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ZEROS OF SECTIONS OF THE BINOMIAL EXPANSION*

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Dedicated to Richard S. Varga, on the occasion of his 80th birthday.

Abstract. We examine the asymptotic behavior of the zeros of sections of the binomial expansion, that is, we consider the distribution of zeros of $B_{r,n}(z) = \sum_{k=0}^{r} \binom{n}{k} z^k$, where $1 \le r \le n$.

Key words. Binomial expansion, partial sums, zeros.

AMS subject classifications. 30C15.

1. Preliminaries. A problem of great interest in the classical complex function theory is the following: given a function $f(z) = \sum_{k=0}^{\infty} a_k z^k$, analytic at $z = 0$, determine the asymptotic distribution of the zeros of the *partial sums* $s_n(z) = \sum_{k=0}^{n} a_k z^k$.

Some contributors to this area include Jentzsch [6], who explored the problem for a finite radius of convergence; Szegő [13], who explored the exponential function e^z; Rosenbloom [12], who discussed the angular distribution of zeros using potential theory, and applied his work to the sub-class of confluent hypergeometric functions; Erdős and Turán [4], who used minimization techniques to discuss angular distributions of zeros; Newman and Rivlin [7, 8], who related the work of Szegő to the Central Limit Theorem; Edrei, Saff and Varga [3], who gave a thorough analysis for the family of Mittag-Leffler functions; Carpenter, Varga and Waldvogel [2], who refined the work of Szegő; Norfolk [9, 10], who refined the work of Rosenbloom on the confluent hypergeometric functions and a related set of integral transforms.

In this paper, we will analyze the behavior of the zeros of sections of the binomial expansion, that is

$$B_{r,n}(z) = \sum_{k=0}^{r} \binom{n}{k} z^k, \quad 1 \le r \le n.$$

This investigation not only fits into the general theme of the works cited, but also arises from matroid theory. Specifically (cf. [14]), the *univariate reliability polynomial* for the uniform matroid $U_{r,n}$ is given by

$$\mathrm{Rel}_{r,n}(q) = (1-q)^n B_{r,n}\left(\frac{q}{1-q}\right) = \sum_{k=0}^{r} \binom{n}{k} q^k (1-q)^{n-k},$$

which can be written as $\mathrm{Rel}_{r,n}(q) = (1-q)^{n-r} H_{r,n}(q)$, where

$$H_{r,n}(q) = \sum_{k=0}^{r} \binom{n}{k} q^k (1-q)^{r-k} = (1-q)^r B_{r,n}\left(\frac{q}{1-q}\right). \tag{1.1}$$

Some special cases are easy to analyze, and may thus be dispensed with. In particular,

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1. $B_{1,n}(z) = 1 + nz$, which has its only zero at $z = -\frac{1}{n}$.
2. $B_{n,n}(z) = (1 + z)^n$, which clearly has a zero of multiplicity n at $z = -1$.
3. $B_{n-1,n}(z) = (1+z)^n - z^n$. Noting that this polynomial cannot have positive zeros, we obtain the zeros $z = \frac{\omega^k}{1-\omega^k}$, for $1 \le k \le n-1$, where $\omega = \exp\left(\frac{2\pi i}{n}\right)$ is the principal n-th root of unity, all of which lie on the vertical line $\operatorname{Re} z = -\frac{1}{2}$.

In what follows, we will therefore focus on the cases $1 \le r < n - 1$, and give two collections of results. The first are concerned with bounding regions for the zeros of $B_{r,n}(z)$, the rest with convergence results.

We note that this problem was investigated independently by Ostrovskii [11], who obtained many of the results that we present here. The methods used there involved using a bilinear transformation to convert the problem to an integral formulation. This choice of formulation makes the proofs more involved and requires some additional constraints. By contrast, we claim that our methods descend directly from the structure of the problem, and yield additional results, in terms of additional bounds on the zeros, and limiting cases. The paper [11] also gives a result on the spacing of the zeros on the limit curve, using classical potential-theoretic methods. We do not duplicate that result here, but give formulations in terms of specific points on the curve.

Our methods generate a set of constants and two related limit curves for $0 < \alpha < 1$, defined by

$$\frac{1}{2} \le K_\alpha = \alpha^\alpha (1-\alpha)^{1-\alpha} < 1, \tag{1.2}$$

$$C_\alpha = \left\{ z \;:\; \frac{|z|^\alpha}{|1+z|} = K_\alpha,\ |z| \le \frac{\alpha}{1-\alpha} \right\}, \tag{1.3}$$

and

$$C'_\alpha = \left\{ z \;:\; \frac{|z|^\alpha}{|1+z|} = K_\alpha,\ \frac{\alpha}{1-\alpha} \le |z| \right\}. \tag{1.4}$$

The properties of these curves are outlined in Lemma 3.1. Section 3 also presents bounds which are used to simplify the proofs of some of the results presented here.

2. Main Results. As discussed above, we begin with a theorem on bounds of the zeros of $B_{r,n}(z)$, and follow with results on convergence of those zeros.

THEOREM 2.1. *Let r, n be positive integers, with $1 \le r < n - 1$, and let z^* be any zero of $B_{r,n}(z) = \sum_{k=0}^{r} \binom{n}{k} z^k$. Then, z^* lies in a region defined by the intersection of two circles, a plane closed curve, and the half plane on the right of a vertical line. Specifically,*

$$|z^*| \le \frac{r}{n+1-r},$$
$$\left| z^* - \frac{\gamma^2}{1-\gamma^2} \right| \le \frac{\gamma}{(1-\gamma^2)}, \quad \textit{where } \gamma = \frac{r}{n-1},$$
$$\operatorname{Re} z^* > -\frac{1}{2},$$

and z^* lies exterior to the curve $C_{r/n}$, defined in (1.2)–(1.3).

Proof. We begin by considering the ratio of coefficients

$$\frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{n-k+1}{k},$$

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which is decreasing in k.

Hence, writing $B_{r,n}\left(\frac{r}{n-r+1}z\right) = \sum_{k=0}^{r} a_k z^k$, we have that

$$\frac{a_k}{a_{k-1}} = \frac{n-k+1}{k} \cdot \frac{r}{n-r+1} \geq 1.$$

That is, $\{a_k\}_{k=0}^{r}$ is non-decreasing, so by the Eneström-Kakeya Theorem ([5], p. 462), the zeros of this polynomial satisfy $|z| \leq 1$. Hence, the zeros of $B_{r,n}(z)$ satisfy $|z| \leq \frac{r}{n-r+1}$.

For the second bounding circle, we refer to Wagner [14], where it is shown, again using the Eneström-Kakeya Theorem, that the zeros of $H_{r,n}(q)$, defined in (1.1), lie in the annulus

$$\frac{1}{n-r} \leq |q| \leq \frac{r}{n-1}.$$

Since $z = -1$ is clearly not a zero of $B_{r,n}(z)$ for $r < n$, we may make the substitution $z = \frac{q}{1-q}$ (or equivalently $q = \frac{z}{1+z}$) in (1.1), which shows immediately that

$$H_{r,n}(q) = (1+z)^{-r} B_{r,n}(z),$$

from which one obtains

$$\left|\frac{z}{1+z}\right| \leq \frac{r}{n-1} =: \gamma. \tag{2.1}$$

Writing this last inequality in terms of the real and imaginary parts of z yields the claimed result. Noting that (2.1) implies that $\left|\frac{z}{1+z}\right| < 1$, yields the half-plane $\mathrm{Re}\, z > -\frac{1}{2}$, as claimed.

For the final bound, we mimic the analysis of Buckholtz [1] on the partial sums of e^z, and write

$$(1+z)^{-n} B_{r,n}(z) = 1 - \frac{z^r}{(1+z)^n} R_{r,n}(z), \tag{2.2}$$

where

$$R_{r,n}(z) = \sum_{k=r+1}^{n} \binom{n}{k} z^{k-r} \;= z^{n-r} B_{n-r-1,n}\left(\frac{1}{z}\right). \tag{2.3}$$

For clarity, we set $\beta = r/n$. Inside and on the curve C_β (1.3), we have $|z| < \frac{\beta}{1-\beta}$ and $\left|\frac{z^r}{(1+z)^n}\right| \leq K_\beta^n$, where K_β is defined in (1.2). This, with the upper bound of Lemma 3.3 yields

$$\left|(1+z)^{-n} B_{r,n}(z)\right| \geq 1 - \left|\frac{z^r}{(1+z)^n}\right| \cdot |R_{r,n}(z)| \; > 1 - K_\beta^n \cdot K_\beta^{-n} = 0, \tag{2.4}$$

which is the desired result. $\square$

Note that the second bounding circle occurring in this result, namely

$$\left|z - \frac{\alpha^2}{1-\alpha^2}\right| = \frac{\alpha}{1-\alpha^2},$$

intersects the negative real axis at $z = -\frac{\alpha}{1+\alpha}$. This circle is contained in the first one, namely $|z| = \frac{\alpha}{1-\alpha}$, and both meet at the common point $z = \frac{\alpha}{1-\alpha}$.

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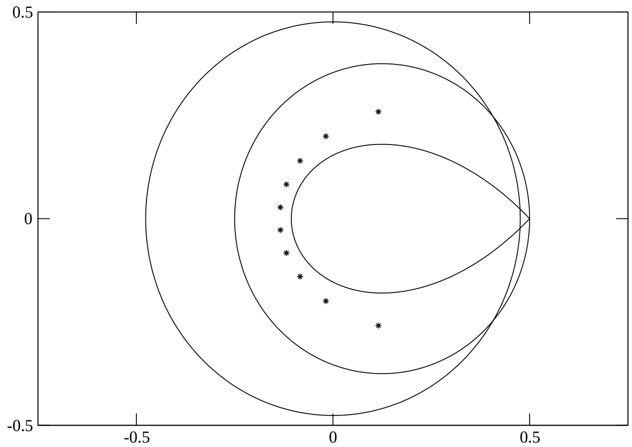

FIGURE 2.1. *The bounding curves and zeros for* $r = 10$, $n = 30$.

The limiting case $|z| = \frac{\alpha}{1-\alpha}$, corresponding to the first bounding circle, and the bounding half-plane $\operatorname{Re} z > -\frac{1}{2}$ both appear in [11], with proofs that require a significantly more detailed derivation. The bounding curves and associated zeros for the case $r = 10$ and $n = 30$ are illustrated in Figure 2.1.

We now use these results, and the bounds from the proof, to discuss some convergence results.

THEOREM 2.2. *Suppose that* $1 \le r_j < n_j - 1$ *for all* j, *that* $\lim_{j\to\infty} n_j = \infty$, *and that*

$$\lim_{j\to\infty} \frac{r_j}{n_j} = \alpha, \quad 0 < \alpha < 1.$$

Then

1. *the zeros of* $\{B_{r_j,n_j}(z)\}$ *converge uniformly to points of the curve* C_α, *i.e.*,

$$\sup_{z: B_{r_j,n_j}(z)=0} d(z, C_\alpha) \longrightarrow 0,$$

where $d(z, C_\alpha) = \inf_{\zeta\in C_\alpha} |z - \zeta|$ *is the distance from* z *to* C_α;

2. *each point of* C_α *is a limit point of zeros of* $\{B_{r_j,n_j}(z)\}_{j=1}^{\infty}$.

Proof. Set $\beta_j = r_j/n_j$, so that $\lim_{j\to\infty} \beta_j = \alpha$. Using (2.2), the zeros of $B_{r_j,n_j}(z)$ then satisfy

$$\frac{z^{r_j}}{(1+z)^{n_j}} \cdot R_{r_j,n_j}(z) = 1. \tag{2.5}$$

Using Theorem 2.1, Lemma 3.1, and Lemma 3.3, these zeros lie outside the curve C_{β_j}, and thus satisfy $\nu\beta_j < X_{\beta_j} \le |z| \le \frac{\beta_j}{1-\beta_j}$, where $-X_{\beta_j}$ is the intersection of the curve C_{β_j} with the negative real axis, and ν is the unique positive solution to $xe^{1+x} = 1$.

Hence,

$$\frac{\nu r_j}{n_j(r_j+1)} \le \frac{K_{\beta_j}^{n_j} \left|R_{r_j,n_j}(z)\right|}{\sum_{k=r_j+1}^{n_j} \binom{n_j}{k} {\beta_j}^k (1-\beta_j)^{n_j-k}} \le 1, \tag{2.6}$$

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for this region. Note that the sum in the denominator above converges to $1/2$ by the Central Limit Theorem.

Consequently, $\lim_{j\to\infty}\left|R_{r_j,n_j}^{1/n_j}(z)\right| = K_\alpha^{-1}$, uniformly on the set in question. Taking moduli and n_j-th roots in (2.5), we observe that the zeros of $B_{r_j,n_j}(z)$ must satisfy

$$\frac{|z|^{\beta_j}}{|1+z|}\left|R_{r_j,n_j}(z)\right|^{1/n_j} = 1. \tag{2.7}$$

Since $\beta_j \to \alpha$, this establishes that every limit point of a sequence of zeros of $B_{r_j,n_j}(z)$ lies on C_α. Since, by Theorem 2.1, the zeros lie in a compact set, it follows that the zeros converge uniformly to points of C_α.

For the second claim, fix any $\zeta \in C_\alpha$ with $\zeta \neq z_\alpha = \alpha/(1-\alpha)$. Then $|\zeta| < z_\alpha$, so we may take a small neighborhood D of ζ such that $0 < |z| < z_\alpha$ for $z \in \overline{D}$. Consequently, for j sufficiently large, $|z| < z_{\beta_j}$ for all $z \in \overline{D}$, and it follows from Lemma 3.3 and the Central Limit Theorem, that

$$\left|R_{r_j,n_j}(z)\right|^{1/n_j} \longrightarrow K_\alpha^{-1},$$

uniformly on $\overline{D}$.

In particular, for large j, $R_{r_j,n_j}(z) \neq 0$ on $\overline{D}$, so we may fix an analytic branch of $R_{r_j,n_j}^{1/n_j}(z)$ in D. Letting $\theta_j = \arg(R_{r_j,n_j}^{1/n_j}(z))$ (with arguments in the range $(0, 2\pi)$), we then have

$$e^{-i\theta_j} R_{r_j,n_j}(z) \longrightarrow K_\alpha^{-1},$$

uniformly on compact subsets of D.

By shrinking D, we may assume that the latter limit holds uniformly on D. Furthermore, we may assume that $0 < \arg(z) < 2\pi$ for $z \in D$, and thus the powers z^{β_j} and z^α are well-defined in D. Hence,

$$\frac{z^{\beta_j}}{1+z} R_{r_j,n_j}^{1/n_j}(z) - \frac{z^\alpha}{1+z} K_\alpha^{-1} e^{i\theta_j} \longrightarrow 0, \tag{2.8}$$

uniformly on D.

Since the mapping $w = \frac{z^\alpha}{1+z}K_\alpha^{-1}$ maps C_α onto an arc of the unit circle, it maps $D \cap C_\alpha$ onto a subarc. Thus, for j sufficiently large, there exists an integer p_j such that $\frac{z^\alpha}{1+z}K_\alpha^{-1}e^{i\theta_j} = e^{2\pi i p_j/n_j}$ for some $z = \zeta_j \in D \cap C_\alpha$. We may further assume that $\zeta_j \to \zeta$. It now follows from Hurwitz' Theorem and (2.8) that, for j sufficiently large,

$$\frac{z^{\beta_j}}{1+z} R_{r_j,n_j}^{1/n_j}(z) - e^{2\pi i p_j/n_j}$$

has a zero $z_j \in D$. Each such zero z_j satisfies (2.5), and so by (2.2) is a zero of $B_{r_j,n_j}(z)$. This proves that every point on C_α is a limit point of zeros of $\left\{B_{r_j,n_j}(z)\right\}$. ∎

We note that, thanks to (2.3), the non-trivial zeros of $R_{r_j,n_j}(z)$ converge uniformly to all points which lie on the curve C'_α, as defined in (1.4).

This result also appears in [11], using more elaborate asymptotics. The analysis presented requires a deletion of a neighborhood of the singular point $z_\alpha = \frac{\alpha}{1-\alpha}$. The results of Lemma 3.3 show that this is not necessary with our methods.

The remaining results presented here do not appear in the literature.

The asymptotic expansions in the proof of Theorem 2.2 immediately give the following result on the rate of convergence. We note that, as shown in [2] in the case of the exponential function, this rate is the best possible.

THEOREM 2.3. *Fix $0<\delta<1$. Then, there exists a constant c, depending only on δ, such that, if r,n are large and $0<\delta<\frac{r}{n}<1-\delta$, for any zero z^* of $B_{r,n}(z)$ we have*

$$\min_{\zeta\in C_{r/n}} |z^*-\zeta| \le \frac{c}{|z^*-\frac{r}{n-r}|}\cdot\frac{\ln n}{n}.$$

Additionally, the proximity to the singular point $z_{r/n}=\frac{r}{n-r}$ is of order $O\left(\frac{1}{\sqrt{n}}\right)$.

Proof. Set $\beta=r/n$. From (2.6), we obtain the approximation

$$\left|R_{r,n}^{1/n}(z)\right|\cdot K_\beta = 1+G_{r,n}(z)\cdot\frac{\ln n}{n}, \tag{2.9}$$

where $G_{r,n}(z)$ is uniformly bounded in a region containing the zeros.

Let z^* be a zero of $B_{r,n}(z)$, and let ζ be the point on C_β closest to z^*. Note that $|\zeta-z^*|=o(1)$, as a consequence of Theorem 2.2, applied to sequences for which β converges. Note that the curve C_β is asymptotically a pair of straight lines at angle $\pi/4$ to the real axis, close to the point $z_\beta=\beta/(1-\beta)$. Hence, if z^* is close to z_β, by Theorem 2.1, it must lie in the wedges between these lines and the vertical line $\operatorname{Re} z=z_\beta$, from which $|z^*-z_\beta|=O(|\zeta-z_\beta|)$.

Note that z^* satisfies (2.7), without the subscript j, and thus, by (2.9), we have

$$\frac{|z^*|}{|1+z^*|}\cdot K_\beta^{-1} = \left(1+G_{r,n}(z)\cdot\frac{\ln n}{n}\right)^{-1}.$$

Expanding $F(z)=\ln(K_\beta^{-1}|z|^\beta/|1+z|)=\operatorname{Re}\ln(K_\beta^{-1}z^\beta/(1+z))$ as a Taylor series centered at ζ (noting that $F(\zeta)=0$), we find that

$$|z^*-\zeta| = O\left(\left|\frac{\zeta(1+\zeta)}{\beta-(1-\beta)\zeta}\cdot G_{r,n}(z)\cdot\frac{\ln n}{n}\right|\right) = O\left(\frac{1}{|z_\beta-\zeta|}\cdot\frac{\ln n}{n}\right).$$

This not only gives the desired result, but shows that, as expected, the rate of convergence is worst for those points closest to the singular point $z_\beta=\frac{\beta}{1-\beta}$.

To discuss the convergence at the singular point, we take an approach similar to that used for the exponential function in [7, 8] and for the Mittag-Leffler functions in [3]. For convenience, we set $\mu=n\beta=r$, and $\sigma^2=n\beta(1-\beta)$. Then,

$$f_{r,n}(w)=(1-\beta)^nB_{r,n}\left(\frac{\beta e^{w/\sigma}}{1-\beta}\right)=\sum_{k=0}^{r}\binom{n}{k}\beta^k(1-\beta)^{n-k}e^{kw/\sigma},$$

which is a truncated moment generating function for a binomial distribution with mean μ and variance σ. Using the Central Limit Theorem,

$$f_{r,n}(w)\approx\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)+\frac{tw}{\sigma}}\,dt.$$

Making the substitution $s=\frac{t-\mu-\sigma w}{\sqrt{2}\sigma}$ yields

$$e^{-\mu w/\sigma-w^2/2}f_{r,n}(w)\approx\frac{1}{\sqrt{\pi}}\int_{-\infty}^{-w/\sqrt{2}} e^{-s^2}\,ds=\frac{1}{2}\operatorname{erfc}\left(\frac{w}{\sqrt{2}}\right),$$

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the complementary error function. Thus, given the zero χ of $\operatorname{erfc}(z)$ which is closest to the origin, there must exist a zero z^* of $B_{r,n}(z)$ for which

$$z^* \approx \frac{\beta e^{\sqrt{2}\chi/\sigma}}{1-\beta} \approx \frac{\beta}{1-\beta} + \sqrt{\frac{2\beta}{(1-\beta)^3}} \cdot \frac{\chi}{\sqrt{n}},$$

the desired result. $\square$

Figures 2.1 and 2.2 show the zeros, bounding curve, and bounding circles, for the cases $r = 10$, $n = 30$ and $r = 30$, $n = 90$, respectively. Since the ratio r/n is the same in both cases, they serve to illustrate both the rate of convergence of the zeros to the limit curve, and the rate of convergence of the bounding circles. Figure 2.3 shows the zeros for the case $r = 40$, $n = 80$, as well as the curve $C_{1/2}$ and the approximating points on the curve.

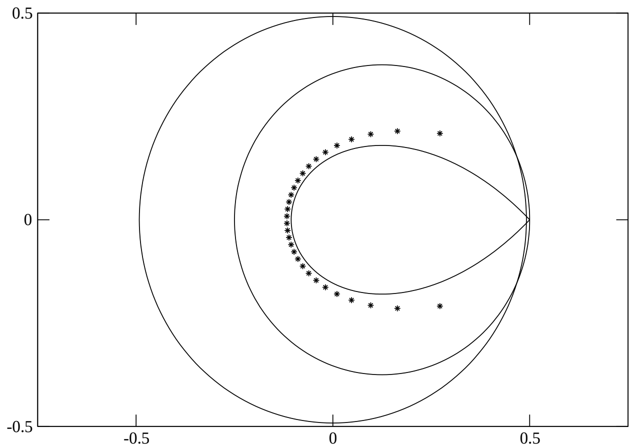

FIGURE 2.2. *The bounding curves and zeros for* $r = 30$, $n = 90$.

It should be noted at this point that, due to the structure of the coefficients of these polynomials, direct computation of the zeros for significantly higher degrees suffers due to numerical instability.

We conclude by considering the limiting cases $\alpha = 0$ and $\alpha = 1$. The trivial result for $\alpha = 0$, given the radius $\frac{r}{n+1-r}$ of the bounding circle, is that all zeros converge uniformly to 0 in this case. However, a slight modification gives a much more interesting result.

THEOREM 2.4. *Suppose that* $\lim_{j\to\infty} r_j = \infty$ *and that* $\lim_{j\to\infty} \frac{r_j}{n_j} = 0$. *Then, the limit points of the zeros of* $\{B_{r_j,n_j}(\frac{r_j z}{n_j - r_j})\}_{j=1}^{\infty}$ *are precisely the points of the Szegő curve* $|ze^{1-z}| = 1$, $|z| \le 1$.

Proof. With the given normalization, the results of Theorem 2.1 yield that the zeros of the normalized polynomial above satisfy

$$1 = \left(\frac{r_j}{n_j - r_j}\right)^{r_j} K_{r_j/n_j}^{-n_j} \frac{z^{r_j}}{\left(1 + \frac{r_j z}{n_j - r_j}\right)^{n_j}} h(z) \quad \text{and} \quad |z| \le 1, \tag{2.10}$$

where

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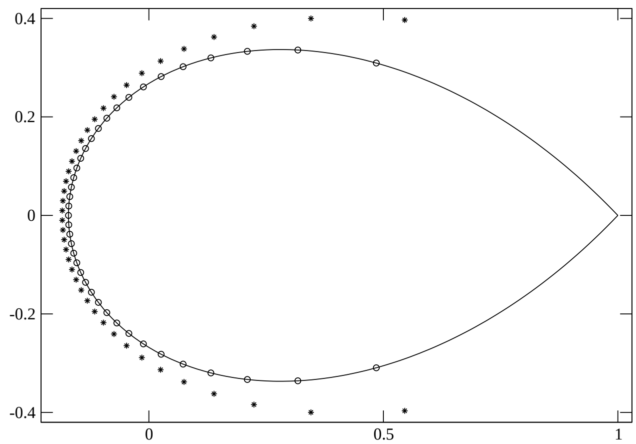

FIGURE 2.3. *The curve* $C_{1/2}$*, the points* $\{\zeta_{p,80}\}$ *and the zeros for* $r = 40$*,* $n = 80$*.*

$$h(z) = \sum_{k=r_j+1}^{n_j} \binom{n_j}{k} \left(\frac{r_j}{n_j}\right)^k \left(1 - \frac{r_j}{n_j}\right)^{n_j - k_j} z^{k-r_j}.$$

Noting that

$$\left(\frac{r_j}{n_j - r_j}\right)^{r_j} K_{r_j/n_j}^{-n_j} = \left(1 - \frac{r_j}{n_j}\right)^{-n_j},$$

we may use standard expansions to convert (2.10) to the form

$$1 = (ze^{1-z+g(z)})^{r_j} h(z),$$

where $|g(z)| \leq \frac{3r}{n}$, uniformly in the unit disk. Considering points inside and on the curve $|ze^{1-z}| = e^{-3r_j/n_j}$, and noting that $|h(z)| \leq h(1) < 1$ on the unit disk, we may repeat the analysis of (2.4) to deduce that the zeros are uniformly bounded away from zero by $|z| \geq \eta > 0$. This implies that we may repeat the bounding process of Lemma 3.3 to deduce that $h^{1/r_j}(z) \to 1$ uniformly in $\eta \leq |z| \leq 1$, defining the roots by a cut along the positive real axis. This establishes the desired result. □

Finally, we consider the other limiting case.

THEOREM 2.5. *Suppose that* $\lim_{j\to\infty} r_j = \infty$ *and* $\lim_{j\to\infty} \frac{r_j}{n_j} = 1$*. Then, the limit points of the zeros of the polynomials* $\left\{B_{r_j,n_j}(z)\right\}_{j=1}^{\infty}$ *are precisely the points of the line* $\operatorname{Re} z = -\frac{1}{2}$.

Proof. As in the previous proofs, we write the equation for the zeros as

$$1 = \frac{z^{r_j}}{(1+z)^{n_j}} R_{r_j,n_j}(z).$$

We again use the bounds of Lemma 3.3 and obtain the desired result, using the fact that $\lim_{\alpha\to 1^-} K_\alpha = 1$. □

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3. Technical Results. Here we give the properties and inequalities necessary for the main results, beginning with the properties of the bounding curves.

LEMMA 3.1. *Fix* $0 < \alpha < 1$, *and let*

$$K_\alpha = \alpha^\alpha (1-\alpha)^{1-\alpha}$$

and

$$C_\alpha = \left\{ z \ : \ \frac{|z|^\alpha}{|1+z|} = K_\alpha, \ |z| \le \frac{\alpha}{1-\alpha} \right\}.$$

Then

1. $\frac{1}{2} \le K_\alpha < 1$, $\lim_{\alpha\to 0^+} K_\alpha = 1$, *and* $\lim_{\alpha \to 1^-} K_\alpha = 1$*;*
2. C_α *is a simple, smooth closed curve, symmetric with respect to the real axis, starlike with respect to* $z = 0$*, which passes through* $z_\alpha = \frac{\alpha}{1-\alpha}$*;*
3. *the intersection of* C_α *with the negative real axis occurs at* $z = -X_\alpha$*, where* $\nu\alpha < X_\alpha < \frac{1}{2}$ *and* $\nu \simeq 0.278$ *is the unique positive root of* $xe^{1+x} = 1$*;*
4. $X_\alpha \le |z|$ *and* $|z| \le z_\alpha$ *for any* $z \in C_\alpha$*, with the latter equality holding only at* $z = z_\alpha$*.*

Proof.

1. A simple calculation gives the limits. Taking derivatives yields

$$\frac{dK_\alpha}{d\alpha} = K_\alpha \ln\left(\frac{\alpha}{1-\alpha}\right),$$

which shows that K_α is decreasing on $\left(0, \frac{1}{2}\right)$ and increasing on $\left(\frac{1}{2}, 1\right)$. Calculating $K_{1/2}$ directly gives the equality.

2. Clearly, the definition shows that C_α is closed and symmetric, and direct calculation shows that it passes through the point $z_\alpha = \alpha/(1-\alpha)$. We write $z = re^{i\theta}$, and set

$$c_\theta(r) = \frac{|z|^\alpha}{|1+z|} = \frac{r^\alpha}{\sqrt{1+2r\cos\theta + r^2}}.$$

Clearly, $c_\theta(0) = 0$ and $\lim_{r\to\infty} c_\theta(r) = 0$. For $\theta = 0$, we have

$$c_0'(r) = \frac{r^{\alpha-1}}{(1+r)^2}[\alpha - (1-\alpha)r],$$

which shows that the given point is the only positive real value satisfying the equation. For $0 < \theta < \pi$, we have

$$c_\theta'(r) = r^{\alpha-1}(1+2r\cos\theta + r^2)^{-3/2}[(\alpha-1)r^2 + (2\alpha-1)r\cos\theta + \alpha].$$

Since $\alpha - 1 < 0$, this derivative has exactly one positive root, which is a maximum of the function. Further, a simple calculation shows that

$$c_\theta\left(\frac{\alpha}{1-\alpha}\right) > K_\alpha,$$

from which each such ray yields exactly one point on the curve, inside the bounding circle $|z| = \frac{\alpha}{1-\alpha}$. Considering the defining function, this value of r is clearly decreasing in $0 \le \theta < \pi$. Hence, the curve is simple and starlike with respect to 0. Finally, for $\theta = \pi$, we have that

$$c_\pi'(r) = \frac{r^{\alpha-1}}{(1-r)^2}[\alpha + (1-\alpha)r] > 0$$

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for $0 < r < 1$, and $\lim_{r\to 1^-} c_\pi(r) = \infty$, which gives exactly one solution in this range. That these points are the only solutions within the bounding circle can be deduced from the fact that $z \in C_\alpha$ if and only if $\frac{1}{z} \in C'_{1-\alpha}$.
Examining the function $w = K_\alpha^{-1}\frac{z^\alpha}{1+z}$, using arguments in the range $(0, 2\pi)$, shows that C_α maps onto the appropriate arc of the unit circle in the w-plane. This mapping is also one-to-one along the arc $0 < \arg w < 2\pi\alpha$, since $w' \neq 0$ on the cut plane. This fact is implicitly used in the calculation of the rate of convergence.

3. The solution on the negative real axis is $-t = -X_\alpha$, and satisfies

$$\frac{t^\alpha}{1-t} = K_\alpha,$$

which we write as

$$f(t) = t^\alpha + \alpha^\alpha(1-\alpha)^{1-\alpha}(t-1) = 0.$$

Now, $f(t)$ is increasing, with $f(0) < 0$, $f(X_\alpha) = 0$, and

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^\alpha - \frac{1}{2}K_\alpha > \frac{1}{2}(1-K_\alpha) > 0,$$

from which $X_\alpha < \frac{1}{2}$ follows immediately.
To show that $\nu\alpha < X_\alpha$, we consider

$$f(\nu\alpha) = \alpha^\alpha(\nu^\alpha - (1-\nu\alpha)(1-\alpha)^{1-\alpha}), \tag{3.1}$$

and set

$$g(\alpha) = \ln((1-\nu\alpha)(1-\alpha)^{1-\alpha}),$$

which satisfies $g(0) = 0$, $g'(0) = -\nu - 1$ and

$$g''(\alpha) = \frac{(\nu\alpha)^2 + (\nu-2)(\nu\alpha) + 1 - \nu^2}{(1-\alpha)(1-\nu\alpha)^2} > 0.$$

The last inequality follows since the numerator has discriminant $\nu^3(5\nu - 4) < 0$, from Lemma 3.1, and so has no real zeros. Hence,

$$e^{g(\alpha)} > e^{-(\nu+1)\alpha} = e^{\alpha\ln\nu} = \nu^\alpha,$$

and thus, by (3.1), $f(\nu\alpha) < 0$ for $0 < \alpha < 1$, as desired. $\square$

We continue with a lemma required for one of the bounds.

LEMMA 3.2. *Let* $f(z) = \sum_{k=0}^\infty b_k z^k$ *satisfy*

$$b_0 > b_1 \geq 0, \quad b_k \geq 0, \quad b_1 b_{k-1} - b_0 b_k \geq 0, \quad \text{for } k \geq 1.$$

Then,

$$|f(z)| \geq \frac{b_0 - b_1}{b_0 + b_1} f(1), \quad \text{for } |z| \leq 1.$$

Proof. The conditions given imply that $\{b_k\}$ is strictly decreasing, unless $b_k = 0$ for $k \geq K$. Let $r = \frac{b_1}{b_0} < 1$. Then, the conditions given show that $b_k \leq r b_{k-1}$ for $k \geq 1$. Hence, $f(z)$ is analytic for $|z| < \frac{1}{r}$, and, in particular, in the closed unit disk. Applying the

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Eneström-Kakaya Theorem to the partial sums $p_n(z) = \sum_{k=0}^{n} b_k z^k$ shows that all have their zeros in the region $|z| > 1$, hence, by Hurwitz' Theorem, $f(z)$ cannot have any zeros inside the unit disk. Thus, applying the Minimum Modulus Theorem, the minimum value of $|f(z)|$ for $|z| \le 1$ must occur on the boundary.

For $|z| = 1$, we have

$$\begin{aligned} |(b_0 - b_1 z) f(z)| &= \left| b_0^2 + \sum_{k=1}^{\infty} (b_0 b_k - b_1 b_{k-1}) z^k \right| \ge b_0^2 - \sum_{k=1}^{\infty} |b_1 b_{k-1} - b_0 b_k| \\ &= b_0^2 - \sum_{k=1}^{\infty} b_1 b_{k-1} + \sum_{k=1}^{\infty} b_0 b_k = b_0^2 - b_1 f(1) + b_0 (f(1) - b_0) \\ &= (b_0 - b_1) f(1). \end{aligned}$$

Hence, we have

$$|f(z)| \ge \frac{(b_0 - b_1) f(1)}{|b_0 - b_1 z|} \ge \frac{(b_0 - b_1) f(1)}{b_0 + b_1},$$

the desired result. $\square$

Finally, we have the estimates of the remainder term.

LEMMA 3.3. *Given integers $1 \le r < n$, we set $\beta = \frac{r}{n}$, and consider the remainder term*

$$R_{r,n}(z) = \sum_{k=r+1}^{n} \binom{n}{k} z^{k-r}.$$

Then, for $|z| \le \frac{\beta}{1-\beta}$, we have

$$|R_{r,n}(z)| \le K_\beta^{-n} \sum_{k=r+1}^{n} \binom{n}{k} \beta^k (1-\beta)^{n-k} \le K_\beta^{-n}$$

and

$$|R_{r,n}(z)| \ge \frac{|z|}{r+1} K_\beta^{-n} \sum_{k=r+1}^{n} \binom{n}{k} \beta^k (1-\beta)^{n-k}.$$

Proof. Given that all coefficients are positive, we use the value of K_β from (1.2) and the bound on $|z|$ to deduce that

$$|R_{r,n}(z)| \le R_{r,n}\left(\frac{\beta}{1-\beta}\right) = K_\beta^{-n} \sum_{k=r+1}^{n} \binom{n}{k} \beta^k (1-\beta)^{n-k}.$$

The latter sum is clearly bounded by 1, using the binomial expansion. In fact, using the Central Limit Theorem, it is asymptotically $1/2$ for both r and $n-r$ large.

For the lower bound, we consider

$$g(z) = \left(\frac{1-\beta}{\beta z}\right) R_{r,n}\left(\frac{\beta z}{1-\beta}\right) = \sum_{k=0}^{n-r-1} b_k z^k,$$

where

$$b_k = \binom{n}{k+r+1} \left(\frac{\beta}{1-\beta}\right)^k.$$

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It is simple to show that $g(z)$ satisfies the conditions of Lemma 3.2, that

$$\frac{b_0 - b_1}{b_0 + b_1} = \frac{2n - r}{2r(n - r) + (2n - 3r)} \geq \frac{1}{r + 1},$$

and finally that

$$g(1) = \left(\frac{1 - \beta}{\beta}\right) K_\beta^{-n} \sum_{k=r+1}^{n} \binom{n}{k} \beta^k (1 - \beta)^{n-k}.$$

Rewriting $R_{r,n}(z)$ in terms of $g(z)$ yields the result. $\square$

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REFERENCES

[1] J. D. BUCKHOLTZ, *A characterization of the exponential series*, Amer. Math. Monthly, 73 (1966), pp. 121–123.
[2] A. J. CARPENTER, R. S. VARGA AND J. WALDVOGEL, *Asymptotics for the zeros of the partial sums of* e^z *I*, Rocky Mountain J. Math., 21 (1991), pp. 99–120.
[3] A. EDREI, E. B. SAFF AND R. S. VARGA, *Zeros of Sections of Power Series*, Springer, Berlin, 1983.
[4] P. ERDŐS AND P. TURÁN, *On the distribution of roots of polynomials*, Ann. of Math., 51 (1950), pp. 105–119.
[5] P. HENRICI, *Applied and Computational Complex Analysis, Vol. I*, John Wiley & Sons, New York, 1974.
[6] R. JENTZSCH, *Untersuchungen zur Theorie der Folgen analytischer Funktionen*, Acta Math., 41 (1917), pp. 219–251.
[7] D. J. NEWMAN AND T. J. RIVLIN, *The zeros of the partial sums of the exponential function*, J. Approx. Theory, 5 (1972), pp. 405–412.
[8] ———, *Correction: The zeros of the partial sums of the exponential function*, J. Approx. Theory, 16 (1976), pp. 299–300.
[9] T. S. NORFOLK, *On the zeros of the partial sums to* ${}_1F_1(1; b; z)$, J. Math. Anal. Appl., 218 (1998), pp. 421–438.
[10] ———, *Asymptotics of the partial sums of a set of integral transforms*, Numer. Algorithms, 25 (2000), pp. 279–291.
[11] I. V. OSTROVSKII, *On a problem of A. Eremenko*, Comput. Methods Funct. Theory, 4 (2004), pp. 275–282.
[12] P. C. ROSENBLOOM, *Distribution of zeros of polynomials*, in Lectures on Functions of a Complex Variable (W. Kaplan, editor), University of Michigan Press, Ann Arbor, Michigan, 1955, pp. 265–275.
[13] G. SZEGŐ, *Über eine Eigenschaft der Exponentialreihe*, Sitzungsber. Berl. Math. Ges., 23 (1924), pp. 50–64.
[14] D. G. WAGNER, *Zeros of reliability polynomials and* f*-vectors of matroids*, Combin. Probab. Comput., 9 (2000), pp. 167–190.