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Weighted majorization theorems via generalization of Taylor's formula

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Abstract
A new generalization of the weighted majorization theorem for *n*-convex functions is given, by using a generalization of Taylor's formula. Bounds for the remainders in new majorization identities are given by using the Čebyšev type inequalities. Mean value theorems and *n*-exponential convexity are discussed for functionals related to the new majorization identities.

MSC: 26D15; 26D20

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1 Introduction

Unless stated otherwise throughout this section I is an interval of $\mathbb{R}$.

Definition 1 A function $f : I \to \mathbb{R}$ is called *convex* if the inequality

$$f\bigl(\lambda x_1 + (1-\lambda)x_2\bigr) \le \lambda f(x_1) + (1-\lambda)f(x_2) \tag{1.1}$$

holds for each $x_1, x_2 \in I$ and $\lambda \in [0,1]$.

Remark 1

(a) If inequality (1.1) is strict for each $x_1 \neq x_2$ and $\lambda \in (0,1)$, then f is called *strictly convex*.

(b) If the inequality in (1.1) is reversed, then f is called *concave*. If it is strict for each $x_1 \neq x_2$ and $\lambda \in (0,1)$, then f is called *strictly concave*.

The following proposition gives us an alternative definition of convex functions [1], p.2.

Proposition 1 *A function $f : I \to \mathbb{R}$ is convex if the inequality*

$$(x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \ge 0$$

holds for each $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$.

The following result can be deduced from Proposition 1.

Proposition 2 *If a function $f : I \to \mathbb{R}$ is convex, then the inequality*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(y_2) - f(y_1)}{y_2 - y_1}$$

holds for each $x_1, x_2, y_1, y_2 \in I$ such that $x_1 \le y_1$, $x_2 \le y_2$, $x_1 \ne x_2$, $y_1 \ne y_2$.

Now we define the generalized convex function which can be found in [2, 3] and [1].

Definition 2 The *nth order divided difference* of a function $f : I \to \mathbb{R}$ at distinct points $x_i, x_{i+1}, \dots, x_{i+n} \in I = [a, b] \subset \mathbb{R}$ for some $i \in \mathbb{N}$ is defined recursively by

$$[x_j; f] = f(x_j), \quad j \in \{i, \dots, i+n\},$$
$$[x_i, \dots, x_{i+n}; f] = \frac{[x_{i+1}, \dots, x_{i+n}; f] - [x_i, \dots, x_{i+n-1}; f]}{x_{i+n} - x_i}.$$

It may easily be verified that

$$[x_i, \dots, x_{i+n}; f] = \sum_{k=0}^{n} \frac{f(x_{i+k})}{\prod_{j=i, j \ne i+k}^{i+n} (x_{i+k} - x_j)}.$$

Remark 2 Let us denote $[x_i, \dots, x_{i+n}; f]$ by $\Delta_{(n)} f(x_i)$. The value $[x_i, \dots, x_{i+n}; f]$ is independent of the order of the points $x_i, x_{i+1}, \dots, x_{i+n}$. We can extend this definition by including the cases in which two or more points coincide by taking respective limits.

Definition 3 A function $f : I \to \mathbb{R}$ is called *convex of order n* or *n-convex* if for all choices of $(n+1)$ distinct points $x_i, \dots, x_{i+n}$ we have $\Delta_{(n)} f(x_i) \ge 0$.

If the nth order derivative $f^{(n)}$ exists, then f is *n-convex* if and only if $f^{(n)} \ge 0$.

Remark 3 For $n = 2$ and $i = 0$, we get the *second order divided difference* of a function $f : I \to \mathbb{R}$, which is defined recursively by

$$\begin{aligned} &[x_j; f] = f(x_j), \quad j \in \{0, 1, 2\}, \\ &[x_j, x_{j+1}; f] = \frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j}, \quad j \in \{0, 1\}, \\ &[x_0, x_1, x_2; f] = \frac{[x_1, x_2; f] - [x_0, x_1; f]}{x_2 - x_0}, \end{aligned} \tag{1.2}$$

for arbitrary points $x_0, x_1, x_2 \in I$. Now, we discuss some limiting cases as follows: taking the limit as $x_1 \to x_0$ in (1.2), we get

$$\lim_{x_1 \to x_0} [x_0, x_1, x_2; f] = [x_0, x_0, x_2; f] = \frac{f(x_2) - f(x_0) - f'(x_0)(x_2 - x_0)}{(x_2 - x_0)^2}, \quad x_2 \ne x_0,$$

provided that $f'(x_0)$ exists. Furthermore, taking the limits as $x_i \to x_0$, $i \in \{1,2\}$ in (1.2), we obtain

$$\lim_{\substack{x_1 \to x_0 \\ x_2 \to x_0}} [x_0, x_1, x_2; f] = [x_0, x_0, x_0; f] = \frac{f''(x_0)}{2},$$

provided that $f''(x_0)$ exists.

For fixed $m \geq 2$, let $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ denote two real m-tuples and $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[m]}$, $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[m]}$ their ordered components.

Definition 4 For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$,

$$\mathbf{x} \prec \mathbf{y} \quad \text{if} \begin{cases} \sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}, & k \in \{1, \ldots, m-1\}, \\ \sum_{i=1}^m x_{[i]} = \sum_{i=1}^m y_{[i]}, \end{cases}$$

when $\mathbf{x} \prec \mathbf{y}$, $\mathbf{x}$ *is said to be majorized by* $\mathbf{y}$ or $\mathbf{y}$ *majorizes* $\mathbf{x}$.

This notion and notation of majorization was introduced by Hardy *et al.* [4]. Now, we state the well-known majorization theorem from the same book [4] as follows.

Proposition 3 *Let* $\mathbf{x}, \mathbf{y} \in [a,b]^m$. *The inequality*

$$\sum_{i=1}^m f(x_i) \leq \sum_{i=1}^m f(y_i) \tag{1.3}$$

holds for every continuous convex function $f : [a,b] \to \mathbb{R}$ *if and only if* $\mathbf{x} \prec \mathbf{y}$. *Moreover, if* f *is a strictly convex function, then equality in* (1.3) *is valid if and only if* $x_{[i]} = y_{[i]}$ *for each* $i \in \{1, \ldots, m\}$.

The following weighted version of the majorization theorem was given by Fuchs in [5] (see also [6], p.580 and [1], p.323).

Proposition 4 *Let* $\mathbf{w} \in \mathbb{R}^m$ *and let* $\mathbf{x}, \mathbf{y} \in [a,b]^m$ *be two decreasing real m-tuples such that*

$$\sum_{i=1}^k w_i x_i \leq \sum_{i=1}^k w_i y_i, \quad k \in \{1, \ldots, m-1\} \quad \textit{and} \tag{1.4}$$

$$\sum_{i=1}^m w_i x_i = \sum_{i=1}^m w_i y_i. \tag{1.5}$$

Then for every continuous convex function $f : [a,b] \to \mathbb{R}$, *the following inequality holds*:

$$\sum_{i=1}^m w_i f(x_i) \leq \sum_{i=1}^m w_i f(y_i). \tag{1.6}$$

Remark 4 Under the assumptions of Proposition 4, for every concave function f the reverse inequality holds in (1.6).

The following proposition is a consequence of Theorem 1 in [7] (see also [1], p.328) and represents an integral majorization result.

Proposition 5 *Let $x, y : [\alpha,\beta] \to I$ be two decreasing continuous functions and $w : [\alpha,\beta] \to \mathbb{R}$ continuous. Then if*

$$\int_\alpha^u w(t)x(t)\,dt \le \int_\alpha^u w(t)y(t)\,dt, \quad \text{for each } u \in (\alpha,\beta), \quad \text{and} \tag{1.7}$$

$$\int_\alpha^\beta w(t)x(t)\,dt = \int_\alpha^\beta w(t)y(t)\,dt, \tag{1.8}$$

hold, then for every continuous convex function $f : I \to \mathbb{R}$ the following inequality holds:

$$\int_\alpha^\beta w(t)f\bigl(x(t)\bigr)\,dt \le \int_\alpha^\beta w(t)f\bigl(y(t)\bigr)\,dt. \tag{1.9}$$

Remark 5 Let $x, y : [\alpha,\beta] \to I$ be two increasing continuous functions and $w : [\alpha,\beta] \to \mathbb{R}$ continuous. If

$$\int_u^\beta w(t)x(t)\,dt \le \int_u^\beta w(t)y(t)\,dt, \quad \text{for each } u \in (\alpha,\beta), \quad \text{and}$$

$$\int_\alpha^\beta w(t)x(t)\,dt = \int_\alpha^\beta w(t)y(t)\,dt,$$

then again inequality (1.9) holds. In this paper we will state our results for decreasing x and y satisfying the assumption of Proposition 5, but they are still valid for increasing x and t satisfying the above condition; see for example [6], p.584.

In paper [8] the following extension of Montgomery identity via Taylor's formula is obtained.

Proposition 6 *Let $n \in \mathbb{N}$, $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Then the following identity holds*:

$$\begin{aligned} f(x) = {}& \frac{1}{b-a}\int_a^b f(t)\,dt + \sum_{k=0}^{n-2}\frac{f^{(k+1)}(a)}{k!(k+2)}\frac{(x-a)^{k+2}}{b-a} - \sum_{k=0}^{n-2}\frac{f^{(k+1)}(b)}{k!(k+2)}\frac{(x-b)^{k+2}}{b-a} \\ & + \frac{1}{(n-1)!}\int_a^b T_n(x,s)f^{(n)}(s)\,ds, \end{aligned} \tag{1.10}$$

where

$$T_n(x,s) = \begin{cases} -\frac{(x-s)^n}{n(b-a)} + \frac{x-a}{b-a}(x-s)^{n-1}, & a \le s \le x, \\ -\frac{(x-s)^n}{n(b-a)} + \frac{x-b}{b-a}(x-s)^{n-1}, & x < s \le b. \end{cases} \tag{1.11}$$

In case $n = 1$ the sum $\sum_{k=0}^{n-2} \cdots$ is empty, so identity (1.10) reduces to the well-known Montgomery identity (see for instance [9])

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \int_a^b P(x,s)f'(s)\,ds,$$

where $P(x,s)$ is the Peano kernel, defined by

$$P(x,s) = \begin{cases} \frac{s-a}{b-a}, & a \le s \le x, \\ \frac{s-b}{b-a}, & x < s \le b. \end{cases}$$

The aim of this paper is to present a new generalization of weighted majorization theorem for n-convex functions, by using generalization of Taylor's formula. We also obtain bounds for the remainders in new majorization identities by using the Čebyšev type inequalities. We give mean value theorems and n-exponential convexity for functionals related to these new majorization identities.

2 Majorization inequality by extension of Montgomery identity via Taylor's formula

Theorem 1 *Suppose all the assumptions from Proposition* 6 *hold. Additionally suppose that* $m \in \mathbb{N}$, $x_i, y_i \in [a,b]$ *and* $w_i \in \mathbb{R}$ *for* $i \in \{1,2,\dots,m\}$. *Then*

$$\begin{aligned} &\sum_{i=1}^{m} w_i f(y_i) - \sum_{i=1}^{m} w_i f(x_i) \\ &\quad = \frac{1}{b-a} \Bigg[\sum_{k=0}^{n-2} \frac{1}{k!(k+2)!} \sum_{i=1}^{m} w_i \big[f^{(k+1)}(a) \big[(y_i - a)^{k+2} - (x_i - a)^{k+2} \big] \\ &\qquad - f^{(k+1)}(b) \big[(y_i - b)^{k+2} - (x_i - b)^{k+2} \big] \big] \Bigg] \\ &\qquad + \frac{1}{(n-1)!} \int_a^b \left(\sum_{i=1}^{m} w_i \big(T_n(y_i,s) - T_n(x_i,s) \big) \right) f^{(n)}(s)\, ds. \end{aligned} \tag{2.1}$$

Proof We take extension of Montgomery identity via Taylor's formula (1.10) to obtain

$$\begin{aligned} &\sum_{i=1}^{m} w_i f(y_i) - \sum_{i=1}^{m} w_i f(x_i) \\ &\quad = \frac{1}{b-a} \int_a^b f(t)\, dt \sum_{i=1}^{m} w_i - \frac{1}{b-a} \int_a^b f(t)\, dt \sum_{i=1}^{m} w_i \\ &\qquad + \sum_{i=1}^{m} w_i \left(\sum_{k=0}^{n-2} \frac{f^{(k+1)}(a)}{k!(k+2)} \frac{(y_i - a)^{k+2}}{b-a} - \sum_{k=0}^{n-2} \frac{f^{(k+1)}(b)}{k!(k+2)} \frac{(y_i - b)^{k+2}}{b-a} \right) \\ &\qquad - \sum_{i=1}^{m} w_i \left(\sum_{k=0}^{n-2} \frac{f^{(k+1)}(a)}{k!(k+2)} \frac{(x_i - a)^{k+2}}{b-a} - \sum_{k=0}^{n-2} \frac{f^{(k+1)}(b)}{k!(k+2)} \frac{(x_i - b)^{k+2}}{b-a} \right) \\ &\qquad + \frac{1}{(n-1)!} \sum_{i=1}^{m} w_i \int_a^b T_n(y_i,s) f^{(n)}(s)\, ds - \frac{1}{(n-1)!} \sum_{i=1}^{m} w_i \int_a^b T_n(x_i,s) f^{(n)}(s)\, ds. \end{aligned}$$

By simplifying this expressions we obtain (2.1). □

We may state its integral version as follows:

Theorem 2 *Let $x, y : [\alpha, \beta] \to [a, b]$ be two functions and $w : [\alpha, \beta] \to \mathbb{R}$ continuous function. Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \in \mathbb{N}$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$, then for all $s \in [a, b]$ we have the following identity*:

$$\begin{aligned}
&\int_\alpha^\beta w(t)f\big(y(t)\big)\,dt - \int_\alpha^\beta w(t)f\big(x(t)\big)\,dt \\
&\quad = \frac{1}{b-a}\Bigg[\sum_{k=0}^{n-2}\frac{1}{k!(k+2)!}\int_\alpha^\beta w(t)\big[f^{(k+1)}(a)\big[\big(y(t)-a\big)^{k+2} - \big(x(t)-a\big)^{k+2}\big] \\
&\qquad - f^{(k+1)}(b)\big[\big(y(t)-b\big)^{k+2} - \big(x(t)-b\big)^{k+2}\big]\big]\,dt\Bigg] \\
&\qquad + \frac{1}{(n-1)!}\int_a^b\bigg(\int_\alpha^\beta w(t)\big(T_n\big(y(t),s\big) - T_n\big(x(t),s\big)\big)\,dt\bigg)f^{(n)}(s)\,ds,
\end{aligned} \tag{2.2}$$

where $T_n(\cdot, s)$ is as defined in Theorem 6.

Proof Our required result is obtained by using extension of Montgomery identity via Taylor's formula (1.10) in the following expression:

$$\int_\alpha^\beta w(t)f\big(y(t)\big)\,dt - \int_\alpha^\beta w(t)f\big(x(t)\big)\,dt$$

and then using Fubini's theorem. □

Now we state the main generalization of the majorization inequality by using the identities just obtained.

Theorem 3 *Let all the assumptions of Theorem* 1 *hold with the additional condition*

$$\sum_{i=1}^m w_i T_n(x_i, s) \le \sum_{i=1}^m w_i T_n(y_i, s), \quad \forall s \in [a, b]. \tag{2.3}$$

Then for every n-convex function $f : I \to \mathbb{R}$ the following inequality holds:

$$\begin{aligned}
&\sum_{i=1}^m w_i f(y_i) - \sum_{i=1}^m w_i f(x_i) \\
&\quad \ge \frac{1}{b-a}\Bigg[\sum_{k=0}^{n-2}\frac{1}{k!(k+2)!}\sum_{i=1}^m w_i\big[f^{(k+1)}(a)\big[(y_i-a)^{k+2} - (x_i-a)^{k+2}\big] \\
&\qquad - f^{(k+1)}(b)\big[(y_i-b)^{k+2} - (x_i-b)^{k+2}\big]\big]\Bigg].
\end{aligned} \tag{2.4}$$

Proof Since the function f is n-convex so we have $f^{(n)} \ge 0$. Using this fact and (2.3) in (2.1) we easily arrive at our required result. □

Remark 6 If reverse inequality holds in (2.3) then reverse inequality holds in (2.4).

Now we state important consequence as follows:

Corollary 1 *Suppose all the assumptions from Theorem* 1 *hold*. *Additionally suppose that* $\mathbf{x}, \mathbf{y} \in [a,b]^m$ *are two decreasing* m-*tuples and* $\mathbf{w} \in \mathbb{R}^m$ *which satisfy conditions* (1.4), (1.5). *If* f *is* $2n$-*convex then the following inequality holds*:

$$\begin{aligned}
&\sum_{i=1}^{m} w_i f(y_i) - \sum_{i=1}^{m} w_i f(x_i) \\
&\quad \geq \frac{1}{b-a}\Bigg[\sum_{k=0}^{2n-2} \frac{1}{k!(k+2)!} \sum_{i=1}^{m} w_i \big[f^{(k+1)}(a)\big[(y_i - a)^{k+2} - (x_i - a)^{k+2}\big] \\
&\qquad - f^{(k+1)}(b)\big[(y_i - b)^{k+2} - (x_i - b)^{k+2}\big]\big]\Bigg]. \qquad (2.5)
\end{aligned}$$

Moreover, *if* $f^{(j)}(a) \geq 0$ *and* $(-1)^j f^{(j)}(b) \geq 0$ *for* $j = 1, \ldots, 2n-1$ *then*

$$\sum_{i=1}^{m} w_i f(y_i) \geq \sum_{i=1}^{m} w_i f(x_i). \qquad (2.6)$$

Proof Since

$$T_n(x,s) = \begin{cases} -\frac{(x-s)^n}{n(b-a)} + \frac{x-a}{b-a}(x-s)^{n-1}, & a \leq s \leq x \leq b, \\ -\frac{(x-s)^n}{n(b-a)} + \frac{x-b}{b-a}(x-s)^{n-1}, & a \leq x < s \leq b. \end{cases}$$

and

$$\frac{d^2}{dx^2} T_n(x,s) = \begin{cases} \frac{n-1}{b-a}[(x-s)^{n-2} + (n-2)(x-a)(x-s)^{n-3}], & a \leq s \leq x \leq b, \\ \frac{n-1}{b-a}[(x-s)^{n-2} + (n-2)(x-b)(x-s)^{n-3}], & a \leq x < s \leq b. \end{cases}$$

$T_n(\cdot, s)$ is continuous for every $n \geq 2$ and convex function for even n. Thus it satisfies inequality (2.3) by weighted majorization theorem (Proposition 4) and hence (2.3) by Theorem 3 provides us (2.4) with $2n$ instead of n. Furthermore, we consider condition $f^{(j)}(a) \geq 0$ and $(-1)^j f^{(j)}(b) \geq 0$ for each $j = 1, \ldots, 2n-1$. By applying Proposition 4 with the continuous convex function $f(x) = (x-a)^{k+2}$, $x \in [a,b]$ we have

$$\sum_{i=1}^{m} w_i (y_i - a)^{k+2} \geq \sum_{i=1}^{m} w_i (x_i - a)^{k+2}.$$

Since continuous function $f(x) = (x-b)^{k+2}$, $x \in [a,b]$ is convex if k is even and concave if k is odd, by the same proposition we have

$$\begin{aligned}
&\sum_{i=1}^{m} w_i (y_i - b)^{k+2} \geq \sum_{i=1}^{m} w_i (x_i - b)^{k+2} \quad \text{if } k \text{ is even}, \\
&\sum_{i=1}^{m} w_i (y_i - b)^{k+2} \leq \sum_{i=1}^{m} w_i (x_i - b)^{k+2} \quad \text{if } k \text{ is odd}.
\end{aligned}$$

Now considering the assumption $f^{(j)}(a) \geq 0$ and $(-1)^j f^{(j)}(b) \geq 0$ for each $j = 1, \dots, 2n-1$ we have

$$\begin{aligned}
&\sum_{i=1}^{m} w_i \big[f^{(k+1)}(a) \big[(y_i - a)^{k+2} - (x_i - a)^{k+2} \big] - f^{(k+1)}(b) \big[(y_i - b)^{k+2} - (x_i - b)^{k+2} \big] \big] \\
&\quad = f^{(k+1)}(a) \sum_{i=1}^{m} w_i \big[(y_i - a)^{k+2} - (x_i - a)^{k+2} \big] - f^{(k+1)}(b) \sum_{i=1}^{m} w_i \big[(y_i - b)^{k+2} - (x_i - b)^{k+2} \big]
\end{aligned}$$

is positive for all $k = 0, 1, \dots, 2n-2$. Thus the right-hand side of (2.5) is positive and (2.6) holds. □

Remark 7 Since in case $a \leq s \leq x \leq b$ $\frac{d^2}{dx^2} T_n(x,s)$ is always positive, $T_n(x,s)$ cannot be concave and reverse inequalities cannot be observed.

Also, if $w_i = 1$, $i = 1, \dots, m$ the result of the previous corollary holds for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{x} \prec \mathbf{y}$.

Its integral analogs are given as follows:

Theorem 4 *Let all the assumptions of Theorem* 2 *hold with the additional condition*

$$\int_{\alpha}^{\beta} w(t) T_n\big(x(t), s\big)\, dt \leq \int_{\alpha}^{\beta} w(t) T_n\big(y(t), s\big)\, dt, \quad \forall s \in [a,b], \tag{2.7}$$

where $T_n(\cdot, s)$ is defined in Proposition 6. *Then for every n-convex function $f : I \to \mathbb{R}$ the following inequality holds*:

$$\begin{aligned}
&\int_{\alpha}^{\beta} w(t) f\big(y(t)\big)\, dt - \int_{\alpha}^{\beta} w(t) f\big(x(t)\big)\, dt \\
&\quad \geq \frac{1}{b-a} \Bigg[\sum_{k=0}^{n-2} \frac{1}{k!(k+2)!} \int_{\alpha}^{\beta} w(t) \big[f^{(k+1)}(a) \big[\big(y(t) - a\big)^{k+2} - \big(x(t) - a\big)^{k+2} \big] \\
&\qquad - f^{(k+1)}(b) \big[\big(y(t) - b\big)^{k+2} - \big(x(t) - b\big)^{k+2} \big] \big]\, dt \Bigg].
\end{aligned} \tag{2.8}$$

Proof Since the function f is n-convex so we have $f^{(n)} \geq 0$. Using this fact and (2.7) in (2.2) we easily arrive at our required result. □

Remark 8 If reverse inequality holds in (2.7) then reverse inequality holds in (2.8).

Corollary 2 *Suppose all the assumptions from Theorem* 2 *hold. Additionally suppose that x and y are decreasing and satisfy conditions* (1.7), (1.8). *If f is $2n$-convex then the following inequality holds*:

$$\begin{aligned}
&\int_{\alpha}^{\beta} w(t) f\big(y(t)\big)\, dt - \int_{\alpha}^{\beta} w(t) f\big(x(t)\big)\, dt \\
&\quad \geq \frac{1}{b-a} \Bigg[\sum_{k=0}^{2n-2} \frac{1}{k!(k+2)!} \int_{\alpha}^{\beta} w(t) \big[f^{(k+1)}(a) \big[\big(y(t) - a\big)^{k+2} - \big(x(t) - a\big)^{k+2} \big] \\
&\qquad - f^{(k+1)}(b) \big[\big(y(t) - b\big)^{k+2} - \big(x(t) - b\big)^{k+2} \big] \big]\, dt \Bigg].
\end{aligned} \tag{2.9}$$

Moreover, if $f^{(j)}(a) \geq 0$ *and* $(-1)^j f^{(j)}(b) \geq 0$ *for* $j = 1, \ldots, 2n-1$ *then*

$$\int_\alpha^\beta w(t) f\big(y(t)\big)\,dt \geq \int_\alpha^\beta w(t) f\big(x(t)\big)\,dt.$$

Proof By using the same arguments as we have given in Corollary 1, we easily arrive at our required results simply by replacing (1.6) by (1.9), (2.3) by (2.7) and (2.4) by (2.8). □

3 Bounds for identities related to generalization of majorization inequality

Let $g, h : [a,b] \to \mathbb{R}$ be two Lebesgue integrable functions. We consider the Čebyšev functional

$$T(g,h) = \frac{1}{b-a}\int_a^b g(x)h(x)\,dx - \left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b h(x)\,dx\right). \tag{3.1}$$

The following results can be found in [10].

Proposition 7 *Let* $g : [a,b] \to \mathbb{R}$ *be a Lebesgue integrable function and* $h : [a,b] \to \mathbb{R}$ *be an absolutely continuous function with* $(\cdot - a)(b - \cdot)[h']^2 \in L[a,b]$. *Then we have the inequality*

$$\big|T(g,h)\big| \leq \frac{1}{\sqrt{2}}\left(\frac{1}{b-a}\big|T(g,g)\big|\int_a^b (x-a)(b-x)\big[h'(x)\big]^2\,dx\right)^{1/2}. \tag{3.2}$$

The constant $\frac{1}{\sqrt{2}}$ *in* (3.2) *is the best possible.*

Proposition 8 *Let* $h : [a,b] \to \mathbb{R}$ *be a monotonic nondecreasing function and let* $g : [a,b] \to \mathbb{R}$ *be an absolutely continuous function such that* $g' \in L_\infty[a,b]$. *Then we have the inequality*

$$\big|T(g,h)\big| \leq \frac{1}{2(b-a)}\big\|g'\big\|_\infty \int_a^b (x-a)(b-x)\,dh(x). \tag{3.3}$$

The constant $\frac{1}{2}$ *in* (3.3) *is the best possible.*

Now by using aforementioned results, we are going to obtain generalizations of the results proved in the previous section.

For m-tuples $w = (w_1, \ldots, w_m)$, $x = (x_1, \ldots, x_m)$, and $y = (y_1, \ldots, y_m)$ with $x_i, y_i \in [a,b]$, $w_i \in \mathbb{R}$ $(i = 1, \ldots, m)$, and the function T_n defined as in (1.11), denote

$$\delta(s) = \sum_{i=1}^m w_i T_n(y_i, s) - \sum_{i=1}^m w_i T_n(x_i, s), \quad \forall s \in [a,b]. \tag{3.4}$$

Similarly for continuous functions $x, y : [\alpha, \beta] \to [a,b]$ and $w : [\alpha, \beta] \to \mathbb{R}$, denote

$$\Delta(s) = \int_\alpha^\beta w(t) T_n\big(y(t), s\big)\,dt - \int_\alpha^\beta w(t) T_n\big(x(t), s\big)\,dt, \quad \forall s \in [a,b]. \tag{3.5}$$

Hence by using these notations we define Čebyšev functionals as follows:

$$T(\delta,\delta) = \frac{1}{b-a}\int_a^b \delta^2(s)\,ds - \left(\frac{1}{b-a}\int_a^b \delta(s)\,ds\right)^2,$$

$$T(\Delta,\Delta)=\frac{1}{b-a}\int_a^b \Delta^2(s)\,ds-\left(\frac{1}{b-a}\int_a^b \Delta(s)\,ds\right)^2.$$

Now, we are ready to state the main results of this section:

Theorem 5 *Let $n\in\mathbb{N}$, $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function with $(\cdot-a)(b-\cdot)[f^{(n+1)}]^2\in L[a,b]$ and $x_i,y_i\in[a,b]$, $w_i\in\mathbb{R}$ $(i=1,2,\ldots,m)$, and let the functions T_n, T, and δ be defined in* (1.11), (3.1) *and* (3.4), *respectively. Then we have*

$$\begin{aligned}
&\sum_{i=1}^m w_i f(y_i)-\sum_{i=1}^m w_i f(x_i)\\
&\quad=\frac{1}{b-a}\sum_{i=1}^m w_i\Biggl[\sum_{k=0}^{n-2}\frac{1}{k!(k+2)!}\bigl[f^{(k+1)}(a)\bigl[(y_i-a)^{k+2}-(x_i-a)^{k+2}\bigr]\\
&\qquad-f^{(k+1)}(b)\bigl[(y_i-b)^{k+2}-(x_i-b)^{k+2}\bigr]\bigr]\Biggr]\\
&\qquad+\frac{[f^{(n-1)}(b)-f^{(n-1)}(a)]}{(n-1)!(b-a)}\int_a^b \delta(s)\,ds+R_n^1(f;a,b),
\end{aligned}\tag{3.6}$$

where the remainder $R_n^1(f;a,b)$ satisfies the estimation

$$\bigl|R_n^1(f;a,b)\bigr|\le\frac{1}{(n-1)!}\left(\frac{b-a}{2}\left|T(\delta,\delta)\int_a^b (s-a)(b-s)\bigl[f^{(n+1)}(s)\bigr]^2\,ds\right|\right)^{1/2}.\tag{3.7}$$

Proof If we apply Proposition 7 for $g\to\delta$ and $h\to f^{(n)}$, then we obtain

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b \delta(s)f^{(n)}(s)\,ds-\left(\frac{1}{b-a}\int_a^b \delta(s)\,ds\right)\left(\frac{1}{b-a}\int_a^b f^{(n)}(s)\,ds\right)\right|\\
&\quad\le\frac{1}{\sqrt{2}}\left(\frac{1}{b-a}\bigl|T(\delta,\delta)\bigr|\int_a^b (s-a)(b-s)\bigl[f^{(n+1)}(s)\bigr]^2\,ds\right)^{1/2}.
\end{aligned}$$

Therefore we have

$$\frac{1}{(n-1)!(b-a)}\int_a^b \delta(s)f^{(n)}(s)\,ds=\frac{[f^{(n-1)}(b)-f^{(n-1)}(a)]}{(n-1)!(b-a)^2}\int_a^b \delta(s)\,ds+\frac{1}{b-a}R_n^1(f;a,b),$$

where $R_n^1(f;a,b)$ satisfies inequality (3.7). Now from identity (2.1) we obtain (3.6). □

Here we state the integral version of the previous theorem.

Theorem 6 *Let $f:[a,b]\to\mathbb{R}$ be such that $f\in C^n[a,b]$ for $n\in\mathbb{N}$ with $(\cdot-a)(b-\cdot)[f^{(n+1)}]^2\in L[a,b]$ and $x,y:[\alpha,\beta]\to[a,b]$ and $w:[\alpha,\beta]\to\mathbb{R}$ and let the functions T_n, T and Δ be defined in* (1.11), (3.1) *and* (3.5), *respectively. Then we have*

$$\begin{aligned}
&\int_\alpha^\beta w(t)f\bigl(y(t)\bigr)\,dt-\int_\alpha^\beta w(t)f\bigl(x(t)\bigr)\,dt\\
&\quad=\frac{1}{b-a}\Biggl[\sum_{k=0}^{n-2}\frac{1}{k!(k+2)!}\int_\alpha^\beta w(t)\bigl[f^{(k+1)}(a)\bigl[\bigl(y(t)-a\bigr)^{k+2}-\bigl(x(t)-a\bigr)^{k+2}\bigr]
\end{aligned}$$

$$- f^{(k+1)}(b)\big[\big(y(t)-b\big)^{k+2} - \big(x(t)-b\big)^{k+2}\big]\big] \, dt \Bigg]$$
$$+ \frac{[f^{(n-1)}(b) - f^{(n-1)}(a)]}{(n-1)!(b-a)} \int_a^b \Delta(s)\, ds + R_n^2(f;a,b), \tag{3.8}$$

where the remainder $R_n^2(f;a,b)$ *satisfies the estimation*

$$\big|R_n^2(f;a,b)\big| \le \frac{1}{(n-1)!}\left(\frac{b-a}{2}\left|T(\Delta,\Delta)\int_a^b (s-a)(b-s)\big[f^{(n+1)}(s)\big]^2\, ds\right|\right)^{1/2}. \tag{3.9}$$

Proof This result easily follows by proceeding as in the proof of previous theorem and by replacing (2.1) by (2.2). □

By using Proposition 8 we obtain the following Grüss type inequality.

Theorem 7 *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f \in C^n[a,b]$ *for* $n \in \mathbb{N}$ *with* $f^{(n+1)} \ge 0$ *on* $[a,b]$ *and let the functions* T *and* δ *be defined in* (3.1) *and* (3.4), *respectively. Then we have the representation* (3.6) *and the remainder* $R_n^1(f;a,b)$ *satisfies the following condition*:

$$\big|R_n^1(f;a,b)\big| \le \frac{1}{(n-1)!}\|\delta'\|_\infty \left[\frac{b-a}{2}\big[f^{(n-1)}(b)+f^{(n-1)}(a)\big] - \big[f^{(n-2)}(b)-f^{(n-2)}(a)\big]\right]. \tag{3.10}$$

Proof If we apply Proposition 8 for $g \to \delta$ and $h \to f^{(n)}$, then we obtain

$$\left|\frac{1}{b-a}\int_a^b \delta(s) f^{(n)}(s)\, ds - \left(\frac{1}{b-a}\int_a^b \delta(s)\, ds\right)\left(\frac{1}{b-a}\int_a^b f^{(n)}(s)\, ds\right)\right|$$
$$\le \frac{1}{2(b-a)}\|\delta'\|_\infty \int_a^b (s-a)(b-s) f^{(n+1)}(s)\, ds.$$

Since

$$\int_a^b (s-a)(b-s) f^{(n+1)}(s)\, ds$$
$$= \int_a^b (2s-a-b) f^{(n)}(s)\, ds$$
$$= (b-a)\big[f^{(n-1)}(b) + f^{(n-1)}(a)\big] - 2\big[f^{(n-2)}(b) - f^{(n-2)}(a)\big]. \tag{3.11}$$

Therefore, by using the identities (2.1) and (3.11) we deduce (3.10). □

Integral version of the above theorem can be given as:

Theorem 8 *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f \in C^n[a,b]$ *for* $n \in \mathbb{N}$ *with* $f^{(n+1)} \ge 0$ *on* $[a,b]$ *and let the functions* T *and* Δ *be defined in* (3.1) *and* (3.5), *respectively. Then we have the representation* (3.8) *and the remainder* $R_n^2(f;a,b)$ *satisfies the following condition*:

$$\big|R_n^2(f;a,b)\big| \le \frac{1}{(n-1)!}\|\Delta'\|_\infty \left[\frac{b-a}{2}\big[f^{(n-1)}(b)+f^{(n-1)}(a)\big] - \big[f^{(n-2)}(b)-f^{(n-2)}(a)\big]\right].$$

Here, the symbol $L_p[a,b]$ ($1 \le p < \infty$) denotes the space of p-power integrable functions on the interval $[a,b]$ equipped with the norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}}$$

and $L_\infty[a,b]$ denotes the space of essentially bounded functions on $[a,b]$ with the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{t \in [a,b]} |f(t)|.$$

Now we state some Ostrowski-type inequalities related to the generalized majorization inequalities.

Theorem 9 *Let all the assumptions of Theorem* 1 *hold. Furthermore, let* (p,q) *be a pair of conjugate exponents, that is,* $1 \le p,q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $f^{(n)} \in L_p[a,b]$ *for some* $n \in \mathbb{N}$, $n > 1$. *Then we have*

$$\begin{aligned} &\left| \sum_{i=1}^m w_i f(y_i) - \sum_{i=1}^m w_i f(x_i) - \frac{1}{b-a} \sum_{i=1}^m w_i \left[\sum_{k=0}^{n-2} \frac{1}{k!(k+2)!} \big[f^{(k+1)}(a) \right.\right. \\ &\quad \left.\left. \times \left[(y_i - a)^{k+2} - (x_i - a)^{k+2} \right] - f^{(k+1)}(b) \left[(y_i - b)^{k+2} - (x_i - b)^{k+2} \right] \big] \right] \right| \\ &\le \frac{1}{(n-1)!} \|f^{(n)}\|_p \left\| \sum_{i=1}^m w_i \big(T_n(y_i, \cdot) - T_n(x_i, \cdot) \big) \right\|_q . \end{aligned} \tag{3.12}$$

The constant on the right-hand side of (3.12) *is sharp for* $1 < p \le \infty$ *and the best possible for* $p = 1$.

Proof Let us denote

$$\lambda(s) = \frac{1}{(n-1)!} \sum_{i=1}^m w_i \big[T_n(y_i, s) - T_n(x_i, s) \big].$$

Now, by using identity (2.1) and applying Hölder's inequality we obtain

$$\begin{aligned} &\left| \sum_{i=1}^m w_i f(y_i) - \sum_{i=1}^m w_i f(x_i) - \frac{1}{b-a} \sum_{i=1}^m w_i \left[\sum_{k=0}^{n-2} \frac{1}{k!(k+2)!} \big[f^{(k+1)}(a) \right.\right. \\ &\quad \left.\left. \times \left[(y_i - a)^{k+2} - (x_i - a)^{k+2} \right] - f^{(k+1)}(b) \left[(y_i - b)^{k+2} - (x_i - b)^{k+2} \right] \big] \right] \right| \\ &= \left| \int_a^b \lambda(s) f^{(n)}(s) \, ds \right| \le \|f^{(n)}\|_p \|\lambda\|_q . \end{aligned} \tag{3.13}$$

For the proof of the sharpness of the constant $(\int_a^b |\lambda(s)|^q \, ds)^{1/q}$, let us find a function f for which the equality in (3.13) is obtained.

For $1 < p < \infty$ take f to be such that

$$f^{(n)}(s) = \operatorname{sgn}\lambda(s)\cdot\left|\lambda(s)\right|^{1/(p-1)}.$$

For $p = \infty$, take f such that

$$f^{(n)}(s) = \operatorname{sgn}\lambda(s).$$

Finally, for $p = 1$, we prove that

$$\left|\int_a^b \lambda(s) f^{(n)}(s)\,ds\right| \leq \max_{s\in[a,b]}\left|\lambda(s)\right| \int_a^b f^{(n)}(s)\,ds \tag{3.14}$$

is the best possible inequality.

Function $T_n(x,\cdot)$ for $n = 1$ has jump of -1 at point x. But for $n \geq 2$ it is continuous, and thus $\lambda(s)$ is continuous. Suppose that $|\lambda(s)|$ attains its maximum at $s_0 \in [a,b]$. First we consider the case $\lambda(s_0) > 0$. For ϵ small enough we define $f_\epsilon(s)$ by

$$f_\epsilon(s) = \begin{cases} 0, & a \leq s \leq s_0, \\ \frac{1}{\epsilon n!}(s-s_0)^n, & s_0 \leq s \leq s_0+\epsilon, \\ \frac{1}{n!}(s-s_0)^{n-1}, & s_0+\epsilon \leq s \leq b. \end{cases} \tag{3.15}$$

So, we have

$$\left|\int_a^b \lambda(s) f_\epsilon^{(n)}(s)\,ds\right| = \left|\int_{s_0}^{s_0+\epsilon} \lambda(s)\frac{1}{\epsilon}\,ds\right| = \frac{1}{\epsilon}\int_{s_0}^{s_0+\epsilon} \lambda(s)\,ds.$$

Now from inequality (3.14) we have

$$\frac{1}{\epsilon}\int_{s_0}^{s_0+\epsilon} \lambda(s)\,ds \leq \lambda(s_0)\frac{1}{\epsilon}\int_{s_0}^{s_0+\epsilon} ds = \lambda(s_0).$$

Since

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{s_0}^{s_0+\epsilon} \lambda(s)\,ds = \lambda(s_0)$$

the statement follows.

In the case $\lambda(s_0) < 0$, we define $f_\epsilon(s)$ by

$$f_\epsilon(s) = \begin{cases} \frac{1}{n!}(s-s_0-\epsilon)^{n-1}, & a \leq s \leq s_0, \\ -\frac{1}{\epsilon n!}(s-s_0-\epsilon)^n, & s_0 \leq s \leq s_0+\epsilon, \\ 0, & s_0+\epsilon \leq s \leq b, \end{cases} \tag{3.16}$$

and the rest of the proof is the same as above. □

The integral case of the above theorem can be given as follows.

Theorem 10 *Let all the assumptions of Theorem* 2 *hold. Furthermore, let* (p,q) *be a pair of conjugate exponents, that is,* $1 \leq p,q \leq \infty$, $\frac{1}{p}+\frac{1}{q}=1$. *Let* $f^{(n)} \in L_p[a,b]$ *for some* $n \in \mathbb{N}$. *Then we have*

$$\begin{aligned}
&\Bigg| \int_\alpha^\beta w(t) f\big(y(t)\big)\,dt - \int_\alpha^\beta w(t) f\big(x(t)\big)\,dt \\
&\quad - \frac{1}{b-a}\Bigg[\sum_{k=0}^{n-2} \frac{1}{k!(k+2)!}\int_\alpha^\beta w(t)\big[f^{(k+1)}(a)\big[\big(y(t)-a\big)^{k+2} - \big(x(t)-a\big)^{k+2}\big] \\
&\quad - f^{(k+1)}(b)\big[\big(y(t)-b\big)^{k+2} - \big(x(t)-b\big)^{k+2}\big]\big]\,dt\Bigg]\Bigg| \\
&\leq \frac{1}{(n-1)!}\big\|f^{(n)}\big\|_p \bigg\| \int_\alpha^\beta w(t)\big(T_n\big(y(t),s\big) - T_n\big(x(t),s\big)\big)\,dt\bigg\|_q. \qquad (3.17)
\end{aligned}$$

The constant on the right-hand side of (3.17) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

For our next two sections, we give here some constructions as follows. Under the assumptions of Theorem 3 using (2.4) and Theorem 4 using (2.7) we define the following functionals, respectively:

$$\begin{aligned}
\Lambda_1(f) &= \sum_{i=1}^m w_i f(y_i) - \sum_{i=1}^m w_i f(x_i) - \frac{1}{b-a}\sum_{i=1}^m w_i \Bigg[\sum_{k=0}^{n-2}\frac{1}{k!(k+2)!}\big[f^{(k+1)}(a) \\
&\quad \times \big[(y_i-a)^{k+2} - (x_i-a)^{k+2}\big] - f^{(k+1)}(b)\big[(y_i-b)^{k+2} - (x_i-b)^{k+2}\big]\big]\Bigg], \qquad (A1)
\end{aligned}$$

$$\begin{aligned}
\Lambda_2(f) &= \int_\alpha^\beta w(t) f\big(y(t)\big)\,dt - \int_\alpha^\beta w(t) f\big(x(t)\big)\,dt \\
&\quad - \frac{1}{b-a}\Bigg[\sum_{k=0}^{n-2}\frac{1}{k!(k+2)!}\int_\alpha^\beta w(t)\big[f^{(k+1)}(a)\big[\big(y(t)-a\big)^{k+2} - \big(x(t)-a\big)^{k+2}\big] \\
&\quad - f^{(k+1)}(b)\big[\big(y(t)-b\big)^{k+2} - \big(x(t)-b\big)^{k+2}\big]\big]\,dt\Bigg]. \qquad (A2)
\end{aligned}$$

4 Mean value theorems

Now we give mean value theorems for Λ_k, $k \in \{1,2\}$. Here $f_0(x) = \frac{x^n}{n!}$.

Theorem 11 *Let* $f \in C^n[a,b]$ *and let* $\Lambda_k : C^n[a,b] \to \mathbb{R}$ *for* $k \in \{1,2\}$ *be linear functionals as defined in* (A1) *and* (A2), *respectively. Then there exists* $\xi_k \in [a,b]$ *for* $k \in \{1,2\}$ *such that*

$$\Lambda_k(f) = f^{(n)}(\xi_k)\Lambda_k(f_0), \quad k \in \{1,2\}. \qquad (4.1)$$

Proof Since $f^{(n)}$ is continuous on $[a,b]$, so $L \leq f^{(n)}(x) \leq M$ for $x \in [a,b]$ where $L = \min_{x\in[a,b]} f^{(n)}(x)$ and $M = \max_{x\in[a,b]} f^{(n)}(x)$.

Therefore the function

$$F(x) = M\frac{x^n}{n!} - f(x) = Mf_0(x) - f(x)$$

gives us

$$F^{(n)}(x) = M - f^{(n)}(x) \geq 0$$

i.e. F is n-convex function. Hence $\Lambda_k(F) \geq 0$ and we conclude that for $k \in \{1, 2\}$

$$\Lambda_k(f) \leq M\Lambda_k(f_0).$$

Similarly, for $k \in \{1, 2\}$ we have

$$L\Lambda_k(f_0) \leq \Lambda_k(f).$$

Combining the two inequalities we get

$$L\Lambda_k(f_0) \leq \Lambda_k(f) \leq M\Lambda_k(f_0),$$

which gives us (4.1). □

Theorem 12 *Let $f, g \in C^n[a, b]$ and let $\Lambda_k : C^n[a, b] \to \mathbb{R}$ for $k \in \{1, 2\}$ be the linear functionals as defined in* (A1) *and* (A2), *respectively. Then there exists $\xi_k \in [a, b]$ for $k \in \{1, 2\}$ such that*

$$\frac{\Lambda_k(f)}{\Lambda_k(g)} = \frac{f^{(n)}(\xi_k)}{g^{(n)}(\xi_k)}$$

assuming that both denominators are non-zero.

Proof Fix $k \in \{1, 2\}$. Let $h \in C^n[a, b]$ be defined as

$$h = \Lambda_k(g)f - \Lambda_k(f)g.$$

Using Theorem 11 there exists ξ_k such that

$$0 = \Lambda_k(h) = h^{(n)}(\xi_k)\Lambda_k(f_0)$$

or

$$\left[\Lambda_k(g)f^{(n)}(\xi_k) - \Lambda_k(f)g^{(n)}(\xi_k)\right]\Lambda_k(f_0) = 0,$$

which gives us the required result. □

Remark 9 If the inverse of $\frac{f^{(n)}}{g^{(n)}}$ exists, then from the above mean value theorems we can give the generalized means,

$$\xi_k = \left(\frac{f^{(n)}}{g^{(n)}}\right)^{-1}\left(\frac{\Lambda_k(f)}{\Lambda_k(g)}\right), \quad k \in \{1, 2\}. \tag{4.2}$$

5 Log-convexity and *n*-exponential convexity

5.1 Logarithmically convex functions

A number of important inequalities arise from the logarithmic convexity of some functions as one can see in [6].

Now, we recall some definitions. The following definition was originally given by Jensen in 1906 [11]. Here I is an interval in $\mathbb{R}$.

Definition 5 A function $f : I \to \mathbb{R}_+$ is called log-convex in the J-sense if the inequality

$$f^2\left(\frac{x_1 + x_2}{2}\right) \leq f(x_1)f(x_2)$$

holds for each $x_1, x_2 \in I$.

Definition 6 ([1], p.7) A function $f : I \to \mathbb{R}_+$ is called log-*convex* if the inequality

$$f\big(\lambda x_1 + (1 - \lambda)x_2\big) \leq \big[f(x_1)\big]^{\lambda}\big[f(x_2)\big]^{(1-\lambda)}$$

holds for each $x_1, x_2 \in I$ and $\lambda \in [0, 1]$.

Remark 10 A function log-convex in the J-sense is log-convex if it is continuous as well.

5.2 *n*-Exponentially convex functions

Bernstein [12] and Widder [13] independently introduced an important sub-class of convex functions, which is called the class of exponentially convex functions on a given open interval, and studied some properties of this newly defined class. Pečarić and Perić in [14] introduced the notion of n-exponentially convex functions, which is in fact a generalization of the concept of exponentially convex functions. In the present subsection, we discus the same notion of n-exponential convexity by describing related definitions and some important results with some remarks from [14].

Definition 7 A function $f : I \to \mathbb{R}$ is *n-exponentially convex* in the J-sense if the inequality

$$\sum_{i,j=1}^{n} u_i u_j f\left(\frac{t_i + t_j}{2}\right) \geq 0$$

holds for each $t_i \in I$ and $u_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$.

Definition 8 A function $f : I \to \mathbb{R}$ is *n-exponentially convex* if it is n-exponentially convex in the J-sense and continuous on I.

Remark 11 We can see from the definition that 1-exponentially convex functions in the J-sense are in fact nonnegative functions. Also, n-exponentially convex functions in the J-sense are k-exponentially convex in the J-sense for every $k \in \mathbb{N}$ such that $k \leq n$.

Definition 9 A function $f : I \to \mathbb{R}$ is *exponentially convex in the J-sense*, if it is n-exponentially convex in the J-sense for each $n \in \mathbb{N}$.

Remark 12 A function $f : I \to \mathbb{R}$ is *exponentially convex* if it is n-exponentially convex in the J-sense and continuous on I.

Proposition 9 *If function $f : I \to \mathbb{R}$ is n-exponentially convex in the J-sense, then the matrix*

$$\left[f\left(\frac{t_i + t_j}{2}\right)\right]_{i,j=1}^{m}$$

is positive-semidefinite. Particularly

$$\det\left[f\left(\frac{t_i + t_j}{2}\right)\right]_{i,j=1}^{m} \geq 0$$

for each $m \in \mathbb{N}$, $m \leq n$ and $t_i \in I$ for $i \in \{1, \ldots, m\}$.

Corollary 3 *If function $f : I \to \mathbb{R}$ is exponentially convex, then the matrix*

$$\left[f\left(\frac{t_i + t_j}{2}\right)\right]_{i,j=1}^{m}$$

is positive-semidefinite. Particularly

$$\det\left[f\left(\frac{t_i + t_j}{2}\right)\right]_{i,j=1}^{m} \geq 0$$

for each $m \in \mathbb{N}$ and $t_i \in I$ for $i \in \{1, \ldots, m\}$.

Corollary 4 *If the function $f : I \to \mathbb{R}_+$ is exponentially convex, then f is* log*-convex.*

Remark 13 A function $f : I \to \mathbb{R}_+$ is log-convex in J-sense if and only if the inequality

$$u_1^2 f(t_1) + 2u_1u_2 f\left(\frac{t_1 + t_2}{2}\right) + u_2^2 f(t_2) \geq 0$$

holds for each $t_1, t_2 \in I$ and $u_1, u_2 \in \mathbb{R}$. It follows that a positive function is log-convex in the J-sense if and only if it is 2-exponentially convex in the J-sense. Also, using basic convexity theory it follows that a positive function is log-convex if and only if it is 2-exponentially convex.

Here, we get our results concerning the n-exponential convexity and exponential convexity for our functionals Λ_k, $k \in \{1, 2\}$, as defined in (A1) and (A2). Throughout the section I is an interval in $\mathbb{R}$.

Theorem 13 *Let $D_1 = \{f_t : t \in I\}$ be a class of functions such that the function $t \mapsto [z_0, z_1, \ldots, z_n; f_t]$ is n-exponentially convex in the J-sense on I for any $n + 1$ mutually distinct points $z_0, z_1, \ldots, z_n \in [a, b]$. Let Λ_k be the linear functionals for $k \in \{1, 2\}$ as defined in* (A1) *and* (A2). *Then the following statements are valid*:

(a) *The function $t \mapsto \Lambda_k(f_t)$ is n-exponentially convex function in the J-sense on I.*
(b) *If the function $t \mapsto \Lambda_k(f_t)$ is continuous on I, then the function $t \mapsto \Lambda_k(f_t)$ is n-exponentially convex on I.*

Proof (a) Fix $k \in \{1,2\}$. Let us define the function ω for $t_i \in I$, $u_i \in \mathbb{R}$, $i \in \{1,\dots,n\}$ as follows:

$$\omega = \sum_{i,j=1}^{n} u_i u_j f_{\frac{t_i+t_j}{2}}.$$

Since the function $t \mapsto [z_0, z_1, \dots, z_n; f_t]$ is n-exponentially convex in the J-sense,

$$[z_0, z_1, \dots, z_n; \omega] = \sum_{i,j=1}^{n} u_i u_j [z_0, z_1, \dots, z_n; f_{\frac{t_i+t_j}{2}}] \geq 0,$$

which implies that ω is n-convex function on I and therefore $\Lambda_k(\omega) \geq 0$. Hence

$$\sum_{i,j=1}^{n} u_i u_j \Lambda_k(f_{\frac{t_i+t_j}{2}}) \geq 0.$$

We conclude that the function $t \mapsto \Lambda_k(f_t)$ is an n-exponentially convex function on I in the J-sense.

(b) This part easily follows from the definition of the n-exponentially convex function. □

As a consequence of the above theorem we give the following corollaries.

Corollary 5 *Let $D_2 = \{f_t : t \in I\}$ be a class of functions such that the function $t \mapsto [z_0, z_1, \dots, z_n; f_t]$ is exponentially convex in the J-sense on I for any $n+1$ mutually distinct points $z_0, z_1, \dots, z_n \in [a,b]$. Let Λ_k be the linear functionals for $k \in \{1,2\}$ as defined in* (A1) *and* (A2). *Then the following statements are valid*:

(a) *The function $t \mapsto \Lambda_k(f_t)$ is exponentially convex in the J-sense on I.*

(b) *If the function $t \mapsto \Lambda_k(f_t)$ is continuous on I, then the function $t \mapsto \Lambda_k(f_t)$ is exponentially convex on I.*

(c) *The matrix $[\Lambda_k(f_{\frac{t_i+t_j}{2}})]_{i,j=1}^{m}$ is positive-semidefinite. Particularly,*

$$\det\bigl[\Lambda_k(f_{\frac{t_i+t_j}{2}})\bigr]_{i,j=1}^{m} \geq 0$$

for each $m \in \mathbb{N}$ and $t_i \in I$ where $i \in \{1,\dots,m\}$.

Proof The proof follows directly from Theorem 13 by using the definition of exponential convexity and Corollary 3. □

Corollary 6 *Let $D_3 = \{f_t : t \in I\}$ be a class of functions such that the function $t \mapsto [z_0, z_1, \dots, z_n; f_t]$ is* 2*-exponentially convex in the J-sense on I for any $n+1$ mutually distinct points $z_0, z_1, \dots, z_n \in [a,b]$. Let Λ_k be the linear functionals for $k \in \{1,2\}$ as defined in* (A1) *and* (A2). *Then the following statements are valid*:

(a) *If the function $t \mapsto \Lambda_k(f_t)$ is continuous on I, then it is* 2*-exponentially convex on I. If the function $t \mapsto \Lambda_k(f_t)$ is additionally positive, then it is also log-convex on I. Moreover, the following Lyapunov inequality holds for $r < s < t$, $r,s,t \in I$:*

$$\bigl[\Lambda_k(f_s)\bigr]^{t-r} \leq \bigl[\Lambda_k(f_r)\bigr]^{t-s}\bigl[\Lambda_k(f_t)\bigr]^{s-r}. \tag{5.1}$$

(b) *If the function $t \mapsto \Lambda_k(f_t)$ is positive and differentiable on I, then for every $s,t,u,v \in I$ such that $s \leq u$ and $t \leq v$, we have*

$$\mu_{s,t}(\Lambda_k, D_3) \leq \mu_{u,v}(\Lambda_k, D_3), \tag{5.2}$$

where $\mu_{s,t}$ is defined as

$$\mu_{s,t}(\Lambda_k, D_3) = \begin{cases} (\frac{\Lambda_k(f_s)}{\Lambda_k(f_t)})^{\frac{1}{s-t}}, & s \neq t, \\ \exp(\frac{\frac{d}{ds}\Lambda_k(f_s)}{\Lambda_k(f_s)}), & s = t, \end{cases} \tag{5.3}$$

for $f_s, f_t \in D_3$.

Proof

(a) It follows directly from Theorem 13 and Remark 13. As the function $t \mapsto \Lambda_k(f_t)$ is log-convex, *i.e.*, $\ln \Lambda_k(f_t)$ is convex, by using Proposition 1, we have

$$\ln\bigl[\Lambda_k(f_s)\bigr]^{t-r} \leq \ln\bigl[\Lambda_k(f_r)\bigr]^{t-s} + \ln\bigl[\Lambda_k(f_t)\bigr]^{s-r}, \quad k \in \{1,2\},$$

which gives us (5.1).

(b) From Proposition 2, for the convex function f, the inequality

$$\frac{f(s)-f(t)}{s-t} \leq \frac{f(u)-f(v)}{u-v} \tag{5.4}$$

holds $\forall s,t,u,v \in I \subset \mathbb{R}$ such that $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$.

Since by (c), $\Lambda(f_t)$ is log-convex, setting $f(t) = \ln \Lambda(f_t)$ in (5.4) we have

$$\frac{\ln \Lambda_k(f_s) - \ln \Lambda_k(f_t)}{s-t} \leq \frac{\ln \Lambda_k(f_u) - \ln \Lambda_k(f_v)}{u-v} \tag{5.5}$$

for $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$, which is equivalent to (5.3). The cases for $s = t$ and/or $u = v$ are easily treated from (5.5) by taking the respective limits. □

Remark 14 The results from Theorem 13 and Corollaries 5 and 6 still hold when any two (all) points $z_0, z_1, \ldots, z_n \in [a,b]$ coincide for a family of differentiable (n times differentiable) functions f_t such that the function $t \mapsto [z_0, z_1, \ldots, z_n; f_t]$ is n-exponentially convex, exponentially convex, and 2-exponentially convex in the J-sense, respectively.

Now, we give two important remarks and one useful corollary from [15], which we will use in some examples in the next section.

Remark 15 To $\mu_{s,t}(\Lambda_k, \Omega)$ defined with (5.3) we will refer as a mean if

$$a \leq \mu_{s,t}(\Lambda_k, \Omega) \leq b$$

for $s,t \in I$ and $k \in \{1,2\}$ where $\Omega = \{f_t : t \in I\}$ is a family of functions and $[a,b] \subset \mathrm{Dom}(f_t)$.

Theorem 13 gives us the following corollary.

Corollary 7 *Let $a, b \in \mathbb{R}$ and Λ_k be linear functionals for $k \in \{1,2\}$. Let $\Omega = \{f_t : t \in I\}$ be a family of functions in $C^2[a,b]$. If*

$$a \leq \left(\frac{\frac{d^2 f_s}{dx^2}}{\frac{d^2 f_t}{dx^2}} \right)^{\frac{1}{s-t}} (\xi) \leq b,$$

for $\xi \in [a,b]$, $s, t \in I$, then $\mu_{s,t}(\Lambda_k, \Omega)$ is a mean for $k \in \{1,2\}$.

Remark 16 In some examples, we will get a mean of this type:

$$\left(\frac{\frac{d^2 f_s}{dx^2}}{\frac{d^2 f_t}{dx^2}} \right)^{\frac{1}{s-t}} (\xi) = \xi, \quad \xi \in [a,b], s \neq t.$$

6 Examples with applications

In this section, we use various classes of functions $\Omega = \{f_t : t \in I\}$ for any open interval $I \subset \mathbb{R}$ to construct different examples of exponentially convex functions and applications to Stolarsky-type means. Let us consider some examples.

Example 1 Let $\Omega_1 = \{\psi_t : \mathbb{R} \to [0,\infty) : t \in \mathbb{R}\}$ be a family of functions defined by

$$\psi_t(x) = \begin{cases} \frac{e^{tx}}{t^n}, & t \neq 0, \\ \frac{x^n}{n!}, & t = 0. \end{cases}$$

Since $\frac{d^n}{dx^n}\psi_t(x) = e^{tx} > 0$, the function $\psi_t(x)$ is n-convex on $\mathbb{R}$ for every $t \in \mathbb{R}$ and $t \to \frac{d^n}{dx^n}\psi_t(x)$ is exponentially convex by definition. Using analogous arguing to the proof of Theorems 13, we see that $t \mapsto [z_0, z_1, \ldots, z_n; \psi_t]$ is exponentially convex (and so exponentially convex in the J-sense). Using Corollary 5 we conclude that $t \mapsto \Lambda_k(\psi_t)$, $k \in \{1,2\}$ are exponentially convex in the J-sense. It is easy to see that these mappings are continuous, so they are exponentially convex.

Assume that $t \mapsto \Lambda_k(\psi_t) > 0$ for $k \in \{1,2\}$. By introducing convex functions ψ_t in (4.2), we obtain the following means: for $k \in \{1,2\}$

$$\mathfrak{M}_{s,t}(\Lambda_k, \Omega_1) = \begin{cases} \frac{1}{s-t} \ln(\frac{\Lambda_k(\psi_s)}{\Lambda_k(\psi_t)}), & s \neq t, \\ \frac{\Lambda_k(id \cdot \psi_s)}{\Lambda_k(\psi_s)} - \frac{n}{s}, & s = t \neq 0, \\ \frac{\Lambda_k(id \cdot \psi_0)}{(n+1)\Lambda_k(\psi_0)}, & s = t = 0, \end{cases}$$

where id stands for the identity function on $\mathbb{R}$. Here $\mathfrak{M}_{s,t}(\Lambda_k, \Omega_1) = \ln(\mu_{s,t}(\Lambda_k, \Omega_1))$, $k \in \{1,2\}$ are in fact means.

Remark 17 We observe here that $(\frac{\frac{d^n \psi_s}{dx^n}}{\frac{d^n \psi_t}{dx^n}})^{\frac{1}{s-t}}(\ln \xi) = \xi$ is a mean for $\xi \in [a,b]$ where $a, b \in \mathbb{R}_+$.

Example 2 Let $\Omega_2 = \{\varphi_t : (0,\infty) \to \mathbb{R} : t \in \mathbb{R}\}$ be a family of functions defined as

$$\varphi_t(x) = \begin{cases} \frac{(x)^t}{t(t-1)\cdots(t-n+1)}, & t \notin \{0, \ldots, n-1\}, \\ \frac{(x)^j \ln(x)}{(-1)^{n-1-j} j!(n-1-j)!}, & t = j \in \{0, \ldots, n-1\}. \end{cases}$$

Since $\varphi_t(x)$ is n-convex function for $x \in (0,\infty)$ and $t \mapsto \frac{d^2}{dx^2}\varphi_t(x)$ is exponentially convex, by the same arguments as given in the previous example we conclude that $\Lambda_k(\varphi_t)$, $k \in \{1,2\}$ are exponentially convex.

We assume that $\Lambda_k(\varphi_t) > 0$ for $k \in \{1,2\}$. For this family of convex functions we obtain the following means: for $k \in \{1,2\}$

$$\mathfrak{M}_{s,t}(\Lambda_k,\Omega_2) = \begin{cases} (\frac{\Lambda_k(\varphi_s)}{\Lambda_k(\varphi_t)})^{\frac{1}{s-t}}, & s \neq t, \\ \exp((-1)^{n-1}(n-1)!\frac{\Lambda_k(\varphi_0\varphi_s)}{\Lambda_k(\varphi_s)} + \sum_{k=0}^{n-1}\frac{1}{k-t}), & s = t \notin \{0,\ldots,n-1\}, \\ \exp((-1)^{n-1}(n-1)!\frac{\Lambda_k(\varphi_0\varphi_s)}{2\Lambda_k(\varphi_s)} + \sum_{k=0,k\neq t}^{n-1}\frac{1}{k-t}), & s = t \in \{0,\ldots,n-1\}. \end{cases}$$

Here $\mathfrak{M}_{s,t}(\Lambda_k,\Omega_2) = \mu_{s,t}(\Lambda_k,\Omega_2)$, $k \in \{1,2\}$, are in fact means.

Remark 18 Further, in this choice of family Ω_2, we have

$$\left(\frac{\frac{d^n\varphi_s}{dx^n}}{\frac{d^n\varphi_t}{dx^n}}\right)^{\frac{1}{s-t}}(\xi) = \xi, \quad \xi \in [a,b], s \neq t, \text{ where } a,b \in (0,\infty).$$

So, using Remark 16 we have the important conclusion that $\mu_{s,t}(\Lambda_k,\Omega_2)$ is in fact a mean for $k \in \{1,2\}$.

Example 3 Let $\Omega_3 = \{\theta_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$ be a family of functions defined by

$$\theta_t(x) = \frac{e^{-x\sqrt{t}}}{t^{n/2}}.$$

Since $t \mapsto \frac{d^n}{dx^n}\theta_t(x) = e^{-x\sqrt{t}}$ is exponentially convex for $x > 0$, being the Laplace transform of a nonnegative function [15], by the same argument as given in Example 1 we conclude that $\Lambda_k(\theta_t)$, $k \in \{1,2\}$ are exponentially convex.

We assume that $\Lambda_k(\theta_t) > 0$ for $k \in \{1,2\}$. For this family of functions we have the following possible cases of $\mu_{s,t}(\Lambda_k,\Omega_3)$: for $k \in \{1,2\}$

$$\mathfrak{M}_{s,t}(\Lambda_k,\Omega_3) = \begin{cases} (\frac{\Lambda_k(\theta_s)}{\Lambda_k(\theta_t)})^{\frac{1}{s-t}}, & s \neq t, \\ \exp(-\frac{\Lambda_k(id\cdot\theta_s)}{2\sqrt{s}\Lambda_k(\theta_s)} - \frac{n}{2s}), & s = t. \end{cases}$$

By (4.2), $\mathfrak{M}_{s,t}(\Lambda_k,\Omega_3) = -(\sqrt{s}+\sqrt{t})\ln\mu_{s,t}(\Lambda_k,\Omega_3)$, $k \in \{1,2\}$, defines a class of means.

Example 4 Let $\Omega_4 = \{\phi_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$ be a family of functions defined by

$$\phi_t(x) = \begin{cases} \frac{t^{-x}}{(\ln t)^n}, & t \neq 1, \\ \frac{x^n}{n}, & t = 1. \end{cases}$$

Since $\frac{d^n}{dx^n}\phi_t(x) = t^{-x} = e^{-x\ln t} > 0$ for $x > 0$, by the same argument as given in Example 1 we conclude that $t \mapsto \Lambda_k(\phi_t)$, $k \in \{1,2\}$, are exponentially convex.

We assume that $\Lambda_k(\phi_t) > 0$ for $k \in \{1,2\}$. For this family of functions we have the following possible cases of $\mu_{s,t}(\Lambda_k, \Omega_4)$: for $k \in \{1,2\}$

$$\mathfrak{M}_{s,t}(\Lambda_k, \Omega_4) = \begin{cases} (\frac{\Lambda_k(\phi_s)}{\Lambda_k(\phi_t)})^{\frac{1}{s-t}}, & s \neq t, \\ \exp(-\frac{\Lambda_k(id\cdot\phi_s)}{s\Lambda_k(\phi_s)} - \frac{n}{s\ln s}), & s = t \neq 1, \\ \exp(-\frac{1}{(n+1)}\frac{\Lambda_k(id\cdot\phi_1)}{\Lambda_k(\phi_1)}), & s = t = 1. \end{cases}$$

By (4.2), $\mathfrak{M}_{s,t}(\Lambda_k, \Omega_4) = -L(s,t)\ln \mu_{s,t}, (\Lambda_k, \Omega_4)$, $k \in \{1,2\}$, defines a class of means, where $L(s,t)$ is the logarithmic mean defined as

$$L(s,t) = \begin{cases} \frac{s-t}{\ln s - \ln t}, & s \neq t, \\ s, & s = t. \end{cases} \tag{6.1}$$

Remark 19 Monotonicity of $\mu_{s,t}(\Lambda_k, \Omega_j)$ follows from (5.2) for $k \in \{1,2\}$, $j \in \{1,2,3,4\}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JP made the main contribution in conceiving the presented research. AAA, ARK, and JP worked jointly on each section, while AAA and ARK drafted the manuscript. All authors read and approved the final manuscript.

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