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STABILITY OF NUMERICAL METHODS FOR ORDINARY STOCHASTIC DIFFERENTIAL EQUATIONS ALONG LYAPUNOV-TYPE AND OTHER FUNCTIONS WITH VARIABLE STEP SIZES*

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Abstract. Some general concepts and theorems on the stability of numerical methods for ordinary stochastic differential equations (SDEs) along Lyapunov-type and other Borel-measurable, nonnegative functions are presented. In particular, we deal with almost sure, moment and weak V-stability, exponential and asymptotic stability of related stochastic difference equations with nonrandom, variable step sizes. The applicability of the main results is explained with the class of balanced implicit methods (i.e. certain stochastic linear-implicit methods with appropriate weights). It is shown that, they are rich enough to provide asymptotically, exponentially and polynomially stable numerical methods discretizing stable continuous time SDEs by controlling the choice of their weights.

Key words. stochastic-numerical approximation, stochastic stability, ordinary stochastic differential equations, numerical methods, drift-implicit Euler methods, balanced implicit methods, Lyapunov-type functions, numerical weak V-stability, stability of moments, a.s. stability, asymptotic stability

AMS subject classifications. 65C20, 65C30, 65C50, 60H10, 37H10, 34F05

1. Introduction.

Convergence and stability are two of the key requirements on numerical methods for approximating systems of stochastic differential equations (SDEs) such as,

$$dX_t = a(t,X_t)\,dt + \sum_{j=1}^{m} b^j(t,X_t)dW_t^j \;=\; \sum_{j=0}^{m} b^j(t,X_t)dW_t^j \tag{1.1}$$

driven by standard one-dimensional Wiener processes $W^j = (W_t^j)_{0\le t\le T}$ and interpreted in Itô sense (for the sake of simplicity of this representation), where $a, b^j \in C^0([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$ and we meet the convention $W_t^0 = t$ and $b^0(t,x) = a(t,x)$ for more compact notation (for theory of SDEs (1.1), see Arnold [2] - [3]). The key role of these concepts is visible from the stochastic Kantorovich-Lax-Richtmeyer principle, as presented in Schurz [44, 45, 46, 47] (For deterministic counterparts and related topics, e.g. see Godunov and Ryabenkii [14], Richtmeyer and Morton [36] or Strikwerda [49]). Roughly speaking, it says that, if some consistency, contractivity and stability requirements are met, one is able to find contractivity constants K_C^X of the exact solution, stability constants K_S^Y of the numerical approximation and contants K_I such that the universal estimates of the weak L^p-error ($p \ge 1$),

$$\varepsilon_p(T) := \sup_{0\le t\le T}\Big(\mathbb{E}\,\|X_t - Y_t^\Delta\|_d^p\Big)^{1/p} \le \exp(K_C^X T)\varepsilon_p(0) + K_I \exp(K_S^Y T)\Delta_{max}^\gamma \tag{1.2}$$

hold, where Δ_{max} denotes the maximum step size, γ the rate of L^p-convergence, $\|.\|_d$ the Euclidean vector norm of $\mathbb{R}^d$, and Y^Δ a suitable (continuous time continuation of) numerical approximation of process $X = (X_t)_{0\le t\le T}$ governed by (1.1). Similar estimates are valid for the strong L^p-error $e_p(T) = (\mathbb{E}\sup_{0\le t\le T}\|X_t - Y_t^\Delta\|_d^p)^{1/p}$ ($p \ge 1$). Also, in view of weak approximations along appropriate classes of functionals of X, the verification of stability of related numerical methods applied to SDEs (1.1) is crucial for the proof of rates of weak convergence, as seen in works of Milstein [30] and Talay [50]. Hence, the important role of stability investigations for discrete stochastic dynamics generated by numerical methods approximating SDEs is apparent. For a more recent overview on aspects of stochastic-numerical analysis

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and interrelations, see also Artemiev and Averina [5] or Schurz [44]. In passing, for technical reasons, we note that both the solution $X = (X_t)_{0 \le t \le T}$, its approximation $Y^\Delta = (Y_n^\Delta)_{n \in \mathbb{N}}$ and the driving independent Wiener processes $W^j = (W_t^j)_{0 \le t \le T}$ are defined on complete, filtered probability spaces $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$ and $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in \mathbb{N}}, \mathbb{P})$, respectively.

This work is a continuation of systematic stability investigations [37] - [43]. Mean square stability of numerical methods for linear SDEs is discussed by many authors, e.g. [4], [5], [8], [9], [16], [17], [18], [33], [35], [37] and [38], and stochastic stability by [6], [17], [18], [7] and [30], whereas expectation stability in [1], [19] and [39]. Additively noised equations and stability is treated in [39]. There is a note [40] on an invariance property of stochastic θ-methods concerning moment stability. Stochastic versions of A-stability are found in [5], [17], [38] and [42]. Numerical stability exponents are investigated in [41]. Contractivity and B-stability of stochastic dynamics (including the drift-implicit Euler-Maruyama method) are discussed in [38], [41] and [42]. However, most of the forementioned works deal with numerical stability when applied to linear, onedimensional SDEs. Here, we are aiming at a presentation of fairly general theorems to control the qualitative behavior with respect to diverse stability concepts of stochastic numerical methods in $\mathbb{R}^d$ along certain nonlinear functions (which play a similar and sometimes more general role as Lyapunov functions) when applied to linear and nonlinear test SDEs (1.1). As an illustrative example, the class of balanced implicit methods with linear-implicit weights as introduced in [31] is thoroughly treated with respect to several stability issues. It is shown that this class can successfully tackle several problems of asymptotic almost sure and moment stability by the help of Lyapunov-type functions. As a major consequence, our analysis contributes to the understanding of constructing both converging and asymptotically stable numerical methods applied to systems of (nonlinear) SDEs with variable step sizes. However, we will interpret the concept of Lyapunov functions in a very generous manner (i.e. along nonnegative Borel-measurable functions with certain equilibria to evaluate the asymptotic behavior of related numerical dynamics, that is why we use "Lyapunov-type" in our title above) compared to its original definition.

As we go on we note that, there exist an incredible rich literature on stability theory of analytic solutions of SDEs. See, e.g. the books of Arnold [2] and [3], Hasḿinskiĭ [15] or Mao [29] in order to mention a few of them. Our aim is to present a fairly general, but still applicable approach, to the stability analysis of stochastic-numerical methods based on Lyapunov-type or similar functions (or functionals), and we understand this work only as a beginning to carry over some of the key concepts known from analytic theory to fastly expanding field of stochastic-numerical analysis. Here, we are mainly interested to study the stability behavior of the numerical methods by its own dynamics. The through comparison to the behavior of the underlying analytic solution is left to future research due to its enormous complexity and many unsolved related problems.

The paper is organized as follows. In Section 2, we report on results with respect to exponential and asymptotic weak V-stability along Lyapunov functions V (i.e. "weak" in the sense of expectations of V). Section 3, deals with asymptotic almost sure stability. Thereafter, we introduce the concept of weak V-stability exponents and study the almost sure stability behavior with their help in Section 4. Finally, section 5, concentrates on fully nonlinear relations involving Lyapunov-type functions and resulting to stability of polynomial type.

2. Exponential and Asymptotic Weak V-Stability. One of the weakest requirements of stability is the stability of the first moments. Such a concept for numerical methods is discussed in [1]. A more general, but still weak concept is given below. Let $Y_{s,x}(t)$, denote the continuous one-step representation of the numerical method $Y = (Y_n)_{n \in \mathbb{N}}$ along discretizations $0 = t_0 \le t_1 \le ... \le t_{n_T} = T$ of time-intervals $[0, T]$, i.e. $Y_{s,x}(t)$ is the value of the numerical approximation at time $t \in [s, T] \subseteq [0, T]$, started at the value x at time $s \in [0, T)$.

2.1. Definition and general theorems. For simplicity, interpret the discrete time numerical method Y as a sequence of values Y_n, along a given adapted discretization of the interval $[0,T]$. We shall also take into account certain standard continuations of those discrete methods to associated continuous time approximations constructed along the adapted sequences of nondecreasing instants $t_n \in [0,T]$ and coinciding with the values Y_n at those instants t_n wherever it is convenient (for simple examples, see below). Let $\mathcal{B}(S)$, denote the σ-algebra of Borel sets of inscribed set S, and μ the Lebesgue-measure.

DEFINITION 2.1. *A random sequence $Y = (Y_n)_{n\in\mathbb{N}}$ of real-valued random variables $Y_n : (\Omega, \mathcal{F}_n, \mathbb{P}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is called* **asymptotically weakly V-stable** *for a nonnegative, Borel-measurable function $V : [t_0,T) \times \mathbb{R}^d \to \mathbb{R}^1_+$ if,*

$$\lim_{t_n \to T-} \mathbb{E}V(t_n, Y_n) \;=\; 0 \tag{2.1}$$

for all finite initial values Y_0 with $\mathbb{E}V(t_0,Y_0) < +\infty$. Moreover, Y is said to be **exponentially weakly V-stable** *for a nonnegative, Borel-measurable function $V : [t_0,T) \times \mathbb{R}^d \to \mathbb{R}^1_+$ if there exist functions $K_S : [0,T) \to \mathbb{R}^1 \in L^1_{loc}([0,T), \mathcal{B}([0,T)), \mu)$ and real constants δ_0 such that,*

$$\mathbb{E}[V(t_{n+1}, Y_{t_n,Y_n}(t_{n+1}))|\mathcal{F}_n] \;\le\; \exp\Big(K_S(t_n)(t_{n+1}-t_n)\Big)V(t_n,Y_n) \tag{2.2}$$

for all finite random variables Y_n which are $(\mathcal{F}_n, \mathcal{B}(\mathbb{R}^d))$-measurable and all $(\mathcal{F}_n)$-predictable discretizations $(t_n)_{n=0,1,\ldots,n_T-1}$ of $[0,T]$ with $\Delta_{max} \le \delta_0$. If $V(x) = \|x\|^p_d$ in the above statements then Y is called **exponentially p-th mean stable** *(in particular, if $p=2$ then* **exponentially mean square stable***).*

Note: The terminologies "asymptotic" and "exponential stability" are interpreted in a fairly wide sense. We incorporate the limit behavior on finite time-intervals $[0,T]$ too, since there are examples such as, the numerical simulation of Brownian bridges, (see e.g. Schurz [38]) where the precise limit behavior at the boundaries on finite time-intervals $[0,T]$ is required on its approximation (or more general in boundary value problems). Moreover, we also allow to observe exponentially increasing behavior by the concept of "exponential stability" in contrast to continuous time standard definitions. The main observation for the term "exponential" is that the numerical method allows estimates of the form (2.2), and hence the behavior of its stability function K_S on $[0,T]$ will mainly determine its limit behavior as seen below. Only the additional words "asymptotic stability" or "asymptotically stable" are exclusively reserved for the convergence of related functional to 0 throughout this paper.

THEOREM 2.2. *Assume that the numerical method Y constructed along any nonrandom time-discretization of $[0,T]$ with maximum step size $\Delta_{max} \le \delta_0$ is exponentially weakly V-stable with δ_0 and nonrandom stability function K_S on $[0,T]$. Then, $\forall n = 1,2,...,n_T$,*

$$\mathbb{E}V(t_n,Y_n) \;\le\; \exp\Big(\sum_{k=0}^{n-1} K_S(t_k)\Delta_k\Big)\mathbb{E}V(0,y_0), \tag{2.3}$$

$$\sup_{0\le t\le T} \mathbb{E}V(t, Y_{0,y_0}(t)) \;\le\; \exp\Big(\sum_{k=0}^{n_T-1} [K_S(t_k)]_+\Delta_k\Big)\mathbb{E}V(0,y_0) \tag{2.4}$$

where $[.]_+$ denotes the positive part of the inscribed expression. Moreover, if Y is a continuous time numerical method and K_S is its nonrandom stability constant (i.e. K_S=constant on $[0,T]$) then,

$$\mathbb{E}V(t, Y_{0,y_0}(t)) \;\le\; \exp(K_S t)\mathbb{E}V(0,y_0), \tag{2.5}$$

$$\sup_{0\le t\le T} \mathbb{E}V(t, Y_{0,y_0}(t)) \;\le\; \exp([K_S]_+ T)\mathbb{E}V(0,y_0). \tag{2.6}$$

Proof. Suppose that $t_k \le t \le t_{k+1}$ with $\Delta_k \le \delta_0$. If $\mathbb{E}\, V(0,y_0) = +\infty$ then there is nothing to prove. Now, suppose that $\mathbb{E}\, V(0,y_0) < +\infty$. We may confine ourselves to the case when K_S is a nonrandom constant on $[0,T]$ and Y is a continuous time numerical method constructed along the adapted nondecreasing instants $t_n \in [0,T]$ since the proof-steps are very similar. Using elementary properties of conditional expectations, we estimate,

$$\begin{aligned}\mathbb{E}\, V(t, Y_{0,y_0}(t)) \;&=\; \mathbb{E}\,\mathbb{E}\,[V(t, Y_{t_k,Y_k}(t))|\mathcal{F}_{t_k}]\\ &\le \exp(K_S(t-t_k)) \cdot \mathbb{E}\, V(t_k, Y_k) = \exp(K_S(t-t_k)) \cdot \mathbb{E}\, V(t_k, Y_{t_{k-1},Y_{k-1}}(t_k)) \le \ldots\\ &\le \exp(K_S t) \cdot \mathbb{E}\, V(0,y_0) \le \exp([K_S]_+ t) \cdot \mathbb{E}\, V(0,y_0) \le \exp([K_S]_+ T) \cdot \mathbb{E}\, V(0,y_0)\end{aligned}$$

by induction. Hence, taking the supremum confirms the claim of Theorem 2.2. $\square$

COROLLARY 2.3. *Assume that, the numerical method Y constructed along any nonrandom time-discretization of $[0,+\infty)$ with maximum step size $\Delta_{max} \le \delta_0$ is exponentially weakly V-stable with constant δ_0 and nonrandom Lesbesgue-integrable stability function $K_S \in L^1_{loc}([0,+\infty), \mathcal{B}([0,+\infty)), \mu)$ satisfying,*

$$\sum_{k=0}^{+\infty} K_S(t_k)\Delta_k = -\infty. \tag{2.7}$$

Then, Y using nonrandom step sizes $(\Delta_k)_{k\in\mathbb{N}}$ is asymptotically weakly V-stable.

Proof. Obvious application of Theorem 2.2 with inequality (2.3) and taking the limit $n \to +\infty$. $\square$

2.2. The example of balanced implicit methods. A fairly easy example of numerical methods for systems of SDEs (1.1), is given by the class of balanced implicit methods (BIMs), as introduced by Milstein, Platen and Schurz [31] and studied in Schurz [38], [43]. These methods follow the iteration scheme,

$$Y_{k+1} = Y_k + \sum_{j=0}^{m} b^j(t_k, Y_k)\Delta W_k^j + \sum_{j=0}^{m} c^j(t_k, Y_k)|\Delta W_k^j|(Y_k - Y_{k+1}) \tag{2.8}$$

where $\Delta W_k^j = W_{t_{k+1}}^j - W_{t_k}^j$, $c^j \in C^0([0,T]\times\mathbb{R}^d, \mathbb{R}^{d\times d})$ (Recall the convention $W_t^0 = t$ and $b^0(t,x) = a(t,x)$). BIMs (2.8) possess the **one-step representations**

$$Y_{s,y}(t) = y + M_{s,y}^{-1}(t)\sum_{j=0}^{m} b^j(s,y)(W_t^j - W_s^j) \quad \text{with} \tag{2.9}$$

$$M_{s,y}(t) = I_d + \sum_{j=0}^{m} c^j(s,y)|W_t^j - W_s^j| \tag{2.10}$$

while assuming the existence of $M_{s,y}^{-1}(t)$ for all $0 \le t-s \le \delta_0 \le T$ and all $y \in \mathbb{R}^d$ and all $s,t \in [0,T]$, where I_d denotes the $d\times d$ unit matrix of $\mathbb{R}^{d\times d}$. Mean square convergence of these linear-implicit methods with rate 0.5 (as the standard Euler-Maruyama methods) has been proven in [31] provided that the coefficients b^j are Lipschitz-continuous and the weights c^j guarantee the uniform boundedness of M^{-1}. Using the one-step representation (2.9), the **continuous polygonal representation** of the scheme (2.8) can recursively be written as,

$$Y_{0,y_0}(t) = Y_k + M_{t_k,Y_k}^{-1}(t)\sum_{j=0}^{m} b^j(t_k,Y_k)(W_t^j - W_{t_k}^j) \quad \text{if} \quad t_k \le t \le t_{k+1} \tag{2.11}$$

for all times $t \in [0,T]$, started at $Y_0 = Y_{0,y_0}(t_0) = y_0 \in \mathbb{R}^d$, where we have the identity $Y_{0,y_0}(t_{k+1}) = Y_{t_k,Y_k}(t_{k+1}) = Y_{k+1}$ for all $k = 0,1,...,n_T - 1$. Let $\|.\|_{d\times d}$ denote a matrix norm which is compatible with the Euclidean vector norm $\|.\|_d$ on $\mathbb{R}^d$.

THEOREM 2.4. *Let $\delta_0 \le \min(1,T)$ and $b^j(t,x) = A^j(t,x)x$. Assume that the stochastic process $X = (X_t)_{0\le t\le T}$ satisfies (a.s.) the Itô SDE*

$$dX_t = A^0(t,X_t)X_t\,dt + \sum_{j=1}^m A^j(t,X_t)X_t\,dW_t^j, \tag{2.12}$$

with nonrandom $\mathbb{R}^{d\times d}$-valued matrix coefficients A^j of Caratheodory-type and there are real constants K_M^S and K_B^j satisfying $\forall t,s \in [0,T] : 0 \le t-s \le \delta_0$ $\forall x \in \mathbb{R}^d$

$$\|A^j(t,x)\|_{d\times d} \le K_B^j, \quad \|(I_d - A^0(t,x)(t-s))^{-1}\|_{d\times d} \le \exp(K_M^S(t-s)) \tag{2.13}$$

Then the drift-implicit BIMs (2.8) applied to SDE (2.12) with weights $c^0(t,x) = -A^0(t,x)$ and $c^j(t,x) = \mathcal{O}$ $(j = 1,2,...,m)$, and step sizes

$$\Delta_k \le \Delta_{max} \le \delta_0 < \min\left\{1, T, \frac{1}{mp(p-1)(K_B^j)^2} : j = 1,2,...,m\right\} \tag{2.14}$$

are exponentially p-th mean stable with $p \ge 2$ and stability constant

$$K_S^{(p)} \le p \cdot \left(m\frac{p-1}{2} \sum_{j=1}^m \frac{(K_B^j)^2}{1 - mp(p-1)(K_B^j)^2 \Delta_{max}} + K_M^S \right) \tag{2.15}$$

and they satisfy global p-th mean stability estimates (2.5) and (2.6) for $p \ge 2$.

Proof. Suppose that $\delta_0 \le \min(1,T)$. Recall that $0 \le t - s \le \delta_0 \le \min(1,T)$. Let $Z(s)$, be any $(\mathcal{F}_s, \mathcal{B}(\mathbb{R}^d))$-measurable random variable. Define the auxiliary quantities $M_{s,x}(t) = I_d - A^0(t,x)(t-s)$ and $\gamma = \sqrt{1/(p-1)}$. Then,

$$\begin{aligned}
\mathbb{E}\,[\|Y_{s,Z(s)}(t)\|_d^p|\mathcal{F}_s] &= \mathbb{E}\,[\|Z(s) + M_{s,Z(s)}^{-1}(t)\sum_{j=0}^m b^j(s,Z(s))(W_t^j - W_s^j)\|_d^p|\mathcal{F}_s] \\
&= \mathbb{E}\,[\|M_{s,Z(s)}^{-1}(t)\Big(I_d + \sum_{j=1}^m A^j(s,Z(s))(W_t^j - W_s^j)\Big)Z(s)\|_d^p|\mathcal{F}_s] \\
&\le \exp(pK_M^S(t-s))\|Z(s)\|_d^p \mathbb{E}\,[\|I_d + \sum_{j=1}^m A^j(s,Z(s))(W_t^j - W_s^j)\|_{d\times d}^p|\mathcal{F}_s] \\
&= \exp(pK_M^S(t-s))\|Z(s)\|_d^p \mathbb{E}\,[\|I_d + \sum_{j=1}^m A^j(s,z)(W_t^j - W_s^j)\|_{d\times d}^p]\Big|_{z=Z(s)}.
\end{aligned}$$

Now, the expectation part at the right hand side is treated as follows. By using an elementary inequality originating from Clarkson [12] and Beckner [10] applied to the Banach space of random matrices with uniformly L^p-integrable coefficients (see Lemma 2.5 in Section 2.3 below) one finds that,

$$\mathbb{E}\,[\|I_d + \sum_{j=1}^m A^j(s,z)(W_t^j - W_s^j)\|_{d\times d}^p]\Big|_{z=Z(s)}$$

$$
\begin{aligned}
&= \frac{1}{2}\mathbb{E}\,[\|I_d + \gamma\frac{1}{\gamma}\sum_{j=1}^{m} A^j(s,z)(W_t^j - W_s^j)\|_{d\times d}^p]\Big|_{z=Z(s)} + \\
&\quad + \frac{1}{2}\mathbb{E}\,[\|I_d - \gamma\frac{1}{\gamma}\sum_{j=1}^{m} A^j(s,z)(W_t^j - W_s^j)\|_{d\times d}^p]\Big|_{z=Z(s)} \\
&\leq \mathbb{E}\left(1 + m\frac{1}{\gamma^2}\sum_{j=1}^{m}\|A^j(s,Z(s))\|_{d\times d}^2 (W_t^j - W_s^j)^2\right)^{p/2} \\
&\leq \mathbb{E}\left(1 + m(p-1)\sum_{j=1}^{m}(K_B^j)^2(W_t^j - W_s^j)^2\right)^{p/2} \\
&\leq \prod_{j=1}^{m}\mathbb{E}\,\exp\left(\frac{1}{2}mp(p-1)(K_B^j)^2(W_t^j - W_s^j)^2\right) \\
&\leq \exp\left(m\frac{p(p-1)}{2}\sum_{j=1}^{m}\frac{(K_B^j)^2}{1 - mp(p-1)(K_B^j)^2\Delta_{max}}(t-s)\right)
\end{aligned}
$$

for $0 \leq t - s \leq \Delta_{max} \leq \delta_0 < \min(1, T, 1/[mp(p-1)(K_B^j)^2])$. Exploiting this fact after returning to the original estimation yields

$$
\begin{aligned}
&\mathbb{E}\,[\|Y_{s,Z(s)}(t)\|_d^p|\mathcal{F}_s] \\
&\leq \exp\left(p[m\frac{p-1}{2}\sum_{j=1}^{m}\frac{(K_B^j)^2}{1 - mp(p-1)(K_B^j)^2\Delta_{max}} + K_M^S](t-s)\right)\cdot\|Z(s)\|_d^p.
\end{aligned}
$$

Therefore, the BIMs (2.8) are exponentially p-th mean stable for $p \geq 2$. It obviously remains to apply Theorem 2.2 in order to complete the proof with $K_S^{(p)}$ as in (2.15). □

Remark. Interestingly, we gain asymptotic p-th mean stability of BIMs provided that $K_M^S < -m(p-1)\sum_{j=1}^{m}(K_B^j)^2/2$ and sufficiently small step sizes. For example, compare with the onedimensional analytic case $dX_t = \alpha X_t dt + \sigma X_t dW_t$ when asymptotic stability of p-th moments can be established under the condition $\alpha + (p-1)\sigma^2/2 < 0$. In passing, one can also show that, the presence of negative semidefinite matrices A^0 in its drift leads to stabilizing effects on the moments of related SDEs (2.12). So some "coincidence" between analytic and numerical behavior is observed. Conditions (2.13) can be guaranteed for negative semidefinite matrices A^0 and uniformly bounded A^j for $j = 1, 2, ..., m$. For practical implementation, one may also take the stabilizing, negative semidefinite part of A^0 as weight matrix c^0 instead of the entire structure of A^0. As seen, drift-implicitness in BIMs (2.8) is sufficient to ensure p-th mean stability - a fact which [38] has already noted for linear systems with $p = 2$.

2.3. Exponential $(1 + \|x\|^2)^{p/2}$-stability.

The stability of numerical methods with respect to Lyapunov functions $V(x) = (1 + \|x\|_d^2)^{p/2}$ is commonly used to prove convergence (i.p. in the weak sense) of numerical methods. We will discuss this issue with the example of balanced implicit methods. For this purpose, we need a series of auxiliary lemmas.

2.3.1. Three auxiliary lemmas.

We begin with a random version of *Clarkson-Beckner inequality* (which even holds for each $\omega \in \Omega$, see proof below).

LEMMA 2.5. *Let X, Y be two elements of a (random) Hilbert space $(H, <.,.>_H)$ equipped with its scalar product $<.,.>_H$, $\mathbb{R}^1$ as its set of scalars and naturally induced norm $\|Z\|_H = (<Z, Z>_H)^{1/2}$. Assume that,*

$$
\mathbb{E}\,[\|X\|_H^p + \|Y\|_H^p] \;<\; +\infty
$$

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for a $p \geq 2$. Then, for that $p \geq 2$, we ω-wisely have

$$\frac{\|X+Y\|_H^p + \|X-Y\|_H^p}{2} \leq \Big(\min\{\|X\|_H^2 + (p-1)\|Y\|_H^2, \|Y\|_H^2 + (p-1)\|X\|_H^2\}\Big)^{p/2}$$

and hence moment-wisely

$$\frac{\mathbb{E}\|X+Y\|_H^p + \mathbb{E}\|X-Y\|_H^p}{2}$$
$$\leq \mathbb{E}\Big(\min\{\|X\|_H^2 + (p-1)\|Y\|_H^2, \|Y\|_H^2 + (p-1)\|X\|_H^2\}\Big)^{p/2}. \tag{2.16}$$

Proof. Define $B := \{X \in (H, <.,.>_H) : \|X\|_B^p = \mathbb{E}(\|X\|_H^p) < +\infty\}$. Then $(B, \|.\|_B)$ forms a Banach space as a subset of H. Suppose that $X, Y \in B \cap H$. Set $\gamma = 1/\sqrt{p-1}$, $z_1 = X + \sqrt{p-1}Y$, $z_2 = X - \sqrt{p-1}Y$, $u_1 = (\|z_1\|_H + \|z_2\|_H)/2$ and $u_2 = |\|z_1\|_H - \|z_2\|_H|/2$. Then, Clarkson-Beckner inequality from [12] and [10] which says that,

$$\Big(\frac{|1+u|^q + |1-u|^q}{2}\Big)^{1/q} \leq \Big(\frac{|1+\sqrt{(q-1)/(p-1)}u|^p + |1-\sqrt{(q-1)/(p-1)}u|^p}{2}\Big)^{1/p}$$

for all numbers $u \geq 0, 1 < p \leq q$ and parallelogram identity on Hilbert spaces imply that

$$\Big(\frac{\|X+Y\|_H^p + \|X-Y\|_H^p}{2}\Big)^{1/p} = \Big(\frac{\|X+\gamma\frac{1}{\gamma}Y\|_H^p + \|X-\gamma\frac{1}{\gamma}Y\|_H^p}{2}\Big)^{1/p}$$
$$\leq \Big(\frac{((1+\gamma)\|z_1\|_H/2 + (1-\gamma)\|z_2\|_H/2)^p + ((1-\gamma)\|z_1\|_H/2 + (1+\gamma)\|z_2\|_H/2)^p}{2}\Big)^{1/p}$$
$$= \Big(\frac{|u_1+\gamma u_2|^p + |u_1-\gamma u_2|^p}{2}\Big)^{1/p} \leq \Big(\frac{|u_1+u_2|^2 + |u_1-u_2|^2}{2}\Big)^{1/2}$$
$$= \Big(\frac{\|z_1\|_H^2 + \|z_2\|_H^2}{2}\Big)^{1/2} = \Big(\|X\|_H^2 + (p-1)\|Y\|_H^2\Big)^{1/2}.$$

Now, it remains to exploit the symmetry of the above expressions with respect to X and Y and to take the p-th power and expectation in order to arrive at (2.16). Thus, the proof is complete. $\square$

Second, observe the following property of moments of Gaussian exponentials.

LEMMA 2.6. *Assume that $X \in \mathcal{N}(0, \Delta)$. Then*

$$\forall \sigma \in \Big(-\frac{1}{\sqrt{2\Delta}}, \frac{1}{\sqrt{2\Delta}}\Big) \quad \mathbb{E}\exp(\sigma^2 X^2) \leq \frac{1}{\sqrt{1-2\sigma^2\Delta}} \leq \exp\Big(\frac{\sigma^2\Delta}{1-2\sigma^2\Delta}\Big). \tag{2.17}$$

Proof. Define $\xi = X/\sqrt{\Delta}$. Note that $\xi \in \mathcal{N}(0,1)$. Calculate

$$\mathbb{E}\exp(\sigma^2 X^2) = \mathbb{E}\exp(\sigma^2\Delta\,\xi^2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\Big(\sigma^2\Delta x^2 - \frac{x^2}{2}\Big)\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\Big(-(1-2\sigma^2\Delta)\frac{x^2}{2}\Big)\,dx = \frac{1}{\sqrt{1-2\sigma^2\Delta}} \leq \exp\Big(\frac{\sigma^2\Delta}{1-2\sigma^2\Delta}\Big)$$

using the elementary inequality $1/(1-z) \leq \exp(z/(1-z))$ for $z = 2\sigma^2\Delta < 1$. Thus, the proof is complete. $\square$

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Third, linear-polynomial boundedness of Lipschitz continuous functions can be established too. Let $C^0_{b(\kappa)}([0,T]\times \mathbb{R}^d,\mathbb{R}^l)$ denote the set of all continuous functions $f : [0,T]\times\mathbb{R}^d \to \mathbb{R}^l$ which are uniformly polynomially bounded such that

$$\forall t\in[0,T]\ \forall x\in\mathbb{R}^d \quad \|f(t,x)\|_l \ \le\ K_f\cdot(1+\|x\|_d^{\kappa}),$$

where $K_f \ge 0$ and $\kappa \ge 0$ are appropriate real constants. ($f : [0,T]\times\mathbb{R}^d\to\mathbb{R}^l$ is called linear-polynomial bounded if it is uniformly polynomially bounded with $\kappa = 1$.)

LEMMA 2.7. *Assume that $f \in C^0_{b(\kappa)}([0,T]\times\mathbb{R}^d,\mathbb{R}^l)$ with constants $\kappa\ge 0$ and K_f is uniformly Lipschitz continuous with constant K_L, i.e.*

$$\forall t\in[0,T]\ \forall x,y\in\mathbb{R}^d \quad \|f(t,x)-f(t,y)\|_l \ \le\ K_L\,\|x-y\|_d. \tag{2.18}$$

Then, there exist real constants $K_{b(p)} = K_{b(p)}(p,T,K_f,K_L)$ such that $\forall t\in[0,T]$

$$\forall x\in\mathbb{R}^d \quad \|f(t,x)\|_l \ \le\ 2^{-(p-1)/p}K_{b(p)}\cdot(1+\|x\|_d) \ \le\ K_{b(p)}\cdot(1+\|x\|_d^p)^{1/p} \tag{2.19}$$

for all $p\ge 1$, where the real constants $K_{b(p)}$ can be estimated by

$$0 \ \le\ K_{b(p)} \ \le\ 2^{(p-1)/p}\cdot\max\{K_f,K_L\}. \tag{2.20}$$

Proof. While using triangle and Hölder inequalities, estimate

$$\begin{aligned} 0 \ &\le\ \|f(t,x)\|_l \ \le\ \|f(t,0)\|_l + \|f(t,x)-f(t,0)\|_l \ \le\ K_f + K_L\|x\|_d \\ &\le\ \max\{K_f,K_L\}(1+\|x\|_d) \ \le\ 2^{(p-1)/p}\max\{K_f,K_L\}(1+\|x\|_d^p)^{1/p}. \end{aligned}$$

Therefore, constant $K_{b(p)}$ can be chosen as in (2.20). Thus, the proof is complete. ☐

Remark. In fact, it suffices that $\sup_{0\le t\le T}\|f(t,x_*)\|_l < +\infty$ for some $x_*\in\mathbb{R}^d$ and f is Lipschitz continuous in $x\in\mathbb{R}^d$ with constant $K_L(t)$ which is uniformly bounded with respect to $t\in[0,T]$. However, $K_{b(p)}$ may depend on κ too.

2.3.2. Exponential weak $(1+\|x\|_d^2)^{p/2}$-stability of BIMs (2.8). Consider BIMs (2.8) with both variable or constant step sizes $\Delta_k \le \Delta_{max}$ where Δ_{max} sufficiently small. Uniform boundedness of p-th moments of these methods can be established as follows.

THEOREM 2.8. *Let $p = 2\kappa$ and $\kappa\in\mathbb{N}\setminus\{0\}$. Assume that BIMs (2.8) with nonrandom step sizes $\Delta_k \le \Delta_{max} \le \delta_0 < \min(1,T)$ and*

$$\forall j = 1,2,...,m: \quad p(p-1)mK_M^2(K^j_{b(2)})^2\Delta_{max} < 1 \tag{2.21}$$

possess real constants $K_M = K_M(T)\ge 0, K_C = K_C(T)\ge 0$ such that, for the chosen weight matrices $c^j\in\mathbb{R}^{d\times d}$ of BIMs (2.8), we have

$$\forall t\in[0,T]\ \forall x\in\mathbb{R}^d \ \sum_{j,k=0}^{m}\|c^k(t,x)b^j(t,x)\|_d^2 \ \le\ (K_C)^2(1+\|x\|_d^2), \tag{2.22}$$

$$\forall s,t : 0\le t-s\le\delta_0 \ \ \forall x\in\mathbb{R}^d \ \exists M^{-1}_{s,x}(t) \text{ with } ||M^{-1}_{s,x}(t)||_{d\times d} \ \le\ K_M, \tag{2.23}$$

$$\forall t\in[0,T]\ \forall x\in\mathbb{R}^d \quad \|b^j(t,x)\|_d^2 \ \le\ (K^j_{b(2)})^2(1+\|x\|_d^2), \tag{2.24}$$

$$\mathbb{E}\|Y_0\|_d^p \ <\ +\infty. \tag{2.25}$$

Then, all p-th moments of BIMs (2.8) are uniformly bounded and, more precisely, for all $k = 0, 1, ..., n_T$ and all $\kappa \in \mathbb{N} \setminus \{0\}$ with $2\kappa \leq p$, we have

$$\begin{aligned} \mathbb{E}\,\|Y_k\|_d^{2\kappa} \;&\leq\; \mathbb{E}\,[1 + \|Y_k\|_d^2]^\kappa \;\leq\; \exp(K_{2\kappa} t_k)\mathbb{E}\,[1 + \|Y_0\|_d^2]^\kappa \\ &\leq\; \exp(K_{2\kappa} T)\mathbb{E}\,[1 + \|Y_0\|_d^2]^\kappa \end{aligned} \tag{2.26}$$

with appropriate real constant

$$K_{2\kappa} \leq \kappa K_M \Big[2K^0_{b(2)} + (2\kappa-1) m K_M \sum_{j=1}^m \frac{(K^j_{b(2)})^2}{1 - 2\kappa(2\kappa-1) m K_M^2 (K^j_{b(2)})^2 \Delta_k} \Big]. \tag{2.27}$$

Proof. Define $v_0(k) := \mathbb{E}\,[\|Y_k\|_d^p]$ for all $k = 0, 1, ..., n_T$. First, note that

$$\begin{aligned} v_0(k+1) \;&=\; \mathbb{E}\,[\|Y_k + M^{-1}_{t_k,Y_k}(t_{k+1}) \sum_{j=0}^m b^j(t_k, Y_k)\Delta W_k^j \|_d^p] \\ &= \frac{1}{2}\mathbb{E}\,[\|Y_k + M^{-1}_{t_k,Y_k}(t_{k+1}) a(t_k, Y_k)\Delta_k + \sum_{j=1}^m M^{-1}_{t_k,Y_k}(t_{k+1}) b^j(t_k, Y_k)\Delta W_k^j \|_d^p] + \\ &\quad + \frac{1}{2}\mathbb{E}\,[\|Y_k + M^{-1}_{t_k,Y_k}(t_{k+1}) a(t_k, Y_k)\Delta_k - \sum_{j=1}^m M^{-1}_{t_k,Y_k}(t_{k+1}) b^j(t_k, Y_k)\Delta W_k^j \|_d^p]. \end{aligned}$$

Second, apply the random version of Clarkson-Beckner inequality (2.16) from Lemma 2.5 and obtain

$$\begin{aligned} &v_0(k+1) \leq \\ &\mathbb{E}\,\Big(\|Y_k + M^{-1}_{t_k,Y_k}(t_{k+1}) a(t_k, Y_k)\Delta_k\|_d^2 + (p-1)\|\sum_{j=1}^m M^{-1}_{t_k,Y_k}(t_{k+1}) b^j(t_k,Y_k)\Delta W_k^j\|_d^2 \Big)^{p/2}. \end{aligned}$$

Thanks to (2.23) and (2.24), this implies

$$v_0(k+1) = \mathbb{E}\,[\|Y_{k+1}\|_d^2]^{p/2}$$

$$\begin{aligned} &\leq \mathbb{E}\,\Big(\|Y_k\|_d^2 + 2K_M \|Y_k\|_d \|a(t_k, Y_k)\|_d \Delta_k + K_M^2 \|a(t_k, Y_k)\|_d^2 \Delta_k^2 + \\ &\quad + (p-1) m K_M^2 \sum_{j=1}^m \|b^j(t_k, Y_k)\|_d^2 (\Delta W_k^j)^2 \Big)^{p/2} \end{aligned} \tag{2.28}$$

$$\begin{aligned} &\leq \mathbb{E}\,\Big(\|Y_k\|_d^2 + (1 + \|Y_k\|_d^2) \Big[2K_M K^0_{b(2)} \Delta_k + K_M^2 (K^0_{b(2)})^2 \Delta_k^2 + \\ &\quad + (p-1) m K_M^2 \sum_{j=1}^m (K^j_{b(2)})^2 (\Delta W_k^j)^2 \Big] \Big)^{p/2}. \end{aligned} \tag{2.29}$$

Third, repeat the previous estimation for all exponents 2κ with $0 < 2\kappa \leq p$ instead of p. This leads to inequalities (2.29) for $2\kappa \leq p$ instead of p. Define $v_p(k) := \mathbb{E}\,[1 + \|Y_k\|^2]_d^{p/2}$ for all $k = 0, 1, ..., n_T$. In particular, we are interested in $v_{2\kappa}(k) = \mathbb{E}\,[1 + \|Y_k\|^2]_d^\kappa$ for all $k = 0, 1, ..., n_T$ and all $\kappa \in [0, p/2]$. For simplicity, suppose that $\kappa \in \mathbb{N} \setminus \{0\}$. Apply the binomial theorem in order to estimate the expression

$$v_{2\kappa}(k+1) = \mathbb{E}\,[1 + \|Y_{k+1}\|_d^2]^\kappa \;=\; \sum_{n=0}^\kappa \binom{\kappa}{n} \mathbb{E}\,[\|Y_{k+1}\|_d^2]^n$$

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for all $\kappa \in (0, p/2] \cap \mathbb{N}$. Adding the inequalities (2.29) for all $2n \leq 2\kappa \leq p$ instead of p, multiplied by the related binomial coefficients, leads to

$$\begin{aligned}
& v_{2\kappa}(k+1) \\
& \leq \mathbb{E}\left((1+\|Y_k\|_d^2)\Big[1+2K_M K^0_{b(2)}\Delta_k + K_M^2 (K^0_{b(2)})^2\Delta_k^2 + \sum_{j=1}^m \frac{\sigma_j^2}{\kappa}(\Delta W_k^j)^2\Big]\right)^\kappa \\
& \leq \mathbb{E}\left((1+\|Y_k\|_d^2)\exp\left(2K_M K^0_{b(2)}\Delta_k + (2\kappa-1)mK_M^2\sum_{j=1}^m (K^j_{b(2)})^2(\Delta W_k^j)^2\right)\right)^\kappa \\
& \leq \mathbb{E}\left((1+\|Y_k\|_d^2)^\kappa \exp\left(2\kappa K_M K^0_{b(2)}\Delta_k\right)\prod_{j=1}^m \mathbb{E}\left[\exp\left(\sigma_j^2(\Delta W_k^j)^2\right)\Big|\mathcal{F}_k\right]\right) \\
& \leq \mathbb{E}\left(1+\|Y_k\|_d^2\right)^\kappa \exp\left(2\kappa K_M K^0_{b(2)}\Delta_k\right)\prod_{j=1}^m \mathbb{E}\left[\exp\left(\kappa(2\kappa-1)mK_M^2(K^j_{b(2)})^2\Delta_k(\xi_k^j)^2\right)\right]
\end{aligned}$$

with i.i.d. $\xi_k^j \in \mathcal{N}(0,1)$, thanks to monotonicity of expectations, tower property of conditional expectations and independence of increments $\Delta W_k^j = \sqrt{\Delta_k}\,\xi_k^j$, where we put $\sigma_j^2 = \kappa(2\kappa-1)mK_M^2(K^j_{b(2)})^2$. Fourth, suppose that the constants σ_j satisfy $2\sigma_j^2\Delta_k < 1$. Apply Lemma 2.6 with σ_j^2 to treat the latter estimate. This implies that

$$0 \leq \mathbb{E}\,\|Y_k\|_d^{2\kappa} < v_{2\kappa}(k+1) \leq v_{2\kappa}(k)\exp\Big(c_H(k)\Big) \tag{2.30}$$

where the coefficients c_H are given by

$$c_H(k) = \kappa K_M\Big(2K^0_{b(2)} + (2\kappa-1)mK_M\sum_{j=1}^m \frac{(K^j_{b(2)})^2}{1-2\kappa(2\kappa-1)mK_M^2(K^j_{b(2)})^2\Delta_k}\Big)\Delta_k.$$

Therefore, $(v_k)_{k=0,1,\ldots,n_T}$ is governed by a linear homogeneous inequality (2.30) whose maximum solution can be estimated by the discrete variation-of-constants formula (which reduces to the discrete Gronwall-Bellman Lemma here) as proven in [38] and applied in [44, 45, 46, 47]. Thus, we arrive at

$$0 \leq \mathbb{E}\,\|Y_k\|_d^{2\kappa} < v_{2\kappa}(k+1) \leq v_{2\kappa}(k)\exp\Big(K_{2\kappa}\Delta_k\Big) \leq v_{2\kappa}(0)\exp\Big(K_{2\kappa}t_{k+1}\Big).$$

This gives the estimates (2.26) with constants $K_{2\kappa}$ estimated as in (2.27). Note that $K_{2\kappa}$ is increasing for increasing κ, hence $K_{2\kappa} \leq K_p$ and the uniform boundedness of all 2κ-moments of BIMs (2.8) is obtained for all $\kappa \in [0, p/2]$ provided that the initial moment $\mathbb{E}\,\|Y_0\|_d^p < +\infty$. Thus, the proof is complete. ∎

3. Asymptotic Almost Sure Stability.

In the following sections, we discuss the almost sure stability behavior of sequences and numerical methods, with both constant and variable step sizes with respect to, the trivial equilibrium $0 \in \mathbb{R}^d$. Let $\|.\|_d$, be a vector norm of $\mathbb{R}^d$ which is compatible with the matrix norm $\|.\|_{d\times d}$ of $\mathbb{R}^{d\times d}$.

3.1. Definition and general theorems.

In the course of our presentation, we identify the stability of equilibria with the stability of related numerical methods, as it is common in numerical analysis.

DEFINITION 3.1. *A random sequence $Y = (Y_n)_{n\in\mathbb{N}}$ of real-valued random variables $Y_n : (\Omega, \mathcal{F}_n, \mathbb{P}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is called* **(globally) asymptotically stable, with probability one** *(or* **(globally) asymptotically a.s. stable***) if,*

$$\lim_{n\to+\infty} \|Y_n\|_d = 0 \; (a.s.)$$

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for all $Y_0 = y_0 \in \mathbb{R}^d \setminus \{0\}$*, where* $y_0 \in \mathbb{R}^d$ *is nonrandom, otherwise* **asymptotically a.s. unstable**.

THEOREM 3.2. *Let* $V = (V(n))_{n\in\mathbb{N}}$ *be a sequence of nonnegative random variables* $V(n) : (\Omega, \mathcal{F}_n, \mathbb{P}) \to \mathbb{R}^1_+$*, with* $V(0) > 0$ *satisfying the recursive scheme*

$$V(n+1) \;=\; V(n)G(n) \tag{3.1}$$

where $G(n) : (\Omega, \mathcal{F}_n, \mathbb{P}) \to \mathbb{R}^1_+$ *are i.i.d. random variables, with the moment property* $\mathbb{E}\,|\ln[G(n)]| < +\infty$*. Then*

V *(globally) asymptotically a.s. stable* *iff* $\mathbb{E}\,\ln[G(0)] < 0$.

Proof. This result is already found in Higham [18] and in Schurz [44]. The main idea is to use the strong law of large numbers (SLLN) in conjunction with the law of iterated logarithm (LIL). Note that V possesses the explicit representation

$$V(n+1) \;=\; \left(\prod_{k=0}^{n} G(k)\right) V(0) \tag{3.2}$$

for all $n \in \mathbb{N}$. Now, suppose that $G(k)$ are i.i.d. random variables and define

$$\mu_0 := \mathbb{E}\,[\ln(G(0))], \quad S_n := \sum_{k=0}^{n-1} \ln G(k),$$

hence $V(n+1) = \exp(S_{n+1})V(0)$ and $\mathbb{E}\,[S_n] = n\mu$ for $n \in \mathbb{N}$. By SLLN, conclude that

$$\lim_{n\to+\infty} \frac{S_n}{n} \;=\; \mu_0 \; (a.s.)$$

thanks to the $\mathbb{P}$-integrability of $G(k)$. This fact implies that if $\mu_0 < 0$ then $S_n \to -\infty$, i.e. $V(n) \to 0$ as n tends to $+\infty$ and if $\mu > 0$ then $S_n \to +\infty$, i.e. $V(n) \to +\infty$ as n tends to $+\infty$. Moreover, in the case $\mu_0 = 0$, we may use LIL (at first, under $\sigma^2 = Var(\ln G(k)) = \mathbb{E}\,[\ln G(k) - \mathbb{E}\,\ln G(k)]^2 < +\infty$, later we may drop $\sigma^2 < +\infty$ by localization procedures) to get

$$\liminf_{n\to+\infty} \frac{S_n}{\sqrt{2n \ln\ln n}} = -|\sigma|, \quad \limsup_{n\to+\infty} \frac{S_n}{\sqrt{2n\ln\ln n}} = |\sigma|,$$

hence S_n oscillates with growing amplitude and $\lim_{n\to+\infty} S_n$ does not exist. Therefore

$$\lim_{n\to+\infty} V(n) \;=\; \lim_{n\to+\infty} \exp(S_n)V(0)$$

does not exist either (a.s.). Thus, $\lim_{n\to+\infty} V(n) \neq 0$ and the proof is complete. ☐

Now, consider the one dimensional test class of pure diffusion equations

$$dX_t \;=\; \sigma X_t \, dW_t \tag{3.3}$$

as suggested by Milstein, Platen and Schurz [31]. Then, the following result provides a mathematical evidence that their numerical experiments for BIMs (2.8) led to the correct observation of numerical stability due to its asymptotic a.s. stability. It extends the results, which are found in [18], [38] and [44].

THEOREM 3.3. *The BIMs (2.8) with scalar weights* $c^0 = 0$ *and* $c^1 = |\sigma|$ *applied to martingale test equations (3.3) for any parameter* $\sigma \in \mathbb{R}^1 \setminus \{0\}$ *with any equidistant step size* Δ *provide (globally) asymptotically a.s. stable sequences* $Y = (Y_n)_{n\in\mathbb{N}}$.

Proof. Suppose $|\sigma| > 0$. Then, the proof is an application of Theorem 3.2. For this purpose, consider the sequence $V = (V(n))_{n\in\mathbb{N}} = (|Y_n|)_{n\in\mathbb{N}}$. Note that $V(n+1) = G(n)V(n)$, $\mathbb{E}\,|\ln G(n)| < +\infty$ and $\mathbb{E}\,[\ln G(n)] < 0$ since

$$\begin{aligned}
&\mathbb{E}\,[|\ln G(n)|] \;\le\; (\mathbb{E}\,[\ln G(n)]^2)^{1/2} \;\le\; \ln(2) + |\sigma|\sqrt{\Delta} \quad \text{and} \\
&\mathbb{E}\,[\ln G(n)] \;=\; \mathbb{E}\left[\ln\left|\frac{1+|\sigma\Delta W_n|+\sigma\Delta W_n}{1+|\sigma\Delta W_n|}\right|\right] \;=\; \mathbb{E}\left[\ln\left|1+\frac{\sigma\Delta W_n}{1+|\sigma\Delta W_n|}\right|\right] \\
&= \frac{1}{2}\mathbb{E}\left[\ln\left|1+\frac{\sigma\Delta W_n}{1+|\sigma\Delta W_n|}\right|\right] + \frac{1}{2}\mathbb{E}\left[\ln\left|1-\frac{\sigma\Delta W_n}{1+|\sigma\Delta W_n|}\right|\right] \\
&= \frac{1}{2}\mathbb{E}\left[\ln\left|1-\left(\frac{\sigma\Delta W_n}{1+|\sigma\Delta W_n|}\right)^2\right|\right] \;<\; -\frac{1}{2}\mathbb{E}\left[\left(\frac{|\sigma\Delta W_n|}{1+|\sigma\Delta W_n|}\right)^2\right] \;<\; 0
\end{aligned}$$

with independently identically Gaussian distributed increments $\Delta W_n \in \mathcal{N}(0,\Delta)$ (In fact, note that, for all $\sigma \neq 0$ and Gaussian ΔW_n, we have

$$0 \;<\; 1-\left(\frac{\sigma\Delta W_n}{1+|\sigma\Delta W_n|}\right)^2 \;<\; 1$$

with probability one, hence, that ΔW_n has a nondegenerate probability distribution with non-trivial support is essential here!). Therefore, the assumptions of Theorem 3.2 are satisfied and an application of Theorem 3.2 yields the claim of Theorem 3.3. Thus, the proof is complete. $\square$

Remark. The increments $\Delta W_n \in \mathcal{N}(0,\Delta_n)$ can also be replaced by multi-point discrete probability distributions such as

$$\mathbb{P}\,\{\Delta W_n = \pm\sqrt{\Delta_n}\} \;=\; \frac{1}{2}$$

$$\text{or}\quad \mathbb{P}\,\{\Delta W_n = 0\} \;=\; \frac{2}{3},\;\; \mathbb{P}\,\{\Delta W_n = \pm\sqrt{3\Delta_n}\} \;=\; \frac{1}{6}$$

as commonly met in weak approximations. In this case, the almost sure stability of the BIMs as chosen by Theorem 3.3 is still guaranteed, as seen by our proof above (due to the inherent symmetry of ΔW_n with respect to 0).

For variable step sizes, we can also formulate and prove a general assertion with respect to asymptotic a.s. stability. Let $Var(Z)$ denote the variance of the inscribed random variable Z.

THEOREM 3.4. *Let $V = (V(n))_{n\in\mathbb{N}}$ be a sequence of nonnegative random variables $V(n) : (\Omega, \mathcal{F}_n, \mathbb{P}) \to \mathbb{R}^1_+$ with $V(0) > 0$ satisfying the recursive scheme*

$$V(n+1) \;=\; V(n)G(n), \tag{3.4}$$

where $G(n) : (\Omega, \mathcal{F}_n, \mathbb{P}) \to \mathbb{R}^1_+$ are independent random variables such that $\exists$ nonrandom sequence $b = (b_n)_{n\in\mathbb{N}}$ with $b_n \to +\infty$ as $n \to +\infty$

$$\sum_{k=0}^{+\infty} \frac{Var(\ln(G(k)))}{b_k^2} \;<\; +\infty, \qquad \lim_{n\to+\infty} \frac{\sum_{k=0}^{n-1} \mathbb{E}\,\ln(G(k))}{b_n} \;<\; 0. \tag{3.5}$$

Then $V = (V(n))_{n\to+\infty}$ is (globally) asymptotically a.s. stable sequence, i.e. we have $\lim_{n\to+\infty} V(n) \;=\; 0$ (a.s.).

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Moreover, if

$$\sum_{k=0}^{+\infty} \frac{Var(\ln(G(k)))}{b_k^2} < +\infty, \quad \lim_{n\to+\infty} \frac{\sum_{k=0}^{n-1} \mathbb{E} \ln(G(k))}{b_n} > 0 \tag{3.6}$$

then $V = (V(n))_{n\to+\infty}$ *is (globally) asymptotically a.s. unstable sequence, i.e. we have* $\lim_{n\to+\infty} V(n) = +\infty$ *(a.s.) for all nonrandom* $y_0 \neq 0$.

Proof. The main idea is to apply Kolmogorov's SLLN, see Shiryaev [48] (p. 389). Recall that V possesses the explicit representation (3.2). Now, define

$$S_n := \sum_{k=0}^{n-1} \ln G(k),$$

hence $V(n+1) = \exp(S_{n+1})V(0)$ for $n \in \mathbb{N}$. By Kolmogorov's SLLN we may conclude that

$$\lim_{n\to+\infty} \frac{S_n}{b_n} = \lim_{n\to+\infty} \frac{\mathbb{E}\, S_n}{b_n} = \lim_{n\to+\infty} \frac{\sum_{k=0}^{n-1} \mathbb{E}\, \ln(G(k))}{b_n} < 0 \ (a.s.)$$

thanks to the assumptions (3.5) of $\mathbb{P}$ -integrability of $G(k)$. This fact together with $b_n \to +\infty$ implies that $S_n \to -\infty$ (a.s.), i.e. $V(n) \to 0$ as n tends to $+\infty$. The reverse direction under (3.6) is proved analogously to previous proof-steps. Thus, the proof is complete. □

Now, let us apply this result to BIMs (2.8) applied to test equation (3.3). For $k = 0, 1, ..., n_T$, define

$$G(k) := \left| \frac{1 + |\sigma \Delta W_k| + \sigma \Delta W_k}{1 + |\sigma \Delta W_k|} \right|. \tag{3.7}$$

THEOREM 3.5. *Assume that* $\exists$ *nonrandom sequence* $b = (b_n)_{n\in\mathbb{N}}$ *with* $b_n \to +\infty$ *as* $n \to +\infty$ *for a fixed choice of step sizes* $\Delta_n > 0$ *such that*

$$\sum_{k=0}^{+\infty} \frac{Var(\ln(G(k)))}{b_k^2} < +\infty, \quad \lim_{n\to+\infty} \frac{\sum_{k=0}^{n-1} \mathbb{E} \ln(G(k))}{b_n} < 0.$$

Then the BIMs (2.8) with scalar weights $c^0 = 0$ *and* $c^1 = |\sigma|$ *applied to martingale test equations (3.3) with parameter* $\sigma \in \mathbb{R}^1 \setminus \{0\}$ *with the fixed sequence of variable step sizes* Δ_n *provide (globally) asymptotically a.s. stable sequences* $Y = (Y_n)_{n\in\mathbb{N}}$.

Proof. We may apply Theorem 3.4 with $V(n) = |Y_n|$ since the assumptions are satisfied for the BIMs (2.8) with scalar weights $c^0 = 0$ and $c^1 = |\sigma|$ applied to martingale test equations (3.3). Hence, the proof is complete. □

THEOREM 3.6. *The BIMs (2.8) with scalar weights* $c^0 = 0$ *and* $c^1 = |\sigma|$ *applied to martingale test equations (3.3) with parameter* $\sigma \in \mathbb{R}^1 \setminus \{0\}$ *with any nonrandom variable step sizes* Δ_k *satisfying* $0 < \Delta_{min} \leq \Delta_k \leq \Delta_{max}$ *provide (globally) asymptotically a.s. stable sequences* $Y = (Y_n)_{n\in\mathbb{N}}$.

Proof. We may again apply Theorem 3.4. with $V(n) = |Y_n|$. For this purpose, we check the assumptions. Define $b_n := n$. Note that the variance $Var(\ln(G(k)))$ is uniformly bounded since $\Delta W_n \in \mathcal{N}(0, \Delta_n)$ and $0 < \Delta_{min} \leq \Delta_k \leq \Delta_{max}$. More precisely, we have

$$\begin{aligned}
&Var(\ln(G(k))) \\
&\leq \mathbb{E}\,[\ln(G(k))]^2 = \mathbb{E}\,[I_{\{\Delta W_n > 0\}} \ln(G(k))]^2 + \mathbb{E}\,[I_{\{\Delta W_n < 0\}} \ln(G(k))]^2 \\
&< p_n[\ln(2)]^2 + \mathbb{E}\,[\ln(1 + |\sigma \Delta W_n|)]^2 \leq p_n[\ln(2)]^2 + \mathbb{E}\,[\ln(\exp(|\sigma \Delta W_n|))]^2 \\
&\leq p_n[\ln(2)]^2 + \mathbb{E}\,[\sigma \Delta W_n]^2 = p_n[\ln(2)]^2 + \sigma^2 \Delta_n \leq p_n[\ln(2)]^2 + \sigma^2 \Delta_{max}
\end{aligned}$$

for $G(k)$ as defined in (3.7), where $I_{\{Q\}}$ denotes the indicator function of the inscribed set Q and $p_n = \sqrt{\mathbb{P}\{\Delta W_n > 0\}}$. Note that $0 < p_n = \sqrt{2}/2 < 1$ if ΔW_n is Gaussian distributed. Therefore, there is a finite real constant $K_2^G < (\ln(2))^2 + \sigma^2 \Delta_{max}$ such that

$$\sum_{k=1}^{+\infty} \frac{Var(\ln(G(k)))}{k^2} \leq \sum_{k=1}^{+\infty} \frac{K_2^G}{k^2} = K_2^G \frac{\pi^2}{6} < +\infty.$$

It remains to check whether

$$\lim_{n \to +\infty} \frac{\sum_{k=1}^n \mathbb{E} \ln(G(k))}{n} < 0.$$

For this purpose, we only note that $\mathbb{E} \ln(G(k))$ is decreasing for increasing $\sqrt{\Delta_k}$ for all $k \in \mathbb{N}$ (see the proof of Theorem 3.3). Therefore, we can estimate this expression by

$$\mathbb{E} \ln(G(k)) \leq \frac{1}{2} \mathbb{E} \left[\ln \left| 1 - \left(\frac{\sigma\sqrt{\Delta_{min}}\xi}{1 + |\sigma\sqrt{\Delta_{min}}\xi|} \right)^2 \right| \right] := K_1^G < 0$$

where $\xi \in \mathcal{N}(0,1)$ is a standard Gaussian distributed random variable and K_1^G the negative real constant as defined above. Thus,

$$\lim_{n \to +\infty} \frac{\sum_{k=1}^n \mathbb{E} \ln(G(k))}{n} \leq K_1^G < 0.$$

Hence, thanks to Theorem 3.4 (or Theorem 3.5), the proof is completed. $\square$

4. V-Stability Exponents and Asymptotic Stability. The estimation of stability exponents and its sign is an important task to measure the convergence or divergence speed of numerical methods towards its equilibria.

4.1. Definition and general estimation theorem. For this purpose, consider the following definition.

DEFINITION 4.1. *The* **upper (forward moment) V-exponent** *of a random sequence $Y = (Y_n)_{n \in \mathbb{N}}$ with values in the domain $\mathbb{D} \subseteq \mathbb{R}^d$ is defined to be*

$$\overline{\lambda}_V := \limsup_{n \to +\infty} \ln \left(\mathbb{E} V(n, (Y_k)_{k \leq n}) \right) \tag{4.1}$$

for a fixed deterministic functional $V = V(n,y) : \mathbb{N} \times (\mathbb{D})^{n+1} \longrightarrow \mathbb{R}_+$ (or positive function). The **lower (forward moment) V-exponent** *of a random sequence $Y = (Y_n)_{n \in \mathbb{N}}$ with values in the domain $\mathbb{D} \subseteq \mathbb{R}^d$ is defined to be*

$$\underline{\lambda}_V := \liminf_{n \to +\infty} \ln \left(\mathbb{E} V(n, (Y_k)_{k \leq n}) \right) \tag{4.2}$$

for a fixed deterministic functional $V = V(n,y) : \mathbb{N} \times (\mathbb{D})^{n+1} \longrightarrow \mathbb{R}_+$ (or positive function).

For the sake of abbreviation, define

$$\Delta \mathbb{E} V_n := \mathbb{E} V(n+1, (Y_k)_{k \leq n+1}) - \mathbb{E} V(n, (Y_k)_{k \leq n})$$

for the discrete time $\mathbb{D}$-valued stochastic process $Y = (Y_n)_{n \in \mathbb{N}}$ on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in \mathbb{N}}, \mathbb{P})$.

THEOREM 4.2. *Assume that $\mathbb{E} V(0, Y_0) < +\infty$ for a nonanticipating functional $V : \mathbb{N} \times \mathbb{D} \times ... \times \mathbb{D} \longrightarrow \mathbb{R}^1_+$ with*

$$\underline{k}_n \mathbb{E} V(n, (Y_k)_{k \leq n}) \leq \Delta \mathbb{E} V_n \leq \overline{k}_n \mathbb{E} V(n, (Y_k)_{k \leq n})$$

for all $n \in \mathbb{N}$*, where* $\underline{k}_i, \overline{k}_i$ *are deterministic, real constants along the dynamics of process* $Y = (Y_n)_{n\in\mathbb{N}}$*, and for all* $n \in \mathbb{N}$

$$1 + \underline{k}_n > 0\,.$$

Then, for all $n \in \mathbb{N}$*, we have*

$$\exp\left(\sum_{i=0}^{n} \frac{\underline{k}_i}{1+\underline{k}_i}\right) \mathbb{E}\, V(0, Y_0) \le \mathbb{E}\, V(n+1, (Y_k)_{k\le n+1}) \le \exp\left(\sum_{i=0}^{n} \overline{k}_i\right) \mathbb{E}\, V(0, Y_0)$$

and, if the limits exist, then

$$\liminf_{n\to+\infty} \frac{\sum_{i=0}^{n-1} \frac{\underline{k}_i}{1+\underline{k}_i}}{t_n} \le \underline{\lambda}_V \le \overline{\lambda}_V \le \limsup_{n\to+\infty} \frac{\sum_{i=0}^{n-1} \overline{k}_i}{t_n}\,.$$

If $\overline{\lambda}_V < 0$ *then* $\lim_{n\to+\infty} V(n, (Y_k)_{k\le n}) = 0$ *(a.s.).*

Proof. First, assume $\Delta \mathbb{E}\, V_n \le \overline{k}_n \mathbb{E}\, V(n, (Y_k)_{k\le n})$ (for all $n \in \mathbb{N}$). Making use of elementary splitting

$$z(n+1) = z(n) + z(n+1) - z(n)$$

with $z(n+1) := \mathbb{E}\, V(n+1, (Y_k)_{k\le n+1})$, one concludes

$$z(n+1) \le z(n)(1+\overline{k}_n) \le z(0) \prod_{i=0}^{n} (1+\overline{k}_i)_+ \le z(0) \exp\left(\sum_{i=0}^{n} \overline{k}_i\right)\,.$$

On the other hand, when $\Delta \mathbb{E}\, V_n \le \underline{k}_n \mathbb{E}\, V(n, (Y_k)_{k\le n})$ and $1 + \underline{k}_n > 0$ (for all $n \in \mathbb{N}$), one recognizes the validity of

$$z(n) \le \frac{z(n+1)}{1+\underline{k}_n} \le z(n+1) \exp\left(\frac{-\underline{k}_n}{1+\underline{k}_n}\right)$$

which implies

$$z(n+1) \ge z(n) \exp\left(\frac{\underline{k}_n}{1+\underline{k}_n}\right) \ge z(0) \exp\left(\sum_{i=0}^{n} \frac{\underline{k}_i}{1+\underline{k}_i}\right)\,,$$

using elementary inequality

$$\frac{1}{1+x} \le \exp(-\frac{x}{1+x})\,.$$

Now one arrives at the second result by taking the exponential logarithm and limit when integration time t_n advances. The remaining almost sure convergence while $\overline{\lambda} < 0$ is concluded by a straightforward application of the well-known Borel-Cantelli-Lemma. Therefore, the proof is complete. $\square$

4.2. The example of discretized damped linear oscillator. For the sake of simple illustration, we consider the stochastic oscillator with multiplicative white noise

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x \;=\; \sigma\dot{x}\,\xi_t \tag{4.3}$$

where $\zeta, \omega > 0$ and the stochastic integration is understood in the sense of Itô. Then the corresponding deterministic equation has an asymptotically stable zero solution if $0 < \zeta < 1$, and does not exponentially grow if $0 \leq \zeta \leq 1$. The stochastic oscillator (4.3) possesses a upper V-stability exponent $\overline{\lambda}_V \leq 0$ with $V(x,y) = y^2 + \omega^2 x^2$ if $0 \leq \sigma^2 \leq 4\zeta\omega$. Let us now look at the discretization of such a equation by numerical methods. Define the numerical Lyapunov function by

$$V(n+1,x,y) := \omega^2 x^2 + (1 + 2\zeta\omega\Delta_n) y^2$$

where $\Delta_n = t_{n+1} - t_n$ is current step size, and $v_{n+1} := \mathbb{E}\, V(n+1, X_{n+1}, Y_{n+1})$.

THEOREM 4.3. *Assume that the stochastic oscillator (4.3) is discretized by the fully drift-implicit Euler method (which can be represented as BIM (2.8) here too) given by*

$$\begin{aligned} X_{n+1} &= X_n + Y_{n+1}\Delta_n \\ Y_{n+1} &= Y_n - (2\zeta\omega Y_{n+1} + \omega^2 X_{n+1})\Delta_n + \sigma Y_n \Delta W_n \end{aligned} \tag{4.4}$$

where $\Delta W_n = W_{t_{n+1}} - W_{t_n}$ along a time-discretization $(t_n)_{n\in\mathbb{N}}$, and

$$\mathbb{E}\,[\omega^2 X_0^2 + Y_0^2] < +\infty.$$

Then, for all $n \in \mathbb{N}$, all $l \in \mathbb{N}$ with $1 \leq l < n$, we have

$$v_l \exp\left(\sum_{i=l}^{n} \frac{\underline{k}_i}{1+\underline{k}_i}\right) \;\leq\; v_{n+1} = \mathbb{E}\, V(n+1, X_{n+1}, Y_{n+1}) \;\leq\; v_l \exp\left(\sum_{i=l}^{n} \overline{k}_i\right)$$

where

$$\overline{k}_i = \frac{-\omega^2\Delta_i^2(1+2\zeta\omega\Delta_{i-1}) + [(\sigma^2 - 2\zeta\omega)\Delta_i - 2\zeta\omega\Delta_{i-1}(1+2\zeta\omega\Delta_i)]_+}{(1+2\zeta\omega\Delta_{i-1})(1+2\zeta\omega\Delta_i + \omega^2\Delta_i^2)}$$

and

$$\underline{k}_i = \frac{-\omega^2\Delta_i^2(1+2\zeta\omega\Delta_{i-1}) - [(\sigma^2 - 2\zeta\omega)\Delta_i - 2\zeta\omega\Delta_{i-1}(1+2\zeta\omega\Delta_i)]_-}{(1+2\zeta\omega\Delta_{i-1})(1+2\zeta\omega\Delta_i + \omega^2\Delta_i^2)}.$$

Furthermore, if $(\Delta_n)_{n\in\mathbb{N}}$ is a deterministic sequence then the V-exponents can be estimated by

$$\liminf_{n\to+\infty} \frac{1}{t_n} \sum_{i=1}^{n-1} \frac{\underline{k}_i}{1+\underline{k}_i} \;\leq\; \underline{\lambda}_V \;\leq\; \overline{\lambda}_V \;\leq\; \limsup_{n\to+\infty} \frac{1}{t_n} \sum_{i=1}^{n-1} \overline{k}_i\,.$$

Additionally, in the following assume that

$$\exists \Delta_a, \Delta_b \in \mathbb{R}_+ \;:\; \forall n \in \mathbb{N} \quad 0 < \Delta_b \leq \Delta_n \leq \Delta_a < +\infty\,. \tag{4.5}$$

If

$$(\sigma^2 - 2\zeta\omega)\Delta_n - 2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n) \;\leq\; 0 \tag{4.6}$$

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for all $n \in I\!\!N$ *then*

$$\overline{\lambda}_V \leq -\frac{\omega^2 \Delta_b}{1+2\zeta\omega\Delta_a+\omega^2\Delta_a^2}$$

and, if additionally $\omega^2 > 0$ *and* $\zeta\omega > 0$, *then* $\lim_{n\to+\infty} V(n+1, X_{n+1}, Y_{n+1}) = 0$ *(a.s.). If*

$$(\sigma^2 - 2\zeta\omega)\Delta_n - 2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n) \;\geq\; 0 \tag{4.7}$$

for all $n \in I\!\!N$ *then*

$$\underline{\lambda}_V \geq -\frac{\omega^2\Delta_a}{1+2\zeta\omega\Delta_b}\,.$$

Proof. First, we equivalently rearrange the scheme (4.4) to an explicit one. Thus, one arrives at

$$\begin{aligned} X_{n+1} &= \frac{1+2\zeta\omega\Delta_n}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}X_n + \frac{(1+\sigma\Delta W_n)\Delta_n}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n \\ Y_{n+1} &= -\frac{\omega^2\Delta_n}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}X_n + \frac{(1+\sigma\Delta W_n)}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n\,. \end{aligned} \tag{4.8}$$

Recall that $v_{n+1} = \mathbb{E}\,[\omega^2 X_{n+1}^2 + (1+2\zeta\omega\Delta_n)Y_{n+1}^2]$. After some elementary calculations we get

$$v_{n+1} = \omega^2\mathbb{E}\left[\frac{1+2\zeta\omega\Delta_n}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}X_n^2\right] + \mathbb{E}\left[\frac{1+\sigma^2\Delta_n}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n^2\right],$$

hence

$$\begin{aligned} &-\frac{\omega^2\Delta_a\Delta_n}{1+2\zeta\omega\Delta_b+\omega^2\Delta_b^2}v_n - \mathbb{E}\left[\frac{(\sigma^2-2\zeta\omega)\Delta_n - 2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n)}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n^2\right]_- \\ &\leq\; \Delta\mathbb{E}\,V_n = -\mathbb{E}\left[\frac{\omega^2\Delta_n^2}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}\omega^2X_n^2\right] \\ &\qquad + \mathbb{E}\left[\frac{(\sigma^2-2\zeta\omega)\Delta_n - \omega^2\Delta_n^2 - 2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2)}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n^2\right] \\ &= -\frac{\omega^2\Delta_n^2}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}v_n + \mathbb{E}\left[\frac{(\sigma^2-2\zeta\omega)\Delta_n - 2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n)}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n^2\right] \\ &\leq -\frac{\omega^2\Delta_b\Delta_n}{1+2\zeta\omega\Delta_a+\omega^2\Delta_a^2}v_n + \mathbb{E}\left[\frac{(\sigma^2-2\zeta\omega)\Delta_n - 2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n)}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n^2\right]_+ \end{aligned}$$

Now, we may choose $\overline{k}_n, \underline{k}_n$ as indicated above, and apply Theorem 4.2 with $\overline{k}_n, \underline{k}_n$. Thus, the proof is complete. ☐

Remark. Most of the clever variable step size algorithms have implemented conditions on the step size selection like that of (4.5). We can conclude from our assertion that the fully drift-implicit Euler method (4.4), applied to stochastic oscillator (4.3) produces damped approximations, particularly in the critical case, (the energy-conservative case) when $\sigma^2 = 4\zeta\omega$ under the condition (4.5). However, the observed effect of numerical stabilization also explains that the requirement (4.5) is meaningful in variable step size algorithms in order to achieve asymptotically stable approximations. Asymptotically considered, when maximum step size Δ_a tends to zero, the V-exponents of the continuous time dynamics are correctly replicated by the discretization method (4.4), as we would naturally expect from a well-behaving and converging numerical method.

5. Fully Nonlinear Weak V-Stability. So far we discussed examples where the stability-controlling function V (or Lyaponuv-type function) is governed by a linear difference inequality (difference inclusion). Now, it is the time for fully nonlinear relations.

5.1. A general theorem on weak V-stability. Define $v_n := \mathbb{E}V(n)$ and $\Delta v_n = v_{n+1} - v_n$ for all $n \in \mathbb{N}$ and V as given below.

THEOREM 5.1. *Let $V : \mathbb{N} \longrightarrow \mathbb{R}^1_+$ be a sequence of random variables $V(n) : (\Omega, \mathcal{F}_n, \mathbb{P}) \to (\mathbb{R}^1_+, \mathcal{B}(\mathbb{R}^1_+))$ satisfying*

$$v_{n+1} = \mathbb{E}[V(n+1)] \leq \mathbb{E}[V(n)] + c(n)g(\mathbb{E}[V(n)]) \tag{5.1}$$

with nonrandom $c(n) \in \mathbb{R}^1$ for all $n \in \mathbb{N}$, where $g : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ is a Borel-measurable, nondecreasing function satisfying the integrability-condition

$$-\infty < G(u) := \int_{v_0}^{u} \frac{dz}{g(z)} < +\infty \tag{5.2}$$

for all $u > 0$ and $(v_n)_{n\in\mathbb{N}}$ is nondecreasing. Assume that $\mathbb{E}V(0) < +\infty$. Then, for all $n \in \mathbb{N}$, we have

$$v_{n+1} = \sup_{k=0,1,\ldots,n} \mathbb{E}V(k) \leq G^{-1}\Big(G(\mathbb{E}[V(0)]) + \sum_{k=0}^{n} c(k)\Big) \tag{5.3}$$

where G^{-1} is the inverse function belonging to G.

Proof. Suppose that $v_0 > 0$, at first. Then, inequality (5.1) implies that

$$\frac{\Delta v_n}{g(v_n)} \leq c(n).$$

Therefore, by simple integration under (5.2), we obtain

$$G(v_{k+1}) - G(v_k) = \int_{v_k}^{v_{k+1}} \frac{dv}{g(v)} \leq \int_{v_k}^{v_{k+1}} \frac{dv}{g(v_k)} = \frac{\Delta v_k}{g(v_k)} \leq c(k)$$

for all $k \in \mathbb{N}$. Summing up these inequalities leads to

$$G(v_{n+1}) - G(v_0) = \sum_{k=0}^{n} G(v_{k+1}) - G(v_k) \leq \sum_{k=0}^{n} c(k)$$

for all $n \in \mathbb{N}$, which is equivalent to

$$G(v_{n+1}) \leq G(v_0) + \sum_{k=0}^{n} c(k).$$

Note that the inverse G^{-1} of G exists and both G and G^{-1} are increasing since G satisfying (5.2) is increasing. Hence, we arrive at

$$v_{n+1} \leq G^{-1}\Big(G(v_0) + \sum_{k=0}^{n} c(k)\Big).$$

Note also that, $c(k) \geq 0$ due to the assumption v is nondecreasing. If $v_0 = 0$ then one can repeat the above calculations, for all $v_0 = \varepsilon > 0$. It just remains to take the limit as ε tends to zero in the obtained estimates. Thus, the proof of (5.3) is complete. $\square$

Remark. Theorem 5.1 can be understood as a discrete version of the continuous time Lemma of Bihari [11].

5.2. The discrete Girsanov-example (Pure diffusions). Girsanov [13] discussed the simplest examples of one dimensional Itô SDEs

$$dX_t = \sigma([X_t]_+)^{\alpha}\, dW_t \tag{5.4}$$

with $X_0 \geq 0$ (a.s.) in view of its solutions and strong uniqueness. Without loss of generality, we may suppose that $\sigma \geq 0$. If $\alpha \in [0,1]$ and $\mathbb{E} X_0^2 < +\infty$ we obtain continuous time solutions which are martingales with respect to the σ-algebra generated by the driving Wiener process $W = (W_t)_{t \geq 0}$. If $\alpha = 0$ or $\alpha \in [1/2, 1]$ these (nonanticipating) martingale-solutions are unique with probability one (by the help of Osgood-Yamada-Watanabe results, cf. Karatzas and Shreve [24]). Due to the pathwise continuity, the nonnegative cone $\mathbb{R}_+ = \{x \in \mathbb{R}^1 : x \geq 0\}$ is left invariant (a.s.). A nontrivial question is whether related numerical approximations are stable, converge to the underlying analytic solution and have the same invariance property. We are able to answer these problems. Here, we are only interested in stability and invariance (for convergence, see a forthcoming paper of the author). For this purpose, consider the balanced implicit methods

$$Y_{n+1} = Y_n + \sigma([Y_n]_+)^{\alpha} \Delta W_n + \sigma([Y_n]_+)^{\alpha-1} |\Delta W_n| (Y_n - Y_{n+1}). \tag{5.5}$$

THEOREM 5.2. *The BIMs (5.5) applied to Girsanov's SDE (5.4) with $0 < \alpha < 1$ leave the nonnegative cone $\mathbb{R}_+$ invariant (a.s.) and provide polynomially stable numerical sequences. More precisely, if $\sigma > 0$, $0 < \mathbb{E} Y_0^2 < +\infty$ and Y_0 is independent of the σ-algebra $\mathcal{F} = \sigma\{W_t : t \geq 0\}$ then their second moments are strictly increasing as n increases and they are governed by*

$$\mathbb{E}[Y_n^2] \leq \Big((\mathbb{E}[Y_0^2])^{1-\alpha} + (1-\alpha)\sigma^2 (t_{n+1} - t_0) \Big)^{1/(1-\alpha)} \tag{5.6}$$

Proof. Suppose that $0 < \alpha < 1$. At first we rewrite (5.5) as the explicit scheme

$$Y_{n+1} = Y_n \frac{1 + \sigma([Y_n]_+)^{\alpha-1} \Delta W_n + \sigma([Y_n]_+)^{\alpha-1} |\Delta W_n|}{1 + \sigma([Y_n]_+)^{\alpha-1} |\Delta W_n|} \tag{5.7}$$

which immediately gives the a.s. invariance with respect to the nonnegative cone $\mathbb{R}_+$, provided that $Y_0 \geq 0$ (a.s.). Therefore, we may drop the taking of positive part by $[.]_+$ in the above form. Now, rewrite (5.7) as

$$Y_{n+1} = Y_n + \frac{\sigma(Y_n)^{\alpha} \Delta W_n}{1 + \sigma(Y_n)^{\alpha-1} |\Delta W_n|}. \tag{5.8}$$

Taking the square and expectation yields

$$v_{n+1} := \mathbb{E}[Y_{n+1}^2] = \mathbb{E}[Y_n]^2 + \mathbb{E}\Big[\frac{\sigma(Y_n)^{\alpha} \Delta W_n}{1 + \sigma(Y_n)^{\alpha-1} |\Delta W_n|}\Big]^2. \tag{5.9}$$

Thus, due to the positivity of all summands at the right hand side, we may conclude that the second moments $(v_n)_{n \in \mathbb{N}}$ are nondecreasing and, in fact if $\sigma > 0$, v_n is strictly increasing. It remains to apply Theorem 5.1. For this purpose, estimate (5.9) by Jensen's inequality for concave functions in order to obtain

$$v_{n+1} \leq v_n + \sigma^2 \mathbb{E}[(Y_n)^{2\alpha} \Delta W_n] \; = \; v_n + \sigma^2 \mathbb{E}[(Y_n)^{2\alpha} \mathbb{E}[\Delta W_n | \mathcal{F}_n]] \tag{5.10}$$

$$= v_n + \sigma^2 \mathbb{E}[(Y_n)^{2\alpha} \mathbb{E}[\Delta W_n]] \; \leq \; v_n + \sigma^2 (v_n)^{\alpha} \Delta_n. \tag{5.11}$$

Therefore, we may take $V(x) = x^2$, $g(z) = z^\alpha$, $c(n) = \sigma^2 \Delta_n$ and apply Theorem 5.1 with the conclusion (5.3) in order to get to (5.6). Note also that

$$G(u) = \int_{v_0}^{u} \frac{dz}{g(z)} = \int_{v_0}^{u} \frac{dz}{z^\alpha} = \frac{z^{1-\alpha}}{1-\alpha}\Big|_{z=v_0}^{z=u} = \frac{u^{1-\alpha} - (v_0)^{1-\alpha}}{1-\alpha},$$
$$G^{-1}(z) = \left(z(1-\alpha) + v_0^{1-\alpha}\right)^{1/(1-\alpha)}.$$

If $v_0 = 0$, then one can repeat the above calculations for $v_0 = \varepsilon > 0$. It just remains to take the limit as ε tends to zero in the obtained estimates. Thus, the proof is complete. ☐

Remark. One could also compare the moment evolutions of the explicit Euler method $Y^E = (Y_n^E)_{n\in\mathbb{N}}$ with that of BIMs $Y^B = (Y_n^B)_{n\in\mathbb{N}}$ governed by (2.8). Then, it is fairly easy to recognize that $\mathbb{E}\,(Y_n^B)^{2\kappa} \le \mathbb{E}\,(Y_n^E)^{2\kappa}$ for all integers $\kappa \in \mathbb{N}$, provided that $\mathbb{E}\,(Y_0^B)^{2\kappa} \le \mathbb{E}\,(Y_0^E)^{2\kappa}$. It is also interesting to note that the explicit Euler methods cannot preserve the a.s. invariance property with respect to the nonnegative cone $\mathbb{R}_+$. In fact, they exit that cone with positive probability, independently of the choice of any nonrandom step sizes Δ_n. Summarizing, the underlying explicit solution to (5.4) has very similar analytic properties as BIMs (5.5).

5.3. Numerical experiments for Girsanov's SDE (5.4). For illustration we conducted numerical experiments in computing trajectories and second moments of solutions to Itô SDE

$$dX_t = 2\sqrt{[X_t]_+}\, dW_t \tag{5.12}$$

with $X_0 = 1.0$. Hence, with $\sigma = 2.0$ and $\alpha = 0.5$, we consider an example for a Girsanov SDE (5.4) and its discretization. Its discretization is done via the balanced implicit method (5.5) along equidistant grids on $[0,1]$ (for the sake of simplicity) with uniform mesh size $h = 2^{-10}$. The Gaussian increments ΔW_n are generated by the well-known Polar-Marsaglia method, where we use a random initialization of the random seed coupled to the internal time clock in order to guarantee randomness of our results based on the built-in pseudo-random number generator for uniform distributed numbers in C. An appropriate C-code (run on a LINUX-operating machine) provides us the simulation results below. All computing was done in double precision and the data are plotted with GNUPLOT.

In figure 5.1, we recognize how the balanced implicit methods can follow the paths and restore nonnegativity pathwisely as the exact solution does. If one repeats simulations then one can confirm that the four trajectories, depicted in this figure, are rather typical for the Girsanov SDE with $\alpha = 0.5$. Some paths might converge to zero, some just fluctuate and some seem to explode. Recall that, the pathwise uniqueness of solutions to this SDE is still guaranteed by an application of the Watanabe-Yamada results, see Karatzas and Shreve [24].

Figure 5.2, shows the graphs of the 2nd moments and its estimate (5.6) from theorem 5.2. Clearly we can see the over estimation from the polynomial bound (5.6) having quadratic growth in t here. Again, we used the Polar-Marsaglia method to generate the Gaussian increments ΔW_n. To check the accuracy of statistical moment estimation, we separated the sampling over the trajectories obtained from the first component and the second one of Polar-Marsaglia pairs (G_1, G_2) (recall that it always generates i.i.d. pairs of Gaussian pseudo-random numbers). The small deviation noticed between both estimates for the 2nd moment results from the use of finite sample sizes (in fact our choice is $N = 10^5$ in the figure 5.2). However, if one repeats for larger sample sizes then this deviation caused by statistical errors will decrease as sample size N increases (with order $O(\sqrt{N})$, cf. standard limit theorems and law of large numbers). We have just used this information on statistical estimation to control the choice of reasonable sample sizes and to evaluate the goodness of our experiments.

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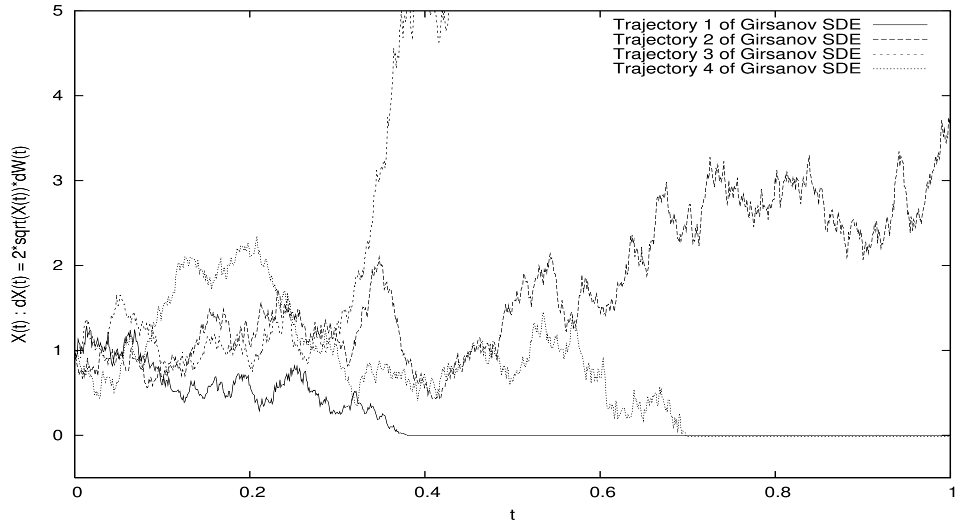

FIG. 5.1. *Four trajectories of Girsanov SDE (5.12).*

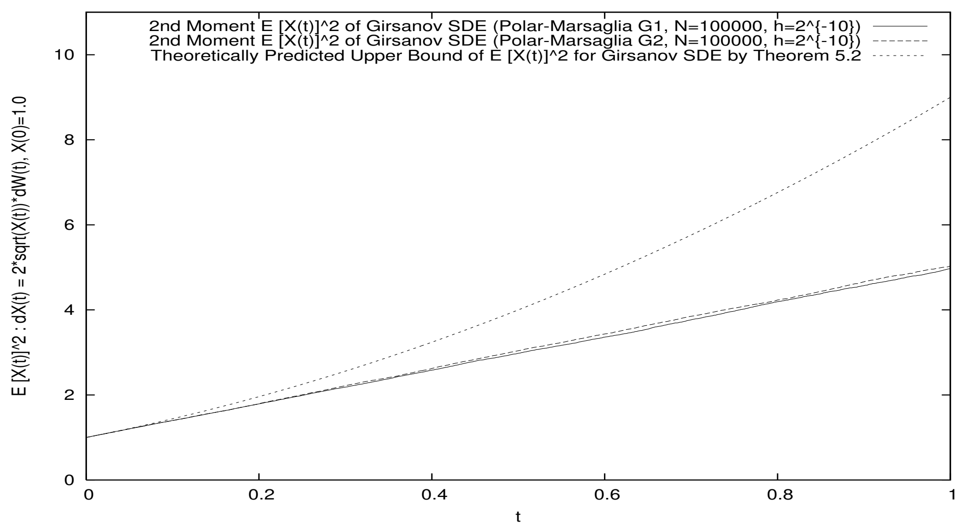

FIG. 5.2. *Estimation of 2nd moments of the discretization of Girsanov SDE (5.12) compared to estimate (5.6).*

A Few Final Remarks. Despite the diversity of stability concepts treated in this paper, our investigations can not be complete. This is due to the large complexity of stochastic stability issues and the absence of the solution of the problem of the relevance of test equations for a significantly larger class of nonlinear SDEs (A thorough treatment of a stochastic version of Dahlquist's stability theory could not be found in the literature so far). Finally, it is also worth noting explicitly that stability investigations are fairly independent of the type of convergence of examined numerical method (the two types of weak and strong convergence have been established as the major ones in stochastic-numerical analysis). However, stability estimates are needed in any refined convergence analysis. So, we hope that we have shown both general theorems concerning several stability issues and its use by fairly simple examples of numerical methods applied to ordinary SDEs - a fact which documents the use of this paper.

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