Huang and Lian *Journal of Inequalities and Applications* (2016) 2016:117
DOI 10.1186/s13660-016-1055-7

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Regularity for compressible isentropic Navier-Stokes equations with cylinder symmetry

Lan Huang* and Ruxu Lian

*Correspondence:
huanglan82@hotmail.com
College of Mathematics and Information Science, North China University of Water Resources and Electric Power, Zhengzhou, 450011, P.R. China

Abstract

This paper is concerned with the regularity of the global solutions in H^4 to the compressible isentropic Navier-Stokes equations with the cylinder symmetry in R^3. Such a circular coaxial cylinder symmetric domain is an unbounded domain, but we assume that the corresponding solution depends only on one radial variable, r in $G = \{r \in R^+, 0 < a \le r \le b < +\infty\}$, in which the related domain G is a bounded domain. Some new ideas and more delicate estimates are introduced to prove these results.

Keywords: compressible Navier-Stokes equations; cylinder symmetry; density-dependent viscosity; regularity

1 Introduction

The compressible isentropic Navier-Stokes equations with density-dependent viscosity coefficients can be written for $t > 0$ as

$$\rho_t + \operatorname{div}(\rho\mathbf{U}) = 0, \tag{1.1}$$

$$(\rho\mathbf{U})_t + \operatorname{div}(\rho\mathbf{U}\otimes\mathbf{U}) - \operatorname{div}\big(\mu(\rho)D(\mathbf{U})\big) - \nabla\big(\lambda(\rho)\operatorname{div}\mathbf{U}\big) + \nabla P(\rho) = 0, \tag{1.2}$$

where $\rho(\mathbf{x},t)$, $\mathbf{U}(\mathbf{x},t)$ and $P(\rho) = \rho^\gamma$ $(\gamma > 1)$ stand for the fluid density, velocity, and pressure, respectively, and

$$D(\mathbf{U}) = \frac{\nabla\mathbf{U} + (\nabla\mathbf{U})^T}{2} \tag{1.3}$$

is the strain tensor and $\mu(\rho)$, $\lambda(\rho)$ are the Lamé viscosity coefficients satisfying

$$\mu(\rho) > 0, \qquad \mu(\rho) + N\lambda(\rho) \ge 0. \tag{1.4}$$

In this paper we establish the regularity of global solutions to the compressible Navier-Stokes equations with cylinder symmetry in R^3. We will pay attention to the flows between two circular coaxial cylinders. We assume that the corresponding solutions depend only on the radial variable $r \in G = \{r | 0 < a \le r \le b < +\infty\}$ and the time variable $t \in [0,T]$. For simplicity, we will take $\mu(\rho) = \rho^\alpha$ and $\lambda(\rho) = \rho\mu'(\rho) - \mu(\rho) = (\alpha-1)\rho^\alpha$ with $\alpha > 1/2$ and $D(\mathbf{U}) = \nabla\mathbf{U}$. Then system (1.1)-(1.2) reduce to the following form:

$$\rho_t + \frac{1}{r}(r\rho u)_r = 0, \tag{1.5}$$

$$\begin{aligned} &r\rho u_t + r\rho u u_r - \rho v^2 + r\big(\rho^\gamma\big)_r - \big(r\rho^\alpha u_r\big)_r \\ &\quad - (\alpha-1)r\left(\frac{1}{r}\rho^\alpha (ru)_r\right)_r + \frac{1}{r}\rho^\alpha u = 0, \end{aligned} \tag{1.6}$$

$$r\rho v_t + r\rho u v_r + \rho u v - \big(r\rho^\alpha v_r\big)_r + \frac{1}{r}\rho^\alpha v = 0, \tag{1.7}$$

$$r\rho w_t + r\rho u w_r - \big(r\rho^\alpha w_r\big)_r = 0. \tag{1.8}$$

We consider the initial boundary value problem (1.5)-(1.8) subject to the following initial and boundary conditions:

$$(\rho, u, v, w)(r,0) = (\rho_0, u_0, v_0, w_0)(r), \quad r \in [a,b], \tag{1.9}$$

$$u(a) = u(b) = 0, \qquad v(a) = v(b) = 0, \qquad w(a) = w(b) = 0. \tag{1.10}$$

First we find it convenient to transfer problems (1.5)-(1.10) into that in Lagrangian coordinates and present the desired results. It is well known that Eulerian coordinates (r,t) are connected to the Lagrangian coordinates (ξ,t) by the following relation:

$$r(\xi,t) = r_0(\xi) + \int_0^t \tilde{u}(\xi,\tau)\,d\tau, \tag{1.11}$$

where $\tilde{u}(\xi,t) = u(r(\xi,t),t)$ and

$$r_0(\xi) = \eta^{-1}(\xi), \qquad \eta(r) = \int_a^r s\rho_0(s)\,ds, \quad r \in G. \tag{1.12}$$

It follows from (1.5) and the boundary condition (1.10) that

$$\frac{\partial}{\partial t}\left(\int_a^{r(\xi,t)} s\rho(s,t)\,ds\right) = \rho r u + \int_a^r s\rho_t\,ds = 0. \tag{1.13}$$

Thus,

$$\int_a^r s\rho(s,t)\,ds = \int_a^{r_0} s\rho_0(s)\,ds = \xi \tag{1.14}$$

and G is transformed to $\Omega = (0,1)$ with

$$1 = \int_a^b s\rho(s,t)\,ds = \int_a^b s\rho_0(s)\,ds. \tag{1.15}$$

Moreover, we have

$$\partial_\xi r(\xi,t) = \big[r(\xi,t)\rho\big(r(\xi,t),t\big)\big]^{-1}. \tag{1.16}$$

For a function $\phi(r,t)$, if we write $\tilde{\phi}(\xi,t) = \phi(r(\xi,t),t)$, by virtue of (1.11), (1.16), and the chain rule, we have

$$\partial_t\tilde{\phi}(\xi,t) = \partial_t\phi(r,t) + v\partial_r\phi(r,t), \tag{1.17}$$

$$\partial_\xi\tilde{\phi}(\xi,t) = \partial_r\phi(r,t)\partial_\xi r(\xi,t) = \bigl(r^2\rho(r,t)\bigr)^{-1}\partial_r\phi(r,t). \tag{1.18}$$

In the following, without danger of confusion we denote $(\tilde{\rho},\tilde{u},\tilde{v},\tilde{w})$ still by (ρ,u,v,w) and (ξ,t) by (x,t). Therefore, (1.5)-(1.8) in the Eulerian coordinates can be written in the Lagrangian coordinates in the new variables (x,t) as follows (see also [1]):

$$\rho_t + \rho^2(ru)_x = 0, \tag{1.19}$$

$$\frac{u_t}{r} - \frac{v^2}{r^2} + \bigl(\rho^\gamma\bigr)_x = \alpha\bigl(\rho^{\alpha+1}(ru)_x\bigr)_x - \frac{(\rho^\alpha)_x u}{r}, \tag{1.20}$$

$$\frac{v_t}{r} + \frac{uv}{r^2} = \bigl(\rho^{\alpha+1}(rv)_x\bigr)_x - \frac{(\rho^\alpha)_x v}{r}, \tag{1.21}$$

$$w_t - \bigl(r^2\rho^{1+\alpha}w_x\bigr)_x = 0, \tag{1.22}$$

subject to the following initial and boundary conditions:

$$(\rho,u,v,w)(x,0) = (\rho_0,u_0,v_0,w_0)(x), \quad x\in\Omega, \tag{1.23}$$

$$(u,v,w)(0,t) = (u,v,w)(1,t) = 0, \quad t\geq 0, \tag{1.24}$$

where $r(x,t)$ is determined by

$$\begin{aligned} &r_t(x,t) = u(x,t), \qquad r(x,t)r_x(x,t) = \frac{1}{\rho}, \\ &r|_{t=0} = r_0(x) = \left[a^2 + 2\int_0^x \frac{1}{\rho_0(y,t)}\,dy\right]^{1/2}. \end{aligned} \tag{1.25}$$

The compressible Navier-Stokes system has been noticed academically by physicists and mathematicians for a relatively long time. We are interested in the case that the viscosity is density-dependent. Now let us first recall the related results in this direction. In the one-dimensional case, for the initial boundary value problems in a bounded domain, there have been many works (see, *e.g.*, [2–11]) on the existence, uniqueness, and asymptotic behavior of weak solutions, based on the initial finite mass and the flow density being connected with the infinite vacuum either continuously or by a jump discontinuity. For the one-dimensional Cauchy problem, see, *e.g.*, [12] and references therein for the results on global existence based on the density-dependent viscosity.

In two or three dimensions, the global existence and large time behavior of solution to compressible Navier-Stokes equations (1.1)-(1.2) with constant viscosity and density-dependent viscosity have been investigated for initial boundary value problems, we refer the reader to [1, 13–27] and references therein. Among them, Bresch and Desjardins [13, 14] obtained the global existence of 2D shallow water equations. For the spherically symmetric problem to a three-dimensional compressible isentropic Navier-Stokes problem, Guo *et al.* [19] analyzed the structure of the solution; Huang *et al.* [28] studied the global well-posedness of the classical solutions with large oscillations and vacuum; Lian *et al.* [22] obtained the global existence of spherically symmetric solution for the exterior problem and the initial boundary value problem; Zhang and Fang [27] investigated the global existence and uniqueness of the weak solution without a solid core. For the cylindrically symmetric problem to the three-dimensional compressible Navier-Stokes equations, when the viscosity coefficients are both constants, the uniqueness of the weak solutions was proved in [17, 18], the global existence of isentropic compressible cylindrically symmetric solution was established in [29]; this result was later generalized to the nonisentropic case in [21]. Recently, Cui and Yao [15] proved the asymptotic behavior of a compressible pth power Newtonian fluid with cylinder symmetry; Qin [24] established the exponential stability in H^1 and H^2 for an ideal fluid. Later on, Qin and Jiang [25] proved the global existence and the exponential stability in H^4. Jiang and Zhang [21] established a boundary layer effect and the convergence rate as the shear viscosity μ goes to zero. When the viscosity coefficient $\mu(\rho)$ is density-dependent and $\lambda(\rho)$ is a positive constant, the global existence was obtained in [26]. When viscosity coefficients μ and λ are density-dependent, Liu and Lian [1] established the global existence and asymptotic behavior of cylindrically symmetric solutions, however, there is no result on the regularity for this system.

It is noticed that the above analysis concerns the existence of solution in $H^1[0,1]$, the regularity in $H^4[0,1]$ has never been investigated for the three-dimensional isentropic compressible Navier-Stokes equations. Therefore, we continue the work by Liu and Lian [1] and study the regularity of the solutions in H^4. In order to obtain a higher regularity of global strong solutions, there are many complicated estimates on higher derivations of the solution involved; this is our difficulty. To overcome this difficulty, we shall use some proper embedding theorems, and the interpolation techniques as well as many delicate estimates.

The notation in this paper will be as follows: $L^{\bar{p}}$, $1 \le \bar{p} \le +\infty$, $W^{m,\bar{p}}$, $m \in N$, $H^1 = W^{1,2}$, $H_0^1 = W_0^{1,2}$ denote the usual (Sobolev) spaces on $[0,1]$. In addition, $\|\cdot\|_B$ denotes the norm in the space B; we also put $\|\cdot\| = \|\cdot\|_{L^2}$. Subscripts t and x denote the (partial) derivatives with respect to t and x, respectively. We use C_i ($i = 1, 2, 4$) to denote the generic positive constant depending only on the H^i norm of the initial data (ρ_0, u_0, v_0, w_0) and the variable t.

Before stating the main result, we assume the initial data

$$(\rho_0 - \bar{\rho}, u_0, v_0, w_0) \in \left(H^4[0,1]\right)^4, \tag{1.26}$$

with $\rho_0 > 0$ and $\bar{\rho} = \frac{1}{b-a}\int_a^b \rho_0 r\,dr$, and we define

$$H_0 = \int_0^1 \left(\frac{1}{2}\left(u_0^2 + v_0^2 + w_0^2\right) + \frac{1}{\gamma - 1}\left(\rho_0^{\gamma-1} - \bar{\rho}^{\gamma-1}\right) + \bar{\rho}^{\gamma}\left(\frac{1}{\rho_0} - \frac{1}{\bar{\rho}}\right)\right) dx,$$
$$H_1 = \int_0^1 \left(\frac{1}{2}\left(\left(u_0 + r\left(\rho_0^{\alpha}\right)_x\right)^2 + v_0^2 + w_0^2\right) + \frac{1}{\gamma - 1}\left(\rho_0^{\gamma-1} - \bar{\rho}^{\gamma-1}\right) + \bar{\rho}^{\gamma}\left(\frac{1}{\rho_0} - \frac{1}{\bar{\rho}}\right)\right) dx.$$

Now we are in a position to state our main results.

Theorem 1.1 *Let* $\gamma > 1$, $\alpha > 1/2$. *Assume that the initial data satisfies* (1.26) *and* $H_0(H_0 + H_1) < a^2\alpha^2(2\alpha - 1)^{-2}\bar{\rho}^{\gamma+2\alpha-1}$. *Then there exists a unique generalized global solution*

$(\rho(t),u(t),v(t),w(t))\in(H^4[0,1])^4$ *to the problem* (1.19)-(1.24) *verifying that, for* $T>0$,

$$\rho-\bar\rho\in L^\infty\bigl([0,T],H^4[0,1]\bigr)\cap L^2\bigl([0,T],H^4[0,1]\bigr), \tag{1.27}$$

$$(u,v,w)\in L^\infty\bigl([0,T],H^4[0,1]\bigr)\cap L^2\bigl([0,T],H^5[0,1]\bigr), \tag{1.28}$$

$$(u_t,v_t,w_t)\in L^\infty\bigl([0,T],H^2[0,1]\bigr)\cap L^2\bigl([0,T],H^3[0,1]\bigr), \tag{1.29}$$

$$(u_{tt},v_{tt},w_{tt})\in L^\infty\bigl([0,T],L^2[0,1]\bigr)\cap L^2\bigl([0,T],H^1[0,1]\bigr). \tag{1.30}$$

Corollary 1.1 *Under the assumptions of Theorem* 1.1, (1.27)-(1.28) *implies* $(\rho(t),u(t),$ $v(t),w(t))$ *is the classical solution verifying, for any* $t>0$,

$$\bigl\|\rho(t)-\bar\rho\bigr\|_{C^{3+1/2}}+\bigl\|u(t)\bigr\|_{C^{3+1/2}}+\bigl\|v(t)\bigr\|_{C^{3+1/2}}+\bigl\|w(t)\bigr\|_{C^{3+1/2}}\le C_4. \tag{1.31}$$

The rest of the paper is arranged as follows. Section 2 is concerned with the proof of the regularity for a cylindrically symmetric solution to the compressible Navier-Stokes problem in detail.

2 Proof of Theorem 1.1

We will complete the proof of Theorem 1.1 and assume that the assumptions in Theorem 1.1 are valid. We begin with the following lemma.

Lemma 2.1 *Under the assumptions in Theorem* 1.1, *there exist positive constants* $\rho_*>0$ *and* $\rho^*>0$ *with* $\rho_*<\bar\rho<\rho^*$ *so that the unique global solution* $(\rho(t),u(t),v(t),w(t))$ *to problem* (1.19)-(1.24) *exists and satisfies, for any* $T>0$,

$$0<\rho_*\le\rho(x,t)\le\rho^*, \tag{2.1}$$

$$(\rho-\bar\rho)_x\in L^\infty\bigl([0,T],L^2[0,1]\bigr)\cap L^2\bigl([0,T],L^2[0,1]\bigr), \tag{2.2}$$

$$(u,v,w)\in L^\infty\bigl([0,T],H^2[0,1]\bigr)\cap L^2\bigl([0,T],H^3[0,1]\bigr), \tag{2.3}$$

$$\bigl\|(u_t,v_t,w_t)(t)\bigr\|^2+\int_0^t\bigl\|(u_t,v_t,w_t)(s)\bigr\|_{H^1}^2\,ds\le C_2,\quad\forall t\in[0,T], \tag{2.4}$$

where $\bar\rho$ *is the same as in Theorem* 1.1.

Proof Estimates (2.1)-(2.4) were obtained in Ref. [1], the proof is complete. □

Lemma 2.2 *Under the assumptions in Theorem* 1.1, *the following estimate holds for any* $T>0$:

$$\bigl\|\rho_{xx}(t)\bigr\|^2+\int_0^t\bigl\|\rho_{xx}(s)\bigr\|^2\,ds\le C_2,\quad t\in[0,T]. \tag{2.5}$$

Proof Differentiating (1.20) with respect to x, exploiting (1.19), we have

$$\frac{u_{tx}}{r}=\bigl(-\rho^\gamma+\alpha\rho^{\alpha+1}(ru)_x\bigr)_{xx}+\left(\frac{v^2}{r^2}-\frac{(\rho^\alpha)_x u}{r}\right)_x+\frac{u_t}{\rho r^3}, \tag{2.6}$$

which gives

$$\alpha\big(\rho^{\alpha-1}\rho_{xx}\big)_t + \gamma\rho^{\gamma-1}\rho_{xx} = M_0(x,t) \tag{2.7}$$

with

$$\begin{aligned} M_0(x,t) = {}& \frac{2vv_x}{r^2} - \frac{2v^2 r_x}{r^3} - \alpha(\alpha-1)\big(2\rho^{\alpha-2}\rho_x\rho_{tx} - (\alpha-2)\rho^{\alpha-3}\rho_t\rho_x^2\big) \\ & - \gamma(\gamma-1)\rho^{\gamma-2}\rho_x^2 - \frac{(\rho^\alpha)_{xx}u + (\rho^\alpha)_x u}{r} - \frac{(\rho^\alpha)_x u}{\rho r^3} - \frac{u_{tx}}{r} + \frac{u_t}{\rho r^3}. \end{aligned}$$

Multiplying (2.7) by $\rho^{\alpha-1}\rho_{xx}$, integrating the result over $[0,1]$, we deduce

$$\begin{aligned} & \frac{d}{dt}\big\|\rho^{\alpha-1}\rho_{xx}(t)\big\|^2 + \int_0^1 \gamma\rho^{\gamma+\alpha-2}\rho_{xx}^2\,dx \\ & \quad \le C_1\int_0^1 \|\rho_{xx}\|\big(\|v_x\| + \big\|v^2\big\| + \|\rho_x\rho_{tx}\| + \big\|\rho_t\rho_x^2\big\| + \big\|\rho_x^2\big\| \\ & \qquad + \|\rho_x\| + \|u_{tx}\| + \|u_t\|\big) - \int_0^1 \alpha\frac{u}{r}\rho^{2\alpha-2}\rho_{xx}^2\,dx, \end{aligned}$$

which, by Young's inequality and the interpolation inequality, implies

$$\frac{d}{dt}\big\|\rho^{\alpha-1}\rho_{xx}(t)\big\|^2 \le C_1\big(\|\rho_{xx}\|^2 + \|\rho_x\|^2 + \|u_x\|_{H^1}^2 + \|u_t\|_{H^1}^2\big). \tag{2.8}$$

Integrating (2.8) with respect to t over $[0,T]$, using initial condition (1.26) and Lemma 2.1, we derive

$$\big\|\rho_{xx}(t)\big\|^2 \le C_2 + C_1\int_0^t \big\|\rho_{xx}(s)\big\|^2\,ds, \quad \forall t\in[0,T],$$

which, by virtue of Gronwall's inequality, gives (2.5). The proof is complete. □

Lemma 2.3 *Under the assumptions in Theorem* 1.1, *the following estimates hold for any* $T>0$:

$$\big\|u_{tt}(t)\big\|^2 + \int_0^t \big\|u_{ttx}(s)\big\|^2\,ds \le C_4 + C_2\int_0^t \big(\|u_{txx}\|^2 + \|v_{txx}\|^2\big)(s)\,ds, \quad t\in[0,T], \tag{2.9}$$

$$\big\|v_{tt}(t)\big\|^2 + \int_0^t \big\|v_{ttx}(s)\big\|^2\,ds \le C_4 + C_2\int_0^t \big(\|v_{txx}\|^2 + \|u_{txx}\|^2\big)(s)\,ds, \quad t\in[0,T], \tag{2.10}$$

$$\big\|w_{tt}(t)\big\|^2 + \int_0^t \big\|w_{ttx}(s)\big\|^2\,ds \le C_4, \quad t\in[0,T]. \tag{2.11}$$

Proof We infer from (1.20)-(1.22) and Lemmas 2.1-2.2 that

$$\big\|u_t(t)\big\| \le C_1\big(\big\|\rho_x(t)\big\| + \big\|u_x(t)\big\|_{H^1} + \big\|v_x(t)\big\|\big), \tag{2.12}$$

$$\big\|v_t(t)\big\| \le C_1\big(\big\|\rho_x(t)\big\| + \big\|u_x(t)\big\| + \big\|v_x(t)\big\|_{H^1}\big), \tag{2.13}$$

$$\big\|w_t(t)\big\| \le C_1\big(\big\|w_x(t)\big\|_{H^1} + \big\|\rho_x(t)\big\|\big). \tag{2.14}$$

Differentiating (1.20)-(1.22) with respect to x, respectively, and exploiting Lemmas 2.1-2.2, we have

$$\|u_{tx}(t)\| \le C_1\big(\|\rho_x(t)\|_{H^1} + \|v_x(t)\| + \|u_x(t)\|_{H^2}\big), \tag{2.15}$$

$$\|v_{tx}(t)\| \le C_1\big(\|\rho_x(t)\|_{H^1} + \|u_x(t)\| + \|v_x(t)\|_{H^2}\big), \tag{2.16}$$

$$\|w_{tx}(t)\| \le C_1\big(\|\rho_x(t)\|_{H^1} + \|w_x(t)\|_{H^2}\big), \tag{2.17}$$

or

$$\|u_{xxx}(t)\| \le C_1\big(\|\rho_x(t)\|_{H^1} + \|v_x(t)\| + \|u_x(t)\|_{H^1} + \|u_{tx}(t)\|\big), \tag{2.18}$$

$$\|v_{xxx}(t)\| \le C_1\big(\|\rho_x(t)\|_{H^1} + \|u_x(t)\| + \|v_x(t)\|_{H^1} + \|v_{tx}(t)\|\big), \tag{2.19}$$

$$\|w_{xxx}(t)\| \le C_1\big(\|\rho_x(t)\|_{H^1} + \|w_x(t)\|_{H^1} + \|w_{tx}(t)\|\big). \tag{2.20}$$

Differentiating (1.20)-(1.22) with respect to x twice, respectively, using Lemmas 2.1-2.2, we get

$$\|u_{txx}(t)\| \le C_1\big(\|\rho_x(t)\|_{H^2} + \|u_t(t)\|_{H^1} + \|v_x(t)\|_{H^1} + \|u_x(t)\|_{H^3}\big), \tag{2.21}$$

$$\|v_{txx}(t)\| \le C_1\big(\|\rho_x(t)\|_{H^2} + \|v_t(t)\|_{H^1} + \|u_x(t)\|_{H^1} + \|v_x(t)\|_{H^3}\big), \tag{2.22}$$

$$\|w_{txx}(t)\| \le C_1\big(\|\rho_x(t)\|_{H^2} + \|w_x(t)\|_{H^3}\big), \tag{2.23}$$

or

$$\|u_{xxxx}(t)\| \le C_1\big(\|\rho_x(t)\|_{H^2} + \|u_t(t)\|_{H^2} + \|v_x(t)\|_{H^1} + \|u_x(t)\|_{H^2}\big), \tag{2.24}$$

$$\|v_{xxxx}(t)\| \le C_1\big(\|\rho_x(t)\|_{H^2} + \|v_t(t)\|_{H^2} + \|u_x(t)\|_{H^1} + \|v_x(t)\|_{H^2}\big), \tag{2.25}$$

$$\|w_{xxxx}(t)\| \le C_1\big(\|\rho_x(t)\|_{H^2} + \|w_x(t)\|_{H^2} + \|w_{txx}(t)\|\big). \tag{2.26}$$

Differentiating (1.20)-(1.22) with respect to t, respectively, we deduce

$$\|u_{tt}(t)\| \le C_1\big(\|u_t(t)\|_{H^2} + \|v_t(t)\| + \|\rho_x(t)\| + \|u_x(t)\|_{H^1}\big), \tag{2.27}$$

$$\|v_{tt}(t)\| \le C_1\big(\|v_t(t)\|_{H^2} + \|u_t(t)\| + \|\rho_x(t)\| + \|u_x(t)\|_{H^1}\big), \tag{2.28}$$

$$\|w_{tt}(t)\| \le C_1\big(\|u_{tx}(t)\|_{H^1} + \|\rho_x(t)\| + \|u_x(t)\|_{H^1}\big). \tag{2.29}$$

Now differentiating (1.20) with respect to t twice, multiplying the resulting equation by $(\frac{u}{r})_{tt}$ in $L^2[0,1]$, and using integration by parts and (1.19), we conclude

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{u}{r}\right)_{tt}^2 dx &= -\int_0^1 \big(-\rho^\gamma + \alpha\rho^{\alpha+1}(\rho u)_x\big)_{tt}\left(\frac{u}{r}\right)_{ttx} dx \\ &\quad + \int_0^1 \left(\frac{v^2-u^2}{r^2}\right)_{tt}\left(\frac{u}{r}\right)_{tt} dx - \int_0^1 \left(\frac{(\rho^\alpha)_x u}{r}\right)_{tt}\left(\frac{u}{r}\right)_{tt} dx \\ &= A_1 + A_2 + A_3.\end{aligned} \tag{2.30}$$

Employing Lemmas 2.1-2.2, (2.27)-(2.28), (1.19), and the interpolation inequality, we get, for any small $\varepsilon \in (0,1)$,

$$\begin{aligned}
A_1 &= -\int_0^1 \left(-\rho^\gamma + \alpha\rho^{\alpha+1}(\rho u)_x\right)_{tt}\left(\frac{u}{r}\right)_{ttx} dx \\
&\le -\alpha\int_0^1 \frac{\rho^{\alpha+2}}{r}u_{ttx}^2\,dx + \varepsilon\|u_{ttx}(t)\|^2 + C_2\big(\|\rho_{tt}(t)\|_{H^1}^2 + \|\rho_t(t)\|_{H^1}^2 \\
&\quad + \|u_t(t)\|_{H^1}^2 + \|u_{tt}(t)\|^2 + \|u_x(t)\|_{H^1}^2 + \|\rho_x(t)\|_{H^1}^2\big) \\
&\le -C_2^{-1}\|u_{ttx}(t)\|^2 + C_2\big(\|u_x(t)\|_{H^2}^2 + \|\rho_x(t)\|_{H^1}^2 + \|u_t(t)\|_{H^2}^2\big), \qquad (2.31)\\
A_2 &= \int_0^1 \left(\frac{v^2-u^2}{r^2}\right)_{tt}\left(\frac{u}{r}\right)_{tt} dx \\
&\le C_2\big(\|u_t(t)\|_{H^1}^2 + \|v_t(t)\|_{H^1}^2 + \|u_{tt}(t)\|^2 + \|v_{tt}(t)\|^2 + \|u_x(t)\|^2\big) \\
&\le C_2\big(\|u_t(t)\|_{H^2} + \|v_t(t)\|_{H^2} + \|\rho_x(t)\| + \|u_x(t)\|_{H^1}\big), \qquad (2.32)\\
A_3 &= -\int_0^1 \left(\frac{(\rho^\alpha)_x u}{r}\right)_{tt}\left(\frac{u}{r}\right)_{tt} dx \\
&\le C_2\big(\|u_{tt}(t)\|^2 + \|u_t(t)\|_{H^1}^2 + \|\rho_{tx}(t)\|^2 + \|\rho_{ttx}(t)\|^2 + \|\rho_x(t)\|^2\big) \\
&\le C_2\big(\|u_t(t)\|_{H^2}^2 + \|\rho_x(t)\|^2 + \|u_x(t)\|_{H^1}^2\big). \qquad (2.33)
\end{aligned}$$

Integrating (2.30) with respect to t, applying Lemmas 2.1-2.2, initial condition (1.26), and (2.31)-(2.33), we obtain (2.9).

Differentiating (1.21) with respect to t twice, multiplying the resulting equation by $(\frac{v}{r})_{tt}$ in $L^2[0,1]$, and using integration by parts, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{v}{r}\right)_{tt}^2 dx &= -\int_0^1 \left(\rho^{\alpha+1}(\rho v)_x\right)_{tt}\left(\frac{v}{r}\right)_{ttx} dx - \int_0^1 \left(\frac{(\rho^\alpha)_x v}{r}\right)_{tt}\left(\frac{v}{r}\right)_{tt} dx \\
&= B_1 + B_2, \qquad (2.34)
\end{aligned}$$

where

$$B_1(t) = -\int_0^1 \left(\rho^{\alpha+1}(\rho v)_x\right)_{tt}\left(\frac{v}{r}\right)_{ttx} dx, \qquad B_2(t) = -\int_0^1 \left(\frac{(\rho^\alpha)_x v}{r} - \frac{2uv}{r^2}\right)_{tt}\left(\frac{v}{r}\right)_{tt} dx.$$

Using the interpolation inequality, Lemmas 2.1-2.2 and (2.27)-(2.28), we obtain for $\epsilon \in (0,1)$,

$$\begin{aligned}
B_1(t) &\le -\int_0^1 \frac{\rho^{\alpha+2}}{r}v_{ttx}^2\,dx + \epsilon\|v_{ttx}(t)\|^2 + C_2\big(\|u_t(t)\|_{H^1}^2 + \|v_t(t)\|_{H^1}^2 + \|v_{tt}(t)\|^2 \\
&\quad + \|\rho_{tx}(t)\|^2 + \|\rho_{ttx}(t)\|^2 + \|\rho_x(t)\|^2 + \|v_x(t)\|^2\big) \\
&\le -\int_0^1 \frac{\rho^{\alpha+2}}{r}v_{ttx}^2\,dx + \epsilon\|v_{ttx}(t)\|^2 + C_2\big(\|v_x(t)\|_{H^1}^2 + \|v_t(t)\|_{H^1}^2 \\
&\quad + \|u_t(t)\|_{H^2}^2 + \|\rho_x(t)\|_{H^1}^2 + \|u_x(t)\|_{H^1}^2\big), \qquad (2.35)
\end{aligned}$$

$$\begin{aligned}B_2(t) &\le C_2\big(\|v_{tt}(t)\|^2 + \|v_t(t)\|_{H^1}^2 + \|u_t(t)\|_{H^1}^2 + \|u_{tt}(t)\|^2 + \|\rho_{ttx}(t)\|^2 \\ &\quad + \|\rho_{tx}(t)\|^2 + \|\rho_x(t)\|_{H^1}^2 + \|u_x(t)\|_{H^1}^2\big) \\ &\le C_2\big(\|v_t(t)\|_{H^2}^2 + \|u_t(t)\|_{H^2}^2 + \|\rho_x(t)\|_{H^1}^2 + \|u_x(t)\|_{H^1}^2\big).\end{aligned} \tag{2.36}$$

Integrating (2.34) with respect to t, using Lemmas 2.1-2.2 and (2.35)-(2.36), and picking ϵ small enough, we conclude (2.10).

Similarly, differentiating (1.22) with respect to t twice, multiplying the resulting equation by w_{tt} in $L^2[0,1]$, and using integration by parts, we deduce

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 w_{tt}^2(x,t)\,dx \\ &\quad = -\int_0^1 \big(r^2\rho^{1+\alpha}w_x\big)_{tt} w_{ttx}\,dx \\ &\quad \le -\int_0^1 r^2\rho^{1+\alpha}w_{ttx}^2(x,t)\,dx + C_2\big(\|w_x\| + \|u_t w_x\| + \|\rho_t w_x\| + \|\rho_t w_{tx}\| \\ &\qquad + \|\rho_{tt} w_x\| + \|\rho_t w_{tx}\|\big)\|w_{ttx}\| \\ &\quad \le -C_1^{-1}\|w_{ttx}\|^2 + C_2\big(\|w_x(t)\|^2 + \|u_t(t)\|_{H^1}^2 + \|w_t(t)\|_{H^1}^2 + \|u_x(t)\|^2\big).\end{aligned} \tag{2.37}$$

Integrating (2.37) with respect to t, using Lemmas 2.1-2.2, we derive the estimate (2.11). The proof is complete. □

Lemma 2.4 *Under the assumptions in Theorem* 1.1, *the following estimates hold for any* $T > 0$ *and* $\varepsilon \in (0,1)$:

$$\|u_{tx}(t)\|^2 + \int_0^t \|u_{txx}(s)\|^2\,ds \le C_4 + C_2^{-1}\varepsilon^2\int_0^t \|u_{ttx}(s)\|^2\,ds, \quad t \in [0,T], \tag{2.38}$$

$$\|v_{tx}(t)\|^2 + \int_0^t \|v_{txx}(s)\|^2\,ds \le C_4 + C_2^{-1}\varepsilon^2\int_0^t \|v_{ttx}(s)\|^2\,ds, \quad t \in [0,T], \tag{2.39}$$

$$\|w_{tx}(t)\|^2 + \int_0^t \|w_{txx}(s)\|^2\,ds \le C_4, \quad t \in [0,T]. \tag{2.40}$$

Proof Differentiating (1.20) with respect to t and x, then multiplying the result by $(\frac{u}{r})_{tx}$ in $L^2[0,1]$, and integrating by parts, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{u}{r}\right)_{tx}^2 dx \\ &\quad = \big(-\rho^\gamma + \alpha\rho^{\alpha+1}(\rho u)_x\big)_{tx}\left(\frac{u}{r}\right)_{tx}\Big|_0^1 - \int_0^1 \big(-\rho^\gamma + \alpha\rho^{\alpha+1}(\rho u)_x\big)_{tx}\left(\frac{u}{r}\right)_{txx} dx \\ &\qquad + \int_0^1 \left(\frac{v^2-u^2}{r^2} - \frac{(\rho^\alpha)_x u}{r}\right)_{tx}\left(\frac{u}{r}\right)_{tx} dx \\ &\quad = D_0(x,t) + D_1(t) + D_2(t),\end{aligned} \tag{2.41}$$

where

$$D_0(x,t) = \left(-\rho^\gamma + \alpha\rho^{\alpha+1}(\rho u)_x\right)_{tx}\left(\frac{u}{r}\right)_{tx}\Big|_0^1,$$
$$D_1(t) = -\int_0^1 \left(-\rho^\gamma + \alpha\rho^{\alpha+1}(\rho u)_x\right)_{tx}\left(\frac{u}{r}\right)_{txx} dx,$$
$$D_2(t) = \int_0^1 \left(\frac{v^2-u^2}{r^2} - \frac{(\rho^\alpha)_x u}{r}\right)_{tx}\left(\frac{u}{r}\right)_{tx} dx.$$

Now employing Lemmas 2.1-2.2 and the interpolation inequality, using Young's inequality several times, we have, for $\varepsilon \in (0,1)$,

$$\begin{aligned} D_0(x,t) &\le C_2\big(\|\rho_x\|_{L^\infty}^2 + \|\rho_x u_x\|_{L^\infty} + \|u_{xx}\|_{L^\infty} + \left\|\rho_x^2 u_x\right\|_{L^\infty} \\ &\quad + \left\|\rho_x u_x^2\right\|_{L^\infty} + \|u_x u_{xx}\|_{L^\infty} + \|\rho_x u_{xx}\|_{L^\infty} + \|\rho_x u_{tx}\|_{L^\infty} \\ &\quad + \left\|u_x^2\right\|_{L^\infty} + \|u_{txx}\|_{L^\infty}\big)\|u_{tx}\|_{L^\infty} \\ &\le C_2\big(\|u_x\|_{H^2} + \|\rho_x\|_{H^1} + \|u_{tx}\|^{\frac{1}{2}}\|u_{txx}\|^{\frac{1}{2}} \\ &\quad + \|u_{txx}\|^{\frac{1}{2}}\|u_{txxx}\|^{\frac{1}{2}}\big)\|u_{tx}\|^{\frac{1}{2}}\|u_{txx}\|^{\frac{1}{2}} \\ &\le C_2^{-1}\varepsilon^2\big(\|u_{txx}\|^2 + \|u_{txxx}\|^2\big) \\ &\quad + C_2\varepsilon^{-6}\big(\|u_{tx}\|^2 + \|u_x\|_{H^2}^2 + \|\rho_x\|_{H^1}^2\big), \end{aligned} \tag{2.42}$$

$$\begin{aligned} D_1(t) &\le -\alpha\int_0^1 \rho^{\alpha+1}u_{txx}^2\,dx + \varepsilon\left\|u_{txx}(t)\right\|^2 \\ &\quad + C_2\big(\left\|\rho_x(t)\right\|_{H^1}^2 + \left\|u_x(t)\right\|_{H^2}^2 \\ &\quad + \left\|u_{tx}(t)\right\|^2 + \left\|u_t(t)\right\|^2\big), \end{aligned} \tag{2.43}$$

$$\begin{aligned} D_2(t) &\le C_2\big(\|u_x u_t\| + \|u_t\|_{H^1} + \|v_t\|_{H^1} + \|u_x\| + \|v_x\| + \left\|u^3\right\| + \left\|v^2\right\| \\ &\quad + \|\rho_{tx}\|_{H^1} + \|\rho_{tx}u_x\| + \|\rho_{xx}u_t\| + \|\rho_x u_{tx}\| + \|\rho_{xx}\| \\ &\quad + \|\rho_x u_x\|\big)\big(\|u_{tx}\| + \|u_t\| + \|u_x\| + \left\|u^2\right\|\big) \\ &\le C_2\big(\|u_x\|_{H^2}^2 + \|\rho_x\|_{H^1}^2 + \|v_t\|_{H^1}^2 + \|u_t\|_{H^1}^2 + \|v_x\|_{H^1}^2\big). \end{aligned} \tag{2.44}$$

On the other hand, we differentiate (1.20)-(1.22) with respect to x and t, and use Lemmas 2.1-2.2 and (2.12)-(2.29) to conclude

$$\left\|u_{txxx}(t)\right\| \le C_2\big(\left\|u_x(t)\right\|_{H^2} + \left\|\rho_x(t)\right\|_{H^1} + \left\|u_t(t)\right\|_{H^2} + \left\|u_{ttx}(t)\right\|\big), \tag{2.45}$$
$$\left\|v_{txxx}(t)\right\| \le C_2\big(\left\|u_x(t)\right\|_{H^2} + \left\|\rho_x(t)\right\|_{H^1} + \left\|v_t(t)\right\|_{H^2} + \left\|v_{ttx}(t)\right\|\big), \tag{2.46}$$
$$\left\|w_{txxx}(t)\right\| \le C_2\big(\left\|\rho_x(t)\right\|_{H^1} + \left\|w_{tx}(t)\right\|_{H^1} + \left\|w_{ttx}(t)\right\|\big). \tag{2.47}$$

Integrating (2.41) with respect to t, using (2.42)-(2.45) and Lemmas 2.1-2.2, we obtain (2.38).

Analogously, differentiating (1.21) with respect to t and x, then multiplying the resultant by $(\frac{v}{r})_{tx}$ in $L^2[0,1]$, and integrating by parts, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{v}{r}\right)_{tx}^2 dx \\
&= \left(\rho^{\alpha+1}(\rho v)_x\right)_{tx}\left(\frac{v}{r}\right)_{tx}\Big|_0^1 - \int_0^1 \left(\rho^{\alpha+1}(\rho v)_x\right)_{tx}\left(\frac{v}{r}\right)_{txx} dx \\
&\quad + \int_0^1 \left(\frac{2uv}{r^2} - \frac{(\rho^\alpha)_x v}{r}\right)_{tx}\left(\frac{v}{r}\right)_{tx} dx \\
&= E_0(x,t) + E_1(t) + E_2(t),
\end{aligned} \tag{2.48}$$

where

$$\begin{aligned}
&E_0(x,t) = \left(\rho^{\alpha+1}(\rho v)_x\right)_{tx}\left(\frac{v}{r}\right)_{tx}\Big|_0^1, \qquad E_1(t) = -\int_0^1 \left(\rho^{\alpha+1}(\rho v)_x\right)_{tx}\left(\frac{v}{r}\right)_{txx} dx, \\
&E_2(t) = \int_0^1 \left(\frac{2uv}{r^2} - \frac{(\rho^\alpha)_x v}{r}\right)_{tx}\left(\frac{v}{r}\right)_{tx} dx.
\end{aligned}$$

Now we apply the interpolation inequality and Young's inequality to estimate $E_0(x,t)$, $E_1(t)$, $E_2(t)$ for any $\varepsilon > 0$,

$$\begin{aligned}
E_0(x,t) &\le C_2\big(\|v_{txx}\|_{L^\infty} + \|v_{tx}\|_{L^\infty} + \|v_t\rho_x\|_{L^\infty} + \|v_t\|_{L^\infty} + \|u_{xx}\|_{L^\infty} + \|u_x v_x\|_{L^\infty} \\
&\qquad + \|v_{xx}\|_{L^\infty} + \|u_x\|_{L^\infty} + \|\rho_x\|_{L^\infty}\big)\|v_{tx}\|_{L^\infty} \\
&\le C_2\big(\|u_x\|_{H^2} + \|v_x\|_{H^2} + \|\rho_x\|_{H^1} + \|v_t\|_{H^1} \\
&\qquad + \|v_{tx}\|^{\frac{1}{2}}\|v_{txx}\|^{\frac{1}{2}} + \|v_{txx}\|^{\frac{1}{2}}\|v_{txxx}\|^{\frac{1}{2}}\big)\|v_{tx}\|^{\frac{1}{2}}\|v_{txx}\|^{\frac{1}{2}} \\
&\le C_2^{-1}\varepsilon^2\big(\|v_{txx}(t)\|^2 + \|v_{txxx}(t)\|^2\big) + C_2\varepsilon^{-6}\big(\|v_{tx}(t)\|^2 + \|u_x(t)\|_{H^2} \\
&\qquad + \|v_x(t)\|_{H^2} + \|\rho_x(t)\|_{H^1}\big),
\end{aligned} \tag{2.49}$$

$$\begin{aligned}
E_1(t) &\le -\int_0^1 \rho^{\alpha+1}v_{txx}^2\, dx + \frac{1}{2}\int_0^1 \rho^{\alpha+1}v_{txx}^2\, dx + C_2\big(\|u_x\|_{H^2}^2 + \|v_x\|_{H^1}^2 \\
&\qquad + \|\rho_x\|_{H^1}^2 + \|v_t\|^2 + \|v_{tx}\|^2\big),
\end{aligned} \tag{2.50}$$

$$\begin{aligned}
E_2(t) &\le C_2\big(\|u_{tx}\| + \|u_t v_x\| + \|u_x v_t\| + \|v_{tx}\| + \|u_t\| + \|v_t\| + \|u_x\| \\
&\qquad + \|v_x\| + \|\rho_{txx}\| + \|\rho_{xt}v_x\| + \|\rho_{xx}v_t\| + \|\rho_x v_{tx}\| + \|\rho_{xt}\| + \|\rho_x v_t\| \\
&\qquad + \|\rho_{xx}\| + \|\rho_x v_x\| + \|\rho_x u_x\|\big)\big(\|v_{tx}\| + \|v_t\| + \|u_x\| + \|v_x\|\big) \\
&\le C_2\big(\|u_t\|_{H^1}^2 + \|v_t\|_{H^1}^2 + \|u_x\|_{H^1}^2 + \|\rho_x\|_{H^1}^2 + \|v_x\|_{H^1}^2\big).
\end{aligned} \tag{2.51}$$

Inserting (2.46) into (2.49), integrating (2.48) with respect to t, and using (2.49)-(2.51), we can get (2.39).

Differentiating (1.22) with respect to x and t, multiplying the resulting equation by w_{tx} in $L^2[0,1]$, integrating by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w_{tx}\|^2 &= \big(r^2\rho^{1+\alpha}w_x\big)_{tx}w_{tx}\big|_0^1 - \int_0^1 \big(r^2\rho^{1+\alpha}w_x\big)_{tx}w_{txx}\,dx \\ &= F_0(x,t) + F_1(t). \end{aligned} \tag{2.52}$$

Employing Lemmas 2.1-2.2, the interpolation inequality and (2.47), we infer for any $\varepsilon \in (0,1)$ that

$$\begin{aligned} F_0(x,t) &= \big(r^2\rho^{1+\alpha}w_x\big)_{tx}w_{tx}\big|_0^1 \\ &\le C_2\big(\|w_x\|_{L^\infty} + \|w_xu_x\|_{L^\infty} + \|\rho_xw_x\|_{L^\infty} + \|w_{xx}\|_{L^\infty} \\ &\quad + \|w_{tx}\|_{L^\infty} + \|w_{txx}\|_{L^\infty}\big)\|w_{tx}\|_{L^\infty} \\ &\le \varepsilon^2\big(\|w_{txx}\|^2 + \|w_{txxx}\|^2\big) + C_2\big(\|w_x\|_{H^2}^2 + \|\rho_x\|_{H^1}^2 + \|u_x\|_{H^1}^2 + \|w_{tx}\|^2\big) \\ &\le \varepsilon^2\big(\|w_{txx}\|^2 + \|w_{ttx}\|^2\big) + C_2\big(\|w_x\|_{H^2}^2 + \|\rho_x\|_{H^1}^2 + \|u_x\|_{H^1}^2 + \|w_{tx}\|^2\big), \end{aligned} \tag{2.53}$$

$$\begin{aligned} F_1(t) &= -\int_0^1 \big(r^2\rho^{1+\alpha}w_x\big)_{tx}w_{txx}\,dx \\ &\le -\int_0^1 r^2\rho^{1+\alpha}w_{txx}^2\,dx + C_2\big(\|w_x\| + \|w_xu_x\| + \|w_x\rho_x\| + \|w_{xx}\| \\ &\quad + \|w_{xt}\| + \|\rho_xw_{tx}\|\big)\|w_{txx}\| \\ &\le -C_1^{-1}\|w_{txx}\|^2 + C_2\big(\|w_x\|_{H^1}^2 + \|w_t\|_{H^1}^2 + \|\rho_x\|_{H^1}^2\big). \end{aligned} \tag{2.54}$$

Integrating (2.52) with respect to t, picking ε small enough, using Lemmas 2.1-2.3 and (2.53)-(2.54), we derive the estimate (2.40). The proof is complete. □

Lemma 2.5 *Under the assumptions in Theorem* 1.1, *the following estimates hold for any* $T > 0$:

$$\big\|u_{tt}(t)\big\|^2 + \big\|u_{tx}(t)\big\|^2 + \int_0^t \big(\|u_{txx}\|^2 + \|u_{ttx}\|^2\big)(s) \le C_4, \quad t \in [0,T], \tag{2.55}$$

$$\big\|v_{tt}(t)\big\|^2 + \big\|v_{tx}(t)\big\|^2 + \int_0^t \big(\|v_{ttx}\|^2 + \|v_{txx}\|^2\big)(s) \le C_4, \quad t \in [0,T], \tag{2.56}$$

$$\begin{aligned} &\big\|\rho_{xxx}(t)\big\|^2 + \big\|u_{xxx}(t)\big\|^2 + \big\|v_{xxx}(t)\big\|^2 + \big\|w_{xxx}(t)\big\|^2 + \int_0^t \big(\|\rho_{xxx}\|^2 + \|u_{xxxx}\|^2 \\ &\quad + \|v_{xxxx}\|^2 + \|w_{xxxx}\|^2\big)(s)\,ds \le C_4, \quad t \in [0,T]. \end{aligned} \tag{2.57}$$

Proof Adding (2.38) and (2.39), using (2.9)-(2.10) and picking ε small enough, we easily obtain (2.55)-(2.56).

Differentiating (2.7) with respect to x, we have

$$\alpha\big(\rho^{\alpha-1}\rho_{xxx}\big)_t + \gamma\rho^{\gamma-1}\rho_{xxx} = M_1(x,t), \tag{2.58}$$

where

$$M_1(x,t) = M_{0x}(x,t) - (\alpha-1)\big(\rho^{\alpha-2}\rho_x\rho_{xx}\big)_t - \gamma(\gamma-1)\rho_x\rho_{xx}.$$

An easy calculation with the interpolation inequality, Lemmas 2.1-2.2, and (2.55)-(2.56) gives

$$\begin{aligned}\|M_1(t)\| &\le C_2\big(\|M_{0x}\| + \|\rho_x\rho_{xx}\| + \|\rho_{xt}\rho_{xx}\| + \|\rho_x\rho_{txx}\|\big)\\ &\le C_2\big(\|v_x^2\| + \|v_{xx}\| + \|\rho_x\rho_{txx}\| + \|\rho_{xx}\rho_{tx}\| + \|\rho_x^2\rho_{tx}\| + \|\rho_t\rho_x\rho_{xx}\|\\ &\quad + \|\rho_x\rho_{xx}\| + \|\rho_{xxx}\| + \|\rho_{xx}u_x\| + \|u_{tx}\| + \|u_{txx}\| + \|u_t\| + \|u_t\rho_x\|\big)\\ &\le C_2\big(\|\rho_x(t)\|_{H^2}^2 + \|u_x(t)\|_{H^2}^2 + \|v_x(t)\|_{H^1}^2 + \|u_t(t)\|_{H^2}^2\big),\end{aligned}$$

which, along with Lemmas 2.1-2.2 and (2.55)-(2.56), implies

$$\int_0^t \|M_1(s)\|^2\,ds \le C_4 + C_2\int_0^t \|\rho_{xxx}(s)\|^2\,ds, \quad \forall t\in[0,T]. \tag{2.59}$$

Multiplying (2.58) by $\rho^{\alpha-1}\rho_{xxx}$ in $L^2[0,1]$, we deduce

$$\frac{\alpha}{2}\frac{d}{dt}\|\rho^{\alpha-1}\rho_{xxx}\|^2 + \gamma\int_0^1 \rho^{\gamma+\alpha-2}\rho_{xxx}^2\,dx \le C_1\|M_1(t)\|^2, \tag{2.60}$$

which implies

$$\frac{d}{dt}\|\rho^{\alpha-1}\rho_{xxx}\|^2 \le C_1\|M_1(t)\|^2. \tag{2.61}$$

Integrating (2.61) with respect to t, using (2.59), we conclude

$$\|\rho^{\alpha-1}\rho_{xxx}\|^2 \le C_4 + C_2\int_0^t \|\rho_{xxx}(s)\|^2\,ds,$$

which, by virtue of Gronwall's inequality and (2.60), gives

$$\|\rho_{xxx}(t)\|^2 + \int_0^t \|\rho_{xxx}(s)\|^2\,ds \le C_4, \quad \forall t\in[0,T]. \tag{2.62}$$

By (2.18)-(2.20), (2.55)-(2.56), and Lemmas 2.1-2.4, we conclude

$$\|u_{xxx}(t)\|^2 + \|v_{xxx}(t)\|^2 + \|w_{xxx}(t)\|^2 \le C_4, \quad \forall t\in[0,T]. \tag{2.63}$$

By virtue of (2.24)-(2.26), (2.55)-(2.56), (2.62), and Lemmas 2.1-2.4, we can get

$$\int_0^t \big(\|u_{xxxx}\|^2 + \|v_{xxxx}\|^2 + \|w_{xxxx}\|^2\big)(s)\,ds \le C_4, \quad t\in[0,T],$$

which, along with (2.62)-(2.63), gives (2.57). The proof is complete. □

Lemma 2.6 *Under the assumptions in Theorem* 1.1, *the following estimates hold for any* $T > 0$:

$$\|u_{txx}(t)\|^2 + \|v_{txx}(t)\|^2 + \|w_{txx}(t)\|^2 \le C_4, \quad t \in [0,T], \tag{2.64}$$

$$\|\rho_{xxxx}(t)\|^2 + \int_0^t \|\rho_{xxxx}(s)\|^2\,ds \le C_4, \quad t \in [0,T], \tag{2.65}$$

$$\begin{aligned}&\|u_{xxxx}(t)\|^2 + \|v_{xxxx}(t)\|^2 + \|w_{xxxx}(t)\|^2 + \int_0^t \big(\|u_{xxxxx}\|^2 \\ &\quad + \|v_{xxxxx}\|^2 + \|w_{xxxxx}\|^2\big)(s)\,ds \le C_4, \quad t \in [0,T].\end{aligned} \tag{2.66}$$

Proof Differentiating (1.20)-(1.22) with respect to t, respectively, we deduce

$$\|u_{txx}(t)\| \le C_1\big(\|u_t(t)\|_{H^1} + \|v_t(t)\| + \|\rho_x(t)\| + \|u_x(t)\|_{H^1} + \|u_{tt}(t)\|\big), \tag{2.67}$$

$$\|v_{txx}(t)\| \le C_1\big(\|v_t(t)\|_{H^1} + \|u_t(t)\| + \|\rho_x(t)\| + \|u_x(t)\|_{H^1} + \|v_{tt}(t)\|\big), \tag{2.68}$$

$$\|w_{txx}(t)\| \le C_1\big(\|w_{tx}(t)\| + \|\rho_x(t)\| + \|u_x(t)\|_{H^1} + \|w_x(t)\|_{H^1} + \|w_{tt}(t)\|\big). \tag{2.69}$$

By virtue of Lemmas 2.1-2.5 and estimates (2.67)-(2.69), we conclude (2.64).

Differentiating (2.58) with respect to x, we have

$$\alpha\big(\rho^{\alpha-1}\rho_{xxxx}\big)_t + \gamma\rho^{\gamma-1}\rho_{xxxx} = M_2(x,t), \tag{2.70}$$

where

$$M_2(x,t) = M_{1x}(x,t) - (\alpha-1)\big(\rho^{\alpha-2}\rho_x\rho_{xxx}\big)_t - \gamma(\gamma-1)\rho_x\rho_{xxx}$$

and

$$M_{1x}(x,t) = M_{0xx}(x,t) - \gamma(\gamma-1)(\rho_x\rho_{xx})_x - (\alpha-1)\big(\rho^{\alpha-2}\rho_x\rho_{xx}\big)_{tx}.$$

Using the interpolation inequality, and the embedding theorem, Lemmas 2.1-2.5, we can deduce that

$$\|M_2(t)\| \le C_4\big(\|u_x(t)\|_{H^3} + \|\rho_x(t)\|_{H^3} + \|u_{tx}(t)\|_{H^2} + \|v_x(t)\|_{H^2}\big). \tag{2.71}$$

Inserting (2.45) into (2.71), and integrating (2.71) with respect to t over $[0,T]$, using Lemmas 2.1-2.5, we have

$$\int_0^t \|M_2(s)\|^2\,ds \le C_4 + C_2\int_0^t \|\rho_{xxxx}(s)\|^2\,ds, \quad \forall t \in [0,T]. \tag{2.72}$$

Multiplying (2.70) by $\rho^{\alpha-1}\rho_{xxxx}$ in $L^2[0,1]$, we can get

$$\frac{\alpha}{2}\frac{d}{dt}\|\rho^{\alpha-1}\rho_{xxxx}\|^2 + \gamma\int_0^1 \rho^{\gamma+\alpha-2}\rho_{xxxx}^2\,dx \le C_2\|M_2(t)\|^2, \tag{2.73}$$

which implies

$$\frac{d}{dt}\left\|\rho^{\alpha-1}\rho_{xxxx}\right\|^2 \leq C_2\left\|M_2(t)\right\|^2. \tag{2.74}$$

Integrating (2.74) with respect to t over $[0,T]$, using (2.72), we conclude

$$\left\|\rho^{\alpha-1}\rho_{xxxx}\right\|^2 \leq C_4 + C_2\int_0^t\left\|\rho_{xxxx}(s)\right\|^2 ds, \quad t\in[0,T], \tag{2.75}$$

which, by virtue of Gronwall's inequality, gives

$$\left\|\rho_{xxxx}(t)\right\|^2 \leq C_4, \quad t\in[0,T]. \tag{2.76}$$

Thus, we can obtain (2.65) by virtue of (2.75)-(2.76).

By (2.24)-(2.26), (2.64)-(2.65), (2.45)-(2.47), and Lemmas 2.1-2.5, we deduce that

$$\begin{aligned}&\left\|u_{xxxx}(t)\right\|^2 + \left\|v_{xxxx}(t)\right\|^2 + \left\|w_{xxxx}(t)\right\|^2\\ &\quad + \int_0^t\left(\|u_{txxx}\|^2 + \|v_{txxx}\|^2 + \|w_{txxx}\|^2\right)(s)\,ds \leq C_4, \quad t\in[0,T].\end{aligned} \tag{2.77}$$

On the other hand, we differentiate (1.20)-(1.22) with respect to x three times, use Lemmas 2.1-2.5 and (2.64)-(2.65) to conclude, for any $t\in[0,T]$,

$$\left\|u_{xxxxx}(t)\right\| \leq C_4\left(\left\|u_{txxx}(t)\right\| + \left\|u_x(t)\right\|_{H^3} + \left\|\rho_x(t)\right\|_{H^3} + \left\|v_x(t)\right\|_{H^3}\right), \tag{2.78}$$

$$\left\|v_{xxxxx}(t)\right\| \leq C_4\left(\left\|v_{txxx}(t)\right\| + \left\|u_x(t)\right\|_{H^3} + \left\|\rho_x(t)\right\|_{H^3} + \left\|v_x(t)\right\|_{H^3}\right), \tag{2.79}$$

$$\left\|w_{xxxxx}(t)\right\| \leq C_4\left(\left\|w_{txxx}(t)\right\| + \left\|u_x(t)\right\|_{H^3} + \left\|\rho_x(t)\right\|_{H^3} + \left\|w_x(t)\right\|_{H^3}\right). \tag{2.80}$$

Thus we conclude from (2.77)-(2.80), (2.64)-(2.65), and Lemmas 2.1-2.5 that

$$\int_0^t\left(\|u_{xxxxx}\|^2 + \|v_{xxxxx}\|^2 + \|w_{xxxxx}\|^2\right)(s)\,ds \leq C_4, \quad \forall t\in[0,T]$$

which, combined with (2.77), implies (2.66). The proof is complete. □

Proof of Theorem 1.1. Applying Lemmas 2.1-2.6, we readily get estimate (1.27)-(1.30) and complete the proof of Theorem 1.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally.

Acknowledgements
This research was supported in part by NSFC (No. 11501199) and the Natural Science Foundation of Henan Province (No. 14B110037).

Received: 26 January 2016 Accepted: 28 March 2016 Published online: 12 April 2016

References

1. Liu, J, Lian, R: Global existence of the cylindrically symmetric strong solution to compressible Navier-Stokes equations. Abstr. Appl. Anal. **2014**, Article ID 728715 (2014)
2. Fang, D, Zhang, T: Global solutions of the Navier-Stokes equations for compressible flow with density-dependent viscosity and discontinuous initial data. J. Differ. Equ. **222**, 63-94 (2006)
3. Jiang, S, Xin, Z, Zhang, P: Global weak solutions to 1D compressible isentropic Navier-Stokes equations with density-dependent viscosity. Methods Appl. Anal. **12**, 239-252 (2005)
4. Okada, M, Matušu-Nečasová, Š, Makino, T: Free boundary problem for the equation of one-dimensional motion of compressible gas with density-dependent viscosity. Ann. Univ. Ferrara, Sez. 7: Sci. Mat. **48**, 1-20 (2002)
5. Qin, Y: Nonlinear Parabolic-Hyperbolic Coupled Systems and Their Attractors. In: Operator Theory: Advances and Applications. Advances in Partial Differential Equations, vol. 184. Birkhäuser, Basel (2008)
6. Qin, Y, Huang, L, Yao, Z: Regularity of 1D compressible isentropic Navier-Stokes equations with density-dependent viscosity. J. Differ. Equ. **245**, 3956-3973 (2008)
7. Vong, S, Yang, T, Zhu, C: Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum (II). J. Differ. Equ. **192**, 475-501 (2003)
8. Yang, T, Yao, Z, Zhu, C: Compressible Navier-Stokes equations with density-dependent viscosity and vacuum. Commun. Partial Differ. Equ. **26**, 965-981 (2001)
9. Yang, T, Zhao, H: A vacuum problem for the one-dimensional compressible Navier-Stokes equations with density dependent viscosity. J. Differ. Equ. **184**, 163-184 (2002)
10. Yang, T, Zhu, C: Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum. Commun. Math. Phys. **230**, 329-363 (2002)
11. Zhu, C: Asymptotic behavior of compressible Navier-Stokes equations with density-dependent viscosity and vacuum. Commun. Math. Phys. **293**, 279-299 (2010)
12. Lian, R, Liu, J, Li, H, Xiao, L: Cauchy problem for the one-dimensional compressible Navier-Stokes equations. Acta Math. Sci. **32**, 315-324 (2012)
13. Bresch, D, Desjardins, B: Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model. Commun. Math. Phys. **238**, 211-223 (2003)
14. Bresch, D, Desjardins, B: On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier-Stokes models. J. Math. Pures Appl. **86**, 362-368 (2006)
15. Cui, H, Yao, Z: Asymptotic behavior of compressible *p*-th power Newtonian fluid with large initial data. J. Differ. Equ. **258**, 919-953 (2015)
16. Feireisl, E: Dynamics of Viscous Compressible Fluids. Oxford University Press, Oxford (2004)
17. Frid, H, Shelukhin, V: Boundary layers for the Navier-Stokes equations of compressible fluids. Commun. Math. Phys. **208**, 309-330 (1999)
18. Frid, H, Shelukhin, VV: Vanishing shear viscosity in the equations of compressible fluids for the flows with the cylinder symmetry. SIAM J. Math. Anal. **31**, 1144-1156 (2000)
19. Guo, Z, Li, H, Xin, Z: Lagrange structure and dynamics for solutions to the spherically symmetric compressible Navier-Stokes equations. Commun. Math. Phys. **209**, 371-412 (2012)
20. Huang, L, Lian, R: Exponential stability of spherically symmetric solutions for compressible viscous micropolar fluid. J. Math. Phys. **56**, 071503 (2015)
21. Jiang, S, Zhang, J: Boundary layers for the Navier-Stokes equations of compressible heat-conducting flows with cylindrical symmetry. SIAM J. Math. Anal. **41**, 237-268 (2009)
22. Lian, R, Huang, L, Liu, J: Global solutions to the spherically symmetric compressible Navier-Stokes equations with density-dependent viscosity. J. Appl. Math. **2012**, Article ID 395209 (2012)
23. Lions, PL: Mathematical Topics in Fluids Dynamics. Volume 2: Compressible Models. Oxford Lecture Series in Mathematics and Its Applications. Oxford University Press, Oxford (1996)
24. Qin, Y: Exponential stability for the compressible Navier-Stokes equations with the cylinder symmetry in R^3. Nonlinear Anal., Real World Appl. **11**, 3590-3607 (2010)
25. Qin, Y, Jiang, L: Global existence and exponential stability of solutions in H^4 for the compressible Navier-Stokes equations with the cylinder symmetry. J. Differ. Equ. **249**, 1353-1384 (2010)
26. Yao, L, Zhang, T, Zhu, C: Boundary layers for compressible Navier-Stokes equations with density-dependent viscosity and cylindrical symmetry. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **28**, 677-709 (2011)
27. Zhang, T, Fang, D: Global behavior of spherically symmetric Navier-Stokes equations with density-dependent viscosity. J. Differ. Equ. **236**, 293-341 (2007)
28. Huang, X, Li, J, Xin, Z: Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations. Commun. Pure Appl. Math. **65**, 549-585 (2012)
29. Fan, J, Jiang, S: Zero shear viscosity limit for the Navier-Stokes equations of compressible isentropic fluids with cylindric symmetry. Rend. Semin. Mat. (Torino) **65**, 35-52 (2007)