T_2|))}\tau\bigl(|T_2|E_{[x-r,x+r]}\bigl(|T_2|\bigr)\bigr),
\end{aligned}$$

taking supremum, we get

$$MT(x) \le MT_1(x) + MT_2(x),$$

which implies that

$$\|MT\|_{p,q} \le C\bigl(\|MT_1\|_{p,q} + \|MT_2\|_{p,q}\bigr).$$

We estimate each term separately. For the first term, using (3.2) we get

$$\begin{aligned}
\|MT_1\|_{p,q} &= \left\{\int_0^\infty t^{\frac{q}{p}}\bigl(\mu_t(MT_1)\bigr)^q\frac{dt}{t}\right\}^{\frac{1}{q}} \\
&= \left\{\int_0^\infty t^{\frac{q}{p}-\frac{q}{p_0}}\bigl(t^{\frac{1}{p_0}}\mu_t(MT_1)\bigr)^q\frac{dt}{t}\right\}^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
&\le C\left\{\int_0^\infty t^{q(\frac{1}{p}-\frac{1}{p_0})}\|T_1\|_{p_0,m}^q \frac{dt}{t}\right\}^{\frac{1}{q}} \\
&= C\left\{\left[\int_0^\infty t^{-q(\frac{1}{p_0}-\frac{1}{p})-1}\left(\int_0^t v^{\frac{m}{p_0}-1}\mu_v(T)^m\,dv\right)^{\frac{q}{m}} dt\right]^{\frac{m}{q}}\right\}^{\frac{1}{m}}.
\end{aligned}$$

After replacing r and q respectively with $q(\frac{1}{p_0}-\frac{1}{p})$ and $\frac{q}{m}$ in the first inequality in Lemma 2.2, we see that the last expression is estimated as follows:

$$\begin{aligned}
&\le \frac{C}{(\frac{m}{p_0}-\frac{m}{p_1})^{\frac{1}{m}}}\left\{\int_0^\infty \left[v\cdot v^{\frac{m}{p_0}-1}\cdot \mu_v(T)^m\right]^{\frac{q}{m}}\cdot v^{-q(\frac{1}{p_0}-\frac{1}{p})-1}\,dv\right\}^{\frac{1}{q}} \\
&= C\left(\int_0^\infty v^{\frac{q}{p}-1}\mu_v(T)^q\,dv\right)^{\frac{1}{q}} \\
&= C\|T\|_{p,q},
\end{aligned}$$

i.e., $\|MT_1\|_{p,q}\le C\|T\|_{p,q}$. To estimate the second term, by applying (3.3) we obtain

$$\begin{aligned}
\|MT_2\|_{p,q} &= \left\{\int_0^\infty t^{\frac{q}{p}}\left(\mu_t(MT_2)\right)^q\frac{dt}{t}\right\}^{\frac{1}{q}} \\
&= \left\{\int_0^\infty t^{\frac{q}{p}-\frac{q}{p_1}}\left(t^{\frac{1}{p_1}}\mu_t(MT_2)\right)^q\frac{dt}{t}\right\}^{\frac{1}{q}} \\
&\le C\left\{\int_0^\infty t^{q(\frac{1}{p}-\frac{1}{p_1})}\|T_2\|_{p_1,m}^q\frac{dt}{t}\right\}^{\frac{1}{q}} \\
&= C\left\{\left[\int_0^\infty t^{q(\frac{1}{p}-\frac{1}{p_1})-1}\left(\int_0^\infty v^{\frac{m}{p_1}-1}\mu_v(T_2)^m\,dv\right)^{\frac{q}{m}}dt\right]^{\frac{m}{q}}\right\}^{\frac{1}{m}} \\
&\le C\left\{\int_0^\infty t^{q(\frac{1}{p}-\frac{1}{p_1})-1}\left(\int_0^t v^{\frac{m}{p_1}-1}\mu_t(T)^m\,dv\right)^{\frac{q}{m}}dt\right\}^{\frac{1}{q}} \\
&\quad + C\left\{\left[\int_0^\infty t^{q(\frac{1}{p}-\frac{1}{p_1})-1}\left(\int_t^\infty v^{\frac{m}{p_1}-1}\mu_v(T)^m\,dv\right)^{\frac{q}{m}}dt\right]^{\frac{m}{q}}\right\}^{\frac{1}{m}}.
\end{aligned}$$

For the second term $\{[\int_0^\infty t^{q(\frac{1}{p}-\frac{1}{p_1})-1}(\int_t^\infty v^{\frac{m}{p_1}-1}\mu_v(T)^m\,dv)^{\frac{q}{m}}\,dt]^{\frac{m}{q}}\}^{\frac{1}{m}}$, replace r and q respectively with $q(\frac{1}{p}-\frac{1}{p_1})$ and $\frac{q}{m}$ in the second inequality in Lemma 2.2, and we estimate the last expression as follows:

$$\begin{aligned}
&\le C\left\{\frac{p_1}{m}\mu_t(T)^q\int_0^\infty t^{\frac{q}{p}-1}\,dt\right\}^{\frac{1}{q}} + C\left\{\int_0^\infty\left[v\cdot v^{\frac{m}{p_1}-1}\cdot\mu_v(T)^m\right]^{\frac{q}{m}}v^{q(\frac{1}{p}-\frac{1}{p_1})-1}\,dv\right\}^{\frac{1}{q}} \\
&= C\left\{\frac{p_1}{m}\mu_t(T)^q\int_0^\infty t^{\frac{q}{p}-1}\,dt\right\}^{\frac{1}{q}} + C\|T\|_{p,q} \\
&\le C\|T\|_{p,q},
\end{aligned}$$

i.e., $\|MT_2\|_{p,q}\le C\|T\|_{p,q}$.

For the case of $p_0 > p_1$, we may simply reverse the roles of p_0 and p_1 in the above proof.

We have now shown that

$$\|MT\|_{p,q} \le C\|T\|_{p,q}. \qquad \square$$

Theorem 3.2 *Let* $0 < q < \infty$, $1 \le p, p_0, p_1 < \infty$ *and* $p_0 \ne p_1$ *be such that*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{for some } 0 < \theta < 1.$$

Assume that $\mathcal{M}$ *has minimal projections, then there exists a constant* C *such that for all* $T \in L^{p,q}(\mathcal{M})$ *we have*

$$\|MT\|_{p,q} \le C\|T\|_{p,q}.$$

Proof Since $\mathcal{M}$ has minimal projections, we consider the von Neumann algebra tensor product $\mathcal{M}\overline{\otimes}L^\infty([0,1];dm)$ denoted by $\overline{\mathcal{M}}$, equipped with the tensor product trace $\tau \otimes dm$, where dm is the Lebesgue measure on $[0,1]$, then $\overline{\mathcal{M}}$ has no minimal projection.

Let $|T| = \int_{\sigma(|T|)} \lambda \, dE_\lambda(|T|)$ be the spectral decomposition of T. Since

$$\sigma\bigl(|T|\bigr) = \sigma\bigl(|T| \otimes 1\bigr),$$

we have

$$|T \otimes 1| = |T| \otimes 1 = \int_{\sigma(|T|)} \lambda \, d\bigl(E_\lambda\bigl(|T|\bigr) \otimes 1\bigr) = \int_{\sigma(|T|\otimes 1)} \lambda \, d\bigl(E_\lambda\bigl(|T|\bigr) \otimes 1\bigr).$$

It is easy to check that $E_\lambda(|T|) \otimes 1$ is a spectral series for each $\lambda \ge 0$. Hence, for any interval

$$I \subset \sigma\bigl(|T|\bigr) = \sigma\bigl(|T \otimes 1|\bigr) = \sigma\bigl(|T| \otimes 1\bigr),$$

by the uniqueness of the spectral decomposition, we see that

$$E_I\bigl(|T \otimes 1|\bigr) = E_I\bigl(|T|\bigr) \otimes 1.$$

For $\forall r > 0$, since

$$\tau\bigl(E_{[x-r,x+r]}\bigl(|T|\bigr)\bigr) = \int_0^1 \tau\bigl(E_{[x-r,x+r]}\bigl(|T|\bigr)\bigr)\, dm = \tau \otimes dm\bigl(E_{[x-r,x+r]}\bigl(|T|\bigr) \otimes 1\bigr)$$

and

$$\begin{aligned}
&\tau \otimes dm\bigl(|T \otimes 1| E_{[x-r,x+r]}\bigl(|T|\bigr) \otimes 1\bigr) \\
&\quad = \tau \otimes dm\bigl\{\bigl(|T \otimes 1| E_{[x-r,x+r]}\bigl(|T|\bigr) \otimes 1\bigr)^* \bigl(|T \otimes 1| E_{[x-r,x+r]}\bigl(|T|\bigr) \otimes 1\bigr)\bigr\}^{\frac{1}{2}} \\
&\quad = \tau \otimes dm\bigl\{\bigl(E_{[x-r,x+r]}\bigl(|T|\bigr) \otimes 1\bigr)^* |T \otimes 1| \bigl(|T \otimes 1| E_{[x-r,x+r]}\bigl(|T|\bigr) \otimes 1\bigr)\bigr\}^{\frac{1}{2}} \\
&\quad = \tau \otimes dm\bigl\{\bigl(E_{[x-r,x+r]}\bigl(|T|\bigr) \otimes 1\bigr) |T \otimes 1|^2 E_{[x-r,x+r]}\bigl(|T|\bigr) \otimes 1\bigr\}^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \tau \otimes dm\big\{E_{[x-r,x+r]}\big(|T|\big) \otimes 1\big(|T|^2 \otimes 1\big)E_{[x-r,x+r]}\big(|T|\big) \otimes 1\big\}^{\frac{1}{2}} \\
&= \tau \otimes dm\big\{\big(E_{[x-r,x+r]}\big(|T|\big) \otimes 1\big)\big(|T| \otimes 1\big)\big(|T| \otimes 1\big)\big(E_{[x-r,x+r]}\big(|T|\big) \otimes 1\big)\big\}^{\frac{1}{2}} \\
&= \tau \otimes dm\big(|T|E_{[x-r,x+r]}\big(|T|\big) \otimes 1\big) \\
&= \tau\big(|T|E_{[x-r,x+r]}\big(|T|\big)\big),
\end{aligned}
$$

which implies that

$$M(T \otimes 1)(x) = MT(x).$$

By an adaptation of the proof of Lemma 3.1, we deduce that

$$\big\|M(T \otimes 1)(T \otimes 1)\big\|_{p,q} \leq C\|T \otimes 1\|_{p,q}.$$

With the trivial fact $\mu_t(T) = \mu_t(T \otimes 1)$, we know

$$\|T \otimes 1\|_{p,q} = \|T\|_{p,q}.$$

Combing this result with $M(T \otimes 1)(|T \otimes 1|) = MT(|T|)$, we infer that

$$\|MT\|_{p,q} \leq C\|T\|_{p,q}. \qquad \square$$

Competing interests
The author declares that she has no competing interests.

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