Electronic Transactions on Numerical Analysis.
Volume 25, pp. 480-510, 2006.

ISSN 1068-9613.

ETNA
Kent State University
etna@mcs.kent.edu

FROM TAYLOR TO QUADRATIC HERMITE-PADÉ POLYNOMIALS*

HERBERT STAHL†

Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. Taylor polynomials, Padé approximants, and algebraic Hermite-Padé approximants form a hierarchy of approximation concepts of growing complexity. In the present contribution we climb this ladder of concepts by reviewing results about the asymptotic behaviour of polynomials that are connected with the three concepts. In each case the concepts are used for the approximation of the exponential function. The review starts with a classical result by G. Szegö about the asymptotic behaviour of zeros of the Taylor polynomials, it is then continued with asymptotic results by E.B. Saff and R.S. Varga about the asymptotic behaviour of zeros and poles of Padé approximants, and in the last part, analogous results are considered with respect to quadatic Hermite-Padé polynomials. Here, known results are reviewed and some new ones are added. The new results are concerned with the non-diagonal case of quadatic Hermite-Padé polynomials.

Key words. Taylor series, Padé approximants, Hermite-Padé polynomials

AMS subject classifications. 41A21, 41A58, 41A63, 30B10

1. Introduction. The present contribution is an extended version of a talk given at the Conference 'Constructive Functions Tech-04' in honor of Eduard B. Saff's 60th birthday. It is concerned with the asymptotic behavior of Taylor, Padé, and Hermite-Padé polynomials (of type I) associated with the exponential function.

The investigations in all three cases are based on a rescaling method that has been introduced by G. Szegö in [39] in the 20th of the last century for the study of Taylor polynomials. The method has then been extended in [29], [32], [34], and [35] by E.B. Saff and R.S. Varga for the study of Padé polynomials and Padé approximants to the exponential function, and lately the method has also been used by the author of the present contribution in [37] and [38] for an investigation of the asymptotic behavior of quadratic Hermite-Padé polynomials of type I and their zeros in the diagonal case. These last investigations will be extended to non-diagonal ray sequences in Section 6 of the present contribution.

The investigations in [29] - [35] deal with Padé approximants in the whole Padé table. Thus, also non-diagonal ray sequences have been considered, and among many other interesting things, it is quite instructive to see how Szegö's earlier results about Taylor polynomials reappear as a limiting case in [35]. Inspired by this aspect of the investigations in [35], an extension of results from [37] to the non-diagonal case has been included in the present contribution.

In the next section we start with formal definitions of all three types of polynomials. In Section 3 we then turn to Szegö's result, followed by a review of E.B. Saff's and R.S. Varga's results in Section 4. Results from [37] and [38] about the asymptotic behavior of diagonal quadratic Hermite-Padé polynomials of type I are reviewed in Section 5, and in Section 6 results from [37] are extended to non-diagonal ray sequences of quadratic Hermite-Padé polynomials.

2. From Taylor Polynomials to Hermite-Padé Polynomials in Three Steps. Let the function f be given by its power series development

$$f(z) \;=\; c_0 + c_1 z + c_2 z^2 + \ldots$$

*Received September 8, 2005. Accepted for publication December 22, 2005. Recommended by D. Lubinsky.

†TFH-Berlin/FB II; Luxemburger Strasse 10; 13353 Berlin; Germany (stahl@tfh-berlin.de). The research has been supported in part by INTAS Research Network NeCCA 03-51-6637, and by the Deutsche Forschungsgemeinschaft (DFG), grant STA 299/13-1.

ETNA
Kent State University
etna@mcs.kent.edu

at the origin.

2.1. Taylor Polynomials. For each $n \in \mathbb{N}$ the Taylor polynomial of the function f is defined as

$$t_n(z) := c_0 + c_1 z + \ldots + c_n z^n, \tag{2.1}$$

and the remainder term satisfies

$$f(z) - t_n(z) = \mathrm{O}(z^{n+1}) \quad \text{as} \quad z \to 0. \tag{2.2}$$

The polynomial t_n is uniquely determined by relation (2.2).

2.2. Padé Polynomials and Padé Approximants. For each pair $(m, n) \in \mathbb{N}^2$ of indices there exist two polynomials $p_{mn} \in \mathcal{P}_m$ and $q_{mn} \in \mathcal{P}_n \backslash \{0\}$ such that

$$q_{mn}(z) f(z) - p_{mn}(z) = \mathrm{O}(z^{m+n+1}) \quad \text{as} \quad z \to 0, \tag{2.3}$$

and they are called Padé polynomials associated with the function f and the indices m and n. By $\mathcal{P}_n$ we have denoted the set of all polynomials of degree at most $n \in \mathbb{N}$.

The existence of the pair (p_{mn}, q_{mn}) of polynomials follows immediately from considering relation (2.3), which is equivalent to $m+n+1$ homogenous linear equations in the coefficients of the two unknown polynomials p_{mn} and q_{mn}. Padé polynomials are not unique; in any case, they can be multiplied by a non-zero constant, but more substantial non-uniqueness may exist.

Contrary to the Padé polynomials, the Padé approximant

$$[m/n] = [m/n]_f := \frac{p_{mn}}{q_{mn}} \tag{2.4}$$

is unique for each pair of indices (m, n). We note that in general we have

$$f(z) - [m/n](z) \neq \mathrm{O}(z^{m+n+1}) \quad \text{as} \quad z \to 0, \tag{2.5}$$

which shows that relation (2.2) does not generalize directly to Padé approximants. Indeed, (2.5), the proper generalization of (2.2) is relation (2.3), and it is called the linearized version of the error relation.

2.3. Hermite-Padé Polynomials. Hermite-Padé polynomials appear in two versions that are known as type I and type II. There exists a duality between both types. Contrary to Taylor and Padé polynomials, where we deal with a single function f, the Hermite-Padé polynomials are associated with a whole system

$$\mathfrak{f} = (1, f_1, \ldots, f_m), \; m \geq 1, \tag{2.6}$$

of functions. We assume that each of the m functions f_j, $j = 1, \ldots, m$, is analytic in a neighborhood of the origin. Each set of Hermite-Padé polynomials is associated with a multi-index $n = (n_0, \ldots, n_m) \in \mathbb{N}^{m+1}$ with $m+1$ components.

DEFINITION 2.1. ***Hermite-Padé Polynomials of Type I:*** *For any multi-index* $n = (n_0, \ldots, n_m) \in \mathbb{N}^{m+1}$ *there exists a vector of polynomials* $(p_0, \ldots, p_m) \in \mathcal{P}_{n_0-1} \times \mathcal{P}_{n_1-1} \times \ldots \times \mathcal{P}_{n_m-1} \backslash \{(0, \ldots, 0)\}$ *such that*

$$\sum_{j=0}^{m} p_j(z) f_j(z) = O(z^{|n|-1}) \quad \text{as} \quad z \to 0, \tag{2.7}$$

where $|n| := n_0 + \ldots + n_m$. *The vector* $(p_0, \ldots, p_m)$ *is called Hermite-Padé form of type I, and its elements* $p_j = p_{jn}$, $j = 0, \ldots, m$, $n \in \mathbb{N}^{m+1}$, *are the Hermite-Padé polynomials of type I of multi-index* n *associated with the system of functions* $\mathfrak{f}$.

DEFINITION 2.2. ***Hermite-Padé Polynomials of Type II:*** *For any multi-index* $n = (n_0, \ldots, n_m) \in \mathbb{N}^{m+1}$ *there exists a vector of polynomials* $(\mathfrak{p}_0, \ldots, \mathfrak{p}_m) \in \mathcal{P}_{N_0} \backslash \{0\} \times \mathcal{P}_{N_1} \times \ldots \times \mathcal{P}_{N_m}$ *with*

$$N_j := |n| - n_j, \;\; j = 0, \ldots, m, \tag{2.8}$$

such that

$$\mathfrak{p}_0(z) f_j(z) - \mathfrak{p}_j(z) \;=\; O(z^{|n|+1}) \quad as \quad z \to 0 \;\; for \;\; j = 1, \ldots, m. \tag{2.9}$$

The vector $(\mathfrak{p}_0, \ldots, \mathfrak{p}_m)$ *is called Hermite-Padé form of type II, and its elements* $\mathfrak{p}_j = \mathfrak{p}_{jn}$, $j = 0, \ldots, m$, $n \in \mathbb{N}^{m+1}$, *are the Hermite-Padé polynomials of type II of multi-index* n *associated with the system of functions* $\mathfrak{f}$.

Counting the number of linear restrictions implied by relation (2.7) or by the relations in (2.9) shows that Hermite-Padé polynomials of both types always exist. But the polynomials are not unique, the situation is similar to that of Padé polynomials only that now the possibilities for non-uniqueness are more manifold.

From both relations (2.7) and (2.9), one gets the defining relation (2.3) of the Padé polynomials as a special case if one chooses $m = 1$. Indeed, in (2.7) one has to take $m = 1$, $f_1 = -f$, and $(n_0, n_1) = (n+1, m+1)$, while in (2.9) the choice has to be $m = 1$, $f_1 = f$, and $(n_0, n_1) = (m, n)$. Thus, Hermite-Padé polynomials of both types can be seen as generalizations of Padé polynomials in a quite analogous way as Padé polynomials can be seen as a generalization of Taylor polynomials.

2.4. Hermite-Padé Approximants. The ideas behind the definition of the two types of Hermite-Padé polynomials are two different approximation concepts, of which each one uses one of the two types of polynomials as basic building blocks. The two concepts will be introduced next.

Let f be a single function that is assumed to be analytic at the origin. We define a system of functions (2.6) by

$$\mathfrak{f} := (1, f, \ldots, f^m). \tag{2.10}$$

DEFINITION 2.3. ***Algebraic Hermite-Padé Approximants****: For a given multi-index* $n \in \mathbb{N}^{m+1}$ *let* $p_0, \ldots, p_m \in \mathcal{P}_{n_0-1} \times \ldots \times \mathcal{P}_{n_m-1} \backslash \{(0, \ldots, 0)\}$ *be the Hermite-Padé polynomials of type I defined by (2.7) and the system (2.10). Let then the algebraic function* $y = y(z)$ *be defined by*

$$\sum_{j=0}^{m} p_j(z) y(z)^j \;\equiv\; 0. \tag{2.11}$$

From the m *branches of* y *we select the branch* $y = y_n$ *which has the highest contact with* f *at the origin. This branch* y_n *is called the algebraic Hermite-Padé approximant to* f *of degree* m *associated with the multi-index* n.

The Padé approximants defined in (2.4) are algebraic Hermite-Padé approximants of degree $m = 1$, and in an analogous sense the Taylor polynomials can be seen as algebraic approximants of degree $m = 0$. Thus, the different concepts of approximants considered so far form a hierarchy of increasing complexity and generality.

ETNA
Kent State University
etna@mcs.kent.edu

Like the Hermite-Padé polynomials of type I lead to algebraic Hermite-Padé approximants, so in an analogous way the Hermite-Padé polynomials of type II lead to simultaneous rational Hermite-Padé approximants. Now, we start from a system (2.6) with $f_0 \equiv 1$ and m independent functions $f_1, \ldots, f_m$ that are assumed to be analytic in a neighborhood of the origin.

DEFINITION 2.4. ***Simultaneous Rational Hermite-Padé Approximants****: For a given multi-index $n \in \mathbb{N}^{m+1}$ let $\mathfrak{p}_0, \ldots, \mathfrak{p}_m$ be the Hermite-Padé polynomials of type II define by (2.9). Then the vector of rational functions*

$$\left(\frac{\mathfrak{p}_1}{\mathfrak{p}_0}(z), \ldots, \frac{\mathfrak{p}_m}{\mathfrak{p}_0}(z)\right) \tag{2.12}$$

with common denominator polynomial $\mathfrak{p}_0$ is called simultaneous rational (Hermite-Padé) approximant to the (reduced) system of functions $\mathfrak{f}_{red} = (f_1, \ldots, f_m)$ associated with the multi-index n.

For $m = 1$, in (2.12) we have the Padé approximant to f_1 with numerator and denominator degrees (n_0, n_1).

Although algebraic Hermite-Padé and simultaneous rational approximants are two rather different concepts, they have at least one aspect in common: in both cases the approximant has m simultaneous components. In case of simultaneous rational approximants the assertion is immediate; in case of algebraic Hermite-Padé approximants, the components are the m branches of the algebraic function y_n. If the function f in (2.10) has branch points, then a very interesting question is whether and how far the m branches of the algebraic approximants y_n follow branches of the function f around its branch points.

2.5. Historic Remarks. Padé's thesis [21] is generally seen as the birth certificate of Padé approximants. Of course, important aspects of the concept had already been studied earlier in the framework of continued fractions, most notably the investigations by G. Frobenius in [10]. Now-a-days, the situation is somewhat reversed, and large parts of the analytic theory of continued fractions are seen as part of the theory of Padé approximation. A great part of Padé's research were concerned with the exponential function, which incidentally is also the central topic of the present contribution.

The research by G. Frobenius [10] is seen as an important forerunner of Padé's contributions since there already the emphasis has been put on the whole table of approximants, and there the algorithmically important identities between neighboring approximants or between numerators and denominators of neighboring approximants have been studied for the first time.

The introduction of Hermite-Padé polynomials is perhaps most famous for its role in C. Hermite's proof of the transcendency of the number e (cf. [11]). A very readable account of this historic achievement is contained in the appendix of [13]. In the further development of the theory of Hermite-Padé polynomials one can distinguish two different lines of research: One direction is concerned with systems of special functions, like the exponential, the binomial, or the logarithmic function, etc. We mention here the publications [16], [17], and [18] by K. Mahler and the publications [12] and [5] by authors who are closely connected with Mahler's research. In the other direction one is concerned with polynomials associated with general classes of functions and systems of functions. The situation is comparable with that of the theory of orthogonal polynomials, where also a split into the same two orientations can be observed. As representatives for the second direction we mention the studies of Angelesco and Nikishin systems, and for an introduction to this direction of research we mention the book [19] or the survey articles [1].

ETNA
Kent State University
etna@mcs.kent.edu

3. Szergö's Result. In an investigation published in [39] G. Szegö has studied the asymptotic distribution of the zeros of the Taylor polynomials

$$t_n(w) \;=\; 1 + z + \ldots + \frac{w^n}{n!} \tag{3.1}$$

to the exponential function e^w.

Since the exponential function is entire, and since it is different from zero everywhere in $\mathbb{C}$, all zeros of the polynomials t_n tend to infinity as $n \to \infty$, and consequently all informative characteristics of the asymptotic distribution of the zeros gets lost during the process. In order to avoid such an unfavorable effect, G. Szegö reorganized the independent variable w in such a way that the zeros of the transformed polynomials T_n have finite cluster points in $\mathbb{C}$ that are not reduced to a single point. An appropriate transformation is given by

$$z \;=\; z_n \;=\; \frac{w}{n}. \tag{3.2}$$

With (3.2) the Taylor polynomial t_n transforms into

$$T_n(z) \;=\; t_n(nz) \;=\; 1 + n\,z + \ldots + \frac{n^n}{n!} z^n, \tag{3.3}$$

and Stirling's formula shows that the $n-$th root of the leading coefficient of T_n converges to a value different from 0 and ∞. Actually, it converges to e, i.e., we have

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}} \;=\; \lim_{n\to\infty} e \sqrt[2n]{\frac{1}{2n\pi}} \;=\; e. \tag{3.4}$$

Let $Z_n = Z(T_n)$ denote the set of all zeros of the polynomial T_n, and let further W_n denote the set of all zeros of the transformed remainder term

$$E_n(z) := e^{nz} - T_n(z). \tag{3.5}$$

In [39] the following theorem has been proved.

THEOREM 3.1. *Let the curve S_0 be defined by*

$$S_0 := \left\{\, z \in \mathbb{C} \;\middle|\; |z|\, e^{1-\mathrm{Re}(z)} = 1 \,\right\}. \tag{3.6}$$

Then the sets Z_n of zeros of the transformed polynomials T_n cluster on the subcurve $S_0 \cap \overline{\mathbb{D}}$ of S_0 as $n \to \infty$, and the sets W_n of zeros of the remainder terms (3.5) cluster on $S_0 \setminus \mathbb{D}$ as $n \to \infty$.

The curve S_0 is nowadays known as the Szegö curve. It is shown in Figure 3.1 together with the zeros of the polynomial T_{36} and some zeros of the remainder term E_{36}. We note that for zeros of E_{36} close to infinity the asymptotic distance between two neighboring zeros is $2\pi/36 \doteq 0.175$. In Figure 3.1 the 6 zeros are not far enough away from the origin since the distance between them is still somewhat larger than 0.175.

The curve S_0 can be interpreted as the intersection of the plane defined by $\{\,(x,y,z) \in \mathbb{R}^3 \mid z = x - 1\,\}$ in $\mathbb{R}^3$ with the logarithmic cone $\{\,(x,y,z) \in \mathbb{R}^3 \;\middle|\; \log|x^2+y^2| = 2\,z\,\}$. A short proof of Theorem 3.1 can be based on this fact together with a clever use of Cauchy's integral formula. Many related results about the distribution of zeros of sections of power series to the exponential function and other entier functions can be found in [9].

With potential-theoretic means it is possible to describe the asymptotic density of the zeros of T_n; cf. [25], [26]. Estimates of the speed with which the zeros of T_n converge

ETNA
Kent State University
etna@mcs.kent.edu

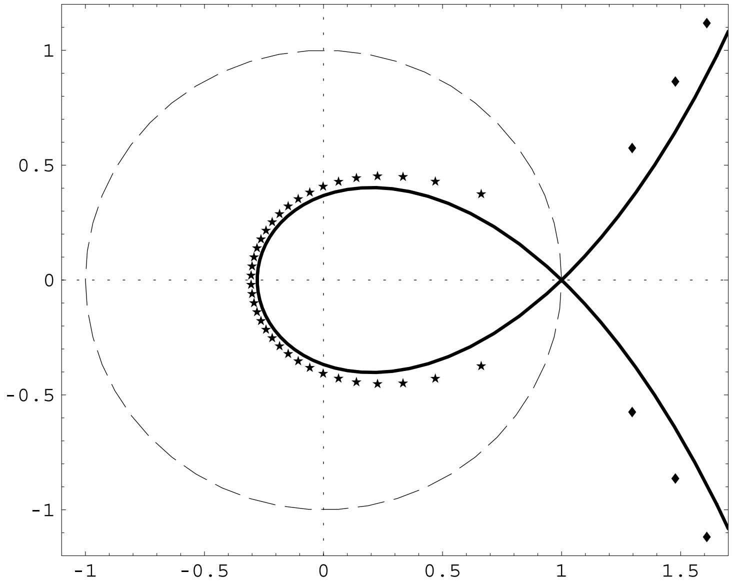

FIG. 3.1. *The Szegö curve S_0 together with the zeros (stars) of the transformed polynomial T_{36} and 6 zeros (diamonds) of the remainder term E_{36} that has been introduced in (3.5).*

towards $S_0 \cap \overline{\mathbb{D}}$ have been proved in [4]. Recently, R.S. Varga has also investigated the discrepancy between the sets Z_n and their asymptotic distribution on $S_0 \cap \overline{\mathbb{D}}$, and he has spoken about this topic at the same conference in honor of E.B. Saff's 60th birthday, to which also the present contribution is dedicated.

Szegö's result, which have just been reviewed, was an important inspiration for the research by E.B. Saff and R.S. Varga about the convergence and asymptotic behavior Padé approximants and Padé polynomials, which will be reviewed in the next Section.

4. E.B. Saff's and R.S. Varga's Results. In Subsection 2.2 it has already been noted that Padé approximants are a natural generalizations of Taylor polynomials into the field of rational functions. The Padé polynomials $p_{mn} \in \mathcal{P}_m$, $q_{mn} \in \mathcal{P}_n$, and the Padé approximants $[m/n]$ have been defined in (2.3) and (2.4), respectively. In case of the exponential function there exist explicit formulae for the Padé polynomials (cf. [35], formula (1.3), or [2], Section 1.2). For instance, for the diagonal approximants $[n/n]$, we have

$$p_{nn}(w) = \binom{2n}{n} + \binom{2n-1}{n-1}\frac{w}{1!} + \ldots + \binom{n}{0}\frac{w^n}{n!}, \quad q_{nn}(w) = p_{nn}(-w). \tag{4.1}$$

Questions about convergence are basic for any type of approximants; for Padé approximants and the exponential function basic answers have already been given in Padé's thesis [21] (cf. also [22], [23], [24]). Locally uniform convergence holds true throughout $\mathbb{C}$ for any sequence of numerator and denominator degrees (m_j, n_j) satisfying $\min(m_j, n_j) \to \infty$ as $j \to \infty$. In case of ray sequences, we have the surprising result that the numerator and denominator polynomials p_{mn} and q_{mn} converge even individually. Indeed, we have

$$\lim_{j\to\infty} \frac{p_{m_j,n_j}(w)}{p_{m_j,n_j}(0)} = e^{\frac{1}{1+\lambda}w}, \quad \lim_{j\to\infty} \frac{q_{m_j,n_j}(w)}{q_{m_j,n_j}(0)} = e^{\frac{-\lambda}{1+\lambda}w} \tag{4.2}$$

ETNA
Kent State University
etna@mcs.kent.edu

for ray sequences $\{(m_j, n_j)\}$ with

$$\frac{n_j}{m_j} \to \lambda \in (0, \infty) \quad \text{as} \quad j \to \infty. \tag{4.3}$$

From (4.2) it follows that all zeros of p_{mn} as well as those of q_{mn} converge to infinity if $\min(m, n)$ tends to infinity. This is an impressive property, which is quite unusual for the general convergence theory of Padé approximants. As in the case of Taylor polynomials, one would like to know more about the specific manner in which poles and zeros of the Padé approximants tend to infinity.

In a series of papers ([29], [30], [32], [33], [34], [35]) E.B. Saff and R.S. Varga have investigated the location of zeros and poles of Taylor polynomials and Padé approximants to the exponential function. Following the example set by G. Szegö with his rescaling method, which has been reviewed in the last section, they carried out an analogous study of the asymptotic behavior of the zeros and poles of Padé approximants. We follow here in our review the treatment in [35].

Since Padé approximants $[m/n]$ contain $m + n + 2$ free parameters, E.B. Saff and R.S. Varga used

$$z = \frac{w}{m+n} \tag{4.4}$$

as new independent variable for the rescaling, which then leads to the transformed (rescaled) Padé polynomials

$$P_{mn}(z) := p_{mn}((m+n)\,z) \quad \text{and} \quad Q_{mn}(z) := q_{mn}((m+n)\,z). \tag{4.5}$$

One of the most interesting results in [35] is an explicit description of the arcs on which the zeros of the rescaled polynomials P_{mn} and Q_{mn} and also the zeros of the transformed remainder term

$$E_{mn}(z) := Q_{mn}(z)e^{(m+n)\,z} - P_{mn}(z) \tag{4.6}$$

cluster if the sequence of indices $\{(m, n)\} = \{(m_j, n_j)\}$ tends to infinity along a ray sequence satisfying (4.3). These arcs are the analogue of the Szegö curve S_0 from (3.6) for the Padé case.

The situation is now somewhat more complicated, and we need some auxiliary definitions before we can specify the arcs of the asymptotic relations. We follow very closely the terminology introduced in Section 1 of [35].

Depending on the parameter λ of the ray sequence in (4.3), the two points $z_\lambda^+, z_\lambda^- \in \partial\mathbb{D}$ are defined as

$$z_\lambda^\pm := \frac{1-\lambda}{1+\lambda} \pm i\,2\frac{+\sqrt{\lambda}}{1+\lambda}, \tag{4.7}$$

and the function g_λ is defined as

$$g_\lambda(z) := \sqrt{1 + z^2 - 2\,z\left(\frac{1-\lambda}{1+\lambda}\right)}. \tag{4.8}$$

The function g_λ has branch points at z_λ^+ and z_λ^-, and its natural domain of definition is the two-sheeted Riemann surface $\mathcal{R}_\lambda$ defined by the relation

$$\mathcal{R}_\lambda : \quad w^2 = 1 + z^2 - 2\,z\left(\frac{1-\lambda}{1+\lambda}\right). \tag{4.9}$$

ETNA
Kent State University
etna@mcs.kent.edu

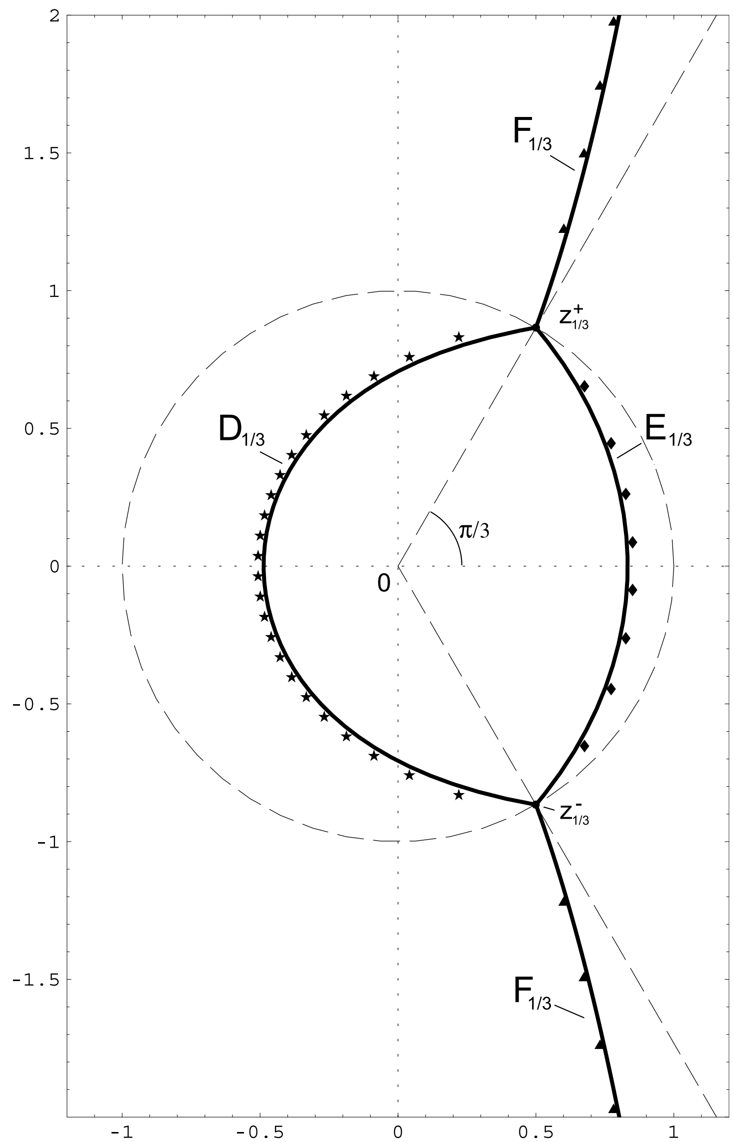

FIG. 4.1. *The rescaled zeros (stars) and poles (diamonds) of the Padé approximant* $[24/8]$ *and several zeros (triangles) of the remainder term (4.6) together with the arcs* $D_{1/3}$*,* $E_{1/3}$*, and* $F_{1/3}$ *defined in (4.12), (4.13), and (4.14), respectively, on which the zeros and poles cluster as* $n \to \infty$*. The figure has been taken from [35], Fig. 2, but its values have been recalculated.*

Let the two half-lines $L^{\pm}_{\lambda}$ be defined by $L^{\pm}_{\lambda} := \{ \ z \in \mathbb{C} \mid \mathrm{Re}(z) = \mathrm{Re}(z^{+}_{\lambda}), \pm \mathrm{Im}(z) \geq \pm \mathrm{Im}(z^{\pm}_{\lambda}) \ \}$, then the Riemann surface $\mathcal{R}_{\lambda}$ can be represented by two sheets S^{+} and S^{-} that are identical copies of $\mathbb{C} \setminus (L^{+}_{\lambda} \cup L^{-}_{\lambda})$ glued together in the usual crosswise fashion along L^{+}_{λ} and L^{-}_{λ}. In the sequel we identify the sheet S^{+} with $\mathbb{C} \setminus (L^{+}_{\lambda} \cup L^{-}_{\lambda})$ and make the assumption that the sign of the square root in (4.8) is positive on S^{+} near $z = 0$, i.e., $g_{\lambda}(0) = 1$. It is not difficult to verify that the two expressions $1 \pm z + g_{\lambda}(z)$ do not vanish in

ETNA
Kent State University
etna@mcs.kent.edu

$\mathbb{C} \setminus (L_\lambda^+ \cup L_\lambda^-)$ for $0 < \lambda < \infty$. Consequently, the function

$$w_\lambda(z) := \frac{4\,\lambda^{\lambda/(1+\lambda)} z\, e^{g_\lambda(z)}}{(1+\lambda)(1+z+g_\lambda(z))^{2/(1+\lambda)}(1-z+g_\lambda(z))^{2\lambda/(1+\lambda)}} \tag{4.10}$$

is well defined for all $0 < \lambda < \infty$ and $z \in \mathbb{C} \setminus (L_\lambda^+ \cup L_\lambda^-)$.

By S_λ we denote the sector

$$S_\lambda = \{\, z \in \mathbb{C} \mid |\arg(z)| > \cos^{-1}\left(\frac{1-\lambda}{1+\lambda}\right) \}. \tag{4.11}$$

In [35] it has been proved that

$$D_\lambda := \{\, z \in \overline{\mathbb{D}} \cap \overline{S}_\lambda \mid \ |w_\lambda(z)| = 1 \,\} \ \text{ and} \tag{4.12}$$

$$E_\lambda := \{\, z \in \overline{\mathbb{D}} \setminus S_\lambda \mid \ |w_\lambda(z)| = 1 \,\} \tag{4.13}$$

are two analytic Jordan arcs, each connecting the two points z_λ^+ and z_λ^- in $\mathbb{D}$, and further that

$$F_\lambda := \{\, z \in \overline{\mathbb{C}} \setminus \mathbb{D} \mid \ |w_\lambda(z)| = 1 \,\} \tag{4.14}$$

consists of two disjoint Jordan arcs each connecting one of the two points z_λ^+ and z_λ^- with infinity. We have $F_\lambda \subset \overline{S}_\lambda \cap \{\, z \in \mathbb{C} \mid \mathrm{Re}(z) \geq \mathrm{Re}(z_\lambda^+) \,\}$ for $0 < \lambda \leq 1$ and $F_\lambda \subset \{\, z \in \mathbb{C} \mid \mathrm{Re}(z) \leq \mathrm{Re}(z_\lambda^+) \,\} \setminus S_\lambda$ for $1 \leq \lambda < \infty$. In Figure 4.1, the two arcs D_λ and E_λ are shown together with pieces of the two arcs that constitute F_λ for $\lambda = 1/3$. In Figure 4.2 the same arcs are shown for the symmetric case $\lambda = 1$.

The result from [35], which is of central interest in the present review, is formulated in the next theorem.

THEOREM 4.1. *([35], Theorem 2.2) Let the ray sequence $\{[m_j/n_j]\}$ of Padé approximants satisfy (4.3) with parameter $\lambda \in (0,\infty)$, then the poles and zeros of the rescaled Padé approximants $[m_j/n_j] = P_{m_j n_j}/Q_{m_j n_j}$ and the zeros of the rescaled remainder terms (4.6) have the following properties:*

(i) All zeros of $[m_j/n_j]$ cluster on D_λ for $j \to \infty$.
(ii) All poles of $[m_j/n_j]$ cluster on E_λ for $j \to \infty$.
(iii) All zeros of the transformed remainder terms $E_{m_j n_j}(z) = Q_{m_j n_j}(z) \times e^{(m_j+n_j)\,z} - P_{m_j n_j}(z)$ introduced in (4.6) cluster on F_λ for $j \to \infty$.

In Figure 4.1 the rescaled poles and zeros of the Padé approximant $[24/8]$ are plotted together with the two arcs $D_{1/3}$ and $E_{1/3}$ on which these objects cluster with growing degrees. Further, those zeros of the remainder term $E_{24,8}$ that fit to the window of Figure 4.1 are plotted together with the parts of the two arcs of $F_{1/3}$ that belong to the same window. Note that the pair of indices $(24, 8)$ is an element of a ray sequence with $\lambda = 1/3$.

The plot in Figure 4.1 shows that already for the degrees $m = 24$ and $n = 8$ a good agreement between the points and their asymptotic loci can be observed. The density of the poles and zeros as well as their asymptotic agreement with the arcs is poorest near the two bifurcation points z_λ^+ and z_λ^-.

With some imagination one can recognize that the configuration shown in Figure 4.1 turns into that of Figure 3.1, i.e., into the Szegö curve S_0 from (3.6), if λ tends to 0. This observation can be made solid by investigating the limit of the expression in (4.10) for $\lambda \to 0$, which has been done in formula ('2.5) of [35]. In this respect, Szegö's result is a special case of the results proved by E.B. Saff and R.S. Varga in [35].

In case of diagonal Padé approximants $[n/n]$, we have $\lambda = 1$, $z_1^{\pm} = \pm i$, and the union $D_1 \cup E_1 \cup F_1$ of the arcs D_1, E_1, and F_1 from (4.12), (4.13), and (4.14) forms a configuration

ETNA
Kent State University
etna@mcs.kent.edu

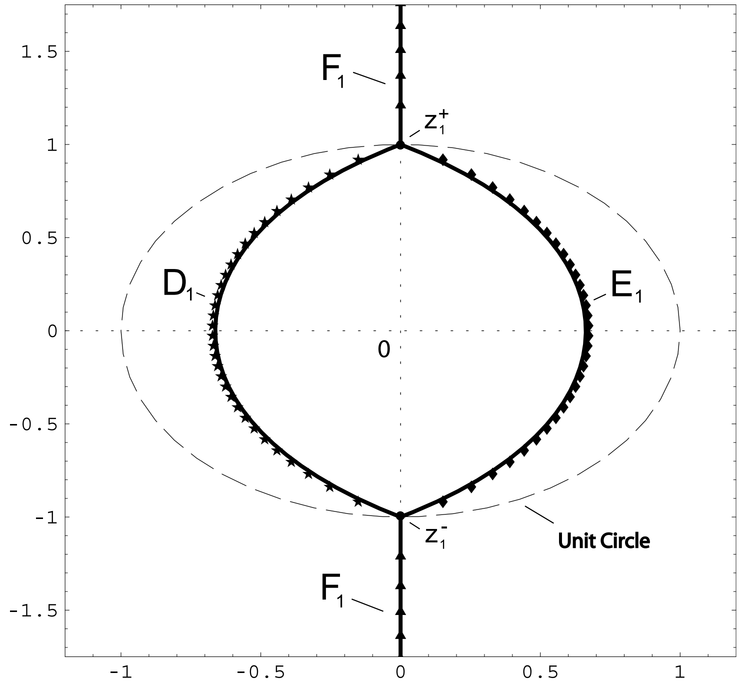

FIG. 4.2. *The zeros (stars) and poles (diamonds) of the rescaled Padé approximant* $[32/32]$ *and some zeros (triangles) of the remainder term (4.6) together with the arcs* D_1*,* E_1*, and* F_1 *defined in (4.12), (4.13), and (4.14), respectively. Note that the scales of the* $x-$ *and* $y-$*axes differ by a factor* 1.5*.*

that is symmetric with respect to the $y-$ and $x-$axis. In Figure 4.2, these arcs are plotted together with the zeros and poles of the rescaled Padé approximants $[32/32]$ and some of the zeros of the rescaled remainder term $E_{32,32}$ from (4.6).

In the symmetric case $\lambda = 1$, the expressions in (4.8) and (4.10) become especially simple. We have

$$g_1(z) = \sqrt{1+z^2}, \quad z_1^{\pm} = \pm i, \tag{4.15}$$

$$w_1(z) = \frac{z}{1+\sqrt{1+z^2}} e^{\sqrt{1+z^2}}. \tag{4.16}$$

Using (4.12), (4.13), and (4.14) as before, one gets the arcs D_1, E_1, and F_1 from (4.15) and (4.16) instead of (4.8) and (4.10).

The expressions in (4.15) and (4.16) show in a transparent way that the asymptotic relations for the Padé polynomials P_{nn} and Q_{nn} contain as a main element an algebraic functions of a second degree, which is in the symmetric case $\lambda = 1$ the rather simply constructed function $g_1(z) = \sqrt{1+z^2}$. In the next section we will see that in case of quadratic Hermite-Padé polynomials the asymptotic relations are again associated with an algebraic functions, but there it is a function of third degree.

5. Quadratic Hermite-Padé Polynomials, the Diagonal Case. We now come to the third stage of our considerations: the asymptotic behavior of quadratic Hermite-Padé polynomials. In Section 2.3 it has been shown that Hermite-Padé polynomials of both types can be seen as generalizations of Padé polynomials like the Padé polynomials are generalizations of Taylor polynomials. Quadratic Hermite-Padé polynomials, which are defined by choosing $m = 2$ in Definition 2.1 and 2.2, are the first step in this process. In [37] and [38] the asymptotic behavior of quadratic Hermite-Padé polynomials of type I has been studied for diagonal sequences of indices. Parts of these results are direct generalizations of the case $\lambda = 1$ in [35].

Let p_n, q_n, and r_n denote the three quadratic Hermite-Padé polynomials $p_{1,\vec{n}}$, $p_{2,\vec{n}}$, $p_{3,\vec{n}} \in \mathcal{P}_{n-1}$ of type I associated with the diagonal multi-index $\vec{n} = (n+1, n+1, n+1) \in \mathbb{N}^3$ and the exponential system $\mathfrak{f} = (1, exp, exp^2)$ as introduced in Definition 2.1. Thus, all three polynomials p_n, q_n, r_n are of degree at most n, and from the defining relation (2.7) in Definition 2.1 we know that they satisfy the relation

$$p_n(w) + q_n(w)\, e^w + r_n(w)\, e^{2w} = \mathrm{O}(w^{3n+2}) \quad \text{as} \quad w \to 0. \tag{5.1}$$

The investigations in [37] and [38] are based on the rescaling method, and

$$z = \frac{w}{3\,n} \tag{5.2}$$

is used as new rescaled independent variable, which then leads to the rescaled Hermite-Padé polynomials

$$P_n(z) := p_n(3n\, z),\, Q_n(z) := q_n(3n\, z), \text{ and } R_n(z) := r_n(3n\, z). \tag{5.3}$$

Note that the three polynomials p_n, q_n, r_n have together $3n + 3$ free parameters. Together with a multiplication by $e^{-3n\, z}$, relation (5.1) transform into

$$E_n(z) := e^{-3n\, z} P_n(z) + Q_n(z) + R_n(z) e^{3n\, z} = \mathrm{O}(z^{3n+2}) \quad \text{as} \quad z \to 0 \tag{5.4}$$

in the new variable z.

In Figure 5.2 the zeros of the rescaled polynomials P_n, Q_n, R_n are shown for $n = 33$ together with those zeros of the remainder term E_{33} from (5.4) that lie within the window of Figure 5.2. In a certain sense the configuration in Figure 5.2 is a doubling of the configuration shown in Figure 4.2 for diagonal Padé polynomials. In Figure 5.2 also the arcs are plotted on which the zeros cluster as $n \to \infty$. The description of these arcs has been one of the main aims of the research in [37].

While the definitions (4.12), (4.13), and (4.14) of the asymptotic arcs for the zeros of the Padé polynomials have been formulated in analogy to Szegö's approach in (3.6) of Theorem 3.1, a different approach had to be chosen in [37]. There, the analysis relies strongly on the geometry of a Riemann surface that is associated with an algebraic function of third degree. Details will be given in Subsection 5.1.

The research in [37] and [38] was initiated by investigations of Hermite-Padé polynomials p_n, q_n, r_n in [3], [7], and [8]. In [3] among other things a 4-term recurrence relation and a very precise asymptotic estimate for the polynomials p_n, q_n, r_n and for the remainder term e_n has been proved. While in [3], like in [37], only the diagonal case has been studied, extensions to the non-diagonal case are contained in [7] and [8]. There, also interesting connections with the theory of special functions have played a prominent role. In both papers great interest has been put on the location of the zeros of the polynomials p_n, q_n, and r_n. Especially, these last aspect has triggered the research in [37] and [38].

Recently, in [14] (cf. also [15]) the asymptotic analysis of quadratic Hermite-Padé polynomials of type I has been incorporated into the Riemann-Hilbert approach for proving strong asymptotic relations for orthogonal polynomials, which is based on Deift's and Zhou's method of steepest descent from [6]. Further, we mention that in [41] and [42] steps of an extension of the analysis to Hermite-Padé polynomials of type I for $m > 2$ have been done.

ETNA
Kent State University
etna@mcs.kent.edu

5.1. The Riemann Surface $\mathcal{R}$ and the Functions ψ and h. In the present subsection we define an algebraic functions ψ of third degree together with its associated Riemann surface $\mathcal{R}$ in a first step, which then is followed by the definition of an harmonic function h, which will be an important element of the asymptotic relations for the rescaled Hermite-Padé polynomials P_n, Q_n, R_n and the remainder term E_n. In a comparison with the results reviewed in Section 4, the algebraic function ψ and the Riemann surface $\mathcal{R}$ are the pendants of the algebraic function g_λ from (4.8) with $\lambda = 1$ and the Riemann surface introduced in (4.9).

DEFINITION 5.1. *The Riemann surface $\mathcal{R}$ together with the bijective mapping $\psi : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$ and the canonical projection $\pi : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$ is defined by the property that the two functions ψ and*

$$z(v) := \frac{1}{3}\left(\frac{1}{v+1} + \frac{1}{v} + \frac{1}{v-1}\right) = \frac{v^2 - 1/3}{v(v^2-1)}, \quad v \in \overline{\mathbb{C}}, \tag{5.5}$$

satisfy the relation

$$z \circ \psi(\zeta) = \pi(\zeta) \quad \textit{for all} \quad \zeta \in \mathcal{R}. \tag{5.6}$$

Relation (5.6) shows that locally the function ψ is the inverse of the rational function (5.5), and therefore $\psi \circ \pi^{-1}$ is an algebraic function of third degree. The surface $\mathcal{R}$ has three sheets and four simple branch points ζ_j, $j = 1, \ldots, 4$, over the four base points

$$z_j = \pi(\zeta_j) := \sqrt[4]{1/3}\, e^{i\varphi_j} \;\text{ with }\; \varphi_j = \frac{5}{12}\pi, \frac{7}{12}\pi, \frac{17}{12}\pi, \frac{19}{12}\pi, \;\; j = 1, \cdots, 4. \tag{5.7}$$

Indeed, the derivative

$$z(v)' = -\frac{v^4 + 1/3}{v^2(v^2-1)^2} \tag{5.8}$$

has four simple zeros at the roots $v_j = \sqrt[4]{-1/3}$, $j = 1, \ldots, 4$, and it then is easy to check that the four points $z_1, \ldots, z_4$ in (5.7) are defined by $z_j = \pi \circ \psi^{-1}(v_j) = z(v_j)$, $j = 1, \ldots, 4$. The Riemann surface $\mathcal{R}$ is of genus 0.

In the sequel, points on $\mathcal{R}$ will be denoted by ζ, while the associated base points $\pi(\zeta) \in \overline{\mathbb{C}}$ will be denoted by z. For shortness we call the four points ζ_j, $j = 1, \ldots, 4$, as well as their base points $z_j = \pi(\zeta_j)$ in (5.7) as branch points of $\mathcal{R}$. From (5.5) and (5.6) one easily deduces that

$$\psi \circ \pi^{-1}(\{0\}) = \left\{-\sqrt{1/3}, \infty, \sqrt{1/3}\right\} \quad \text{and} \quad \psi \circ \pi^{-1}(\{\infty\}) = \{-1, 0, 1\}. \tag{5.9}$$

In the sequel, we assume that the defining relation (5.4) of the Hermite-Padé polynomials P_n, Q_n, R_n is lifted to $\mathcal{R}$ so that a neighborhood of the origin in $\mathbb{C}$ corresponds to a neighborhood of the point ζ_0 on $\mathcal{R}$ defined by

$$\zeta_0 := \psi^{-1}(\infty). \tag{5.10}$$

From the definition of the Riemann surface $\mathcal{R}$ in (5.5) and (5.6) it follows that $\mathcal{R}$ can be constructed by gluing together three sheets $\widetilde{S}_{-1}, \widetilde{S}_0, \widetilde{S}_1 \subset \mathcal{R}$ in the following way: Let $\widetilde{\Gamma}_{-1}$ and $\widetilde{\Gamma}_1 \subset \mathbb{C}$ be two disjoint Jordan arcs with the property that $\widetilde{\Gamma}_1$ connects the two branch points z_1 and z_4, and $\widetilde{\Gamma}_{-1}$ connects the two branch points z_2 and z_3. Let then the three sheets $\widetilde{S}_{-1}$, $\widetilde{S}_0$, and $\widetilde{S}_1$ be copies of $\overline{\mathbb{C}} \setminus \widetilde{\Gamma}_{-1}$, $\overline{\mathbb{C}} \setminus (\widetilde{\Gamma}_{-1} \cup \widetilde{\Gamma}_1)$, and $\overline{\mathbb{C}} \setminus \widetilde{\Gamma}_1$, respectively, assume that the two sheets $\widetilde{S}_1$ and $\widetilde{S}_0$ are glued together along the arc $\widetilde{\Gamma}_1$ in the usual cross-wise fashion, and in the same way assume that the two sheets $\widetilde{S}_{-1}$ and $\widetilde{S}_0$ are glued together along the arc $\widetilde{\Gamma}_{-1}$. The boundary points of $\partial\widetilde{S}_j$, $j = -1, 0, 1$, are attributed to the neighboring sheets in such a way that all three restrictions $\pi|_{\widetilde{S}_j}$, $j = -1, 0, 1$, of the canonical projection π are univalent. At the present moment the two arcs $\widetilde{\Gamma}_{-1}$ and $\widetilde{\Gamma}_1$ are still variable and will be fixed later.

Having defined $\mathcal{R}$, which will serve as domain of definition, we are now ready to define the function h.

DEFINITION 5.2. *Let the functions* $h: \mathcal{R} \longrightarrow \overline{\mathbb{R}}$ *be defined by*

$$h(\zeta) := \mathrm{Re}\; u \circ \psi(\zeta) \quad \textit{for} \quad \zeta \in \mathcal{R} \;\; \textit{with} \tag{5.11}$$

$$u(v) := \frac{2\,v^2}{v^2 - 1} + \log \frac{2}{3\,v\,(v^2 - 1)}. \tag{5.12}$$

It follows from (5.9), (5.10), (5.11), and (5.12) that the function h is harmonic in $\mathcal{R}\diagdown(\{\zeta_0\} \cup\pi^{-1}(\{\infty\}))$ and subharmonic at ζ_0.

5.2. The Definition of Jordan Arcs and a First Result. With the help of the function h from Definition 5.2 we define Jordan arcs that will turn out to be core pieces of the cluster sets of the zeros of the rescaled Hermite-Padé polynomials P_n, Q_n, R_n and the remainder terms E_n if $n \to \infty$.

Starting point for our definition is the function

$$h_{\max}(z) := \max\{\, h(\zeta) \mid \zeta \in \mathcal{R},\, \pi(\zeta) = z \,\} \quad \text{for} \quad z \in \overline{\mathbb{C}}. \tag{5.13}$$

From Definition 5.2 we know that h is harmonic in $\mathcal{R}\diagdown\left(\{\zeta_0\} \cup \pi^{-1}(\{\infty\})\right)$ and subharmonic in a neighborhood of ζ_0. From standard knowledge in potential theory (cf. [27], Chapter 2) it then follows that the function $h_{\max}$, as the maximum of subharmonic functions, is again subharmonic in $\mathbb{C}$. Further, we know from the same background that $h_{\max}$ is harmonic everywhere in $\mathbb{C}$ except for those loci z where at least two different branches of the multi-valued function $h \circ \pi^{-1}$ coincide and are at the same time identical with the maximal value $h_{\max}(z)$.

As a consequence of the subharmonicity of $h_{\max}$ it follows from the Poisson-Jensen formula of potential theory (cf. [27], Theorem 4.5.1) that in any bounded domain in $\mathbb{C}$ the function $h_{\max}$ can be represented as the sum of a harmonic function and a Green potential. These details and some of its immediate consequences are put together in the next lemma.

LEMMA 5.3. *(cf. [37], Lemma 2.6) The function $h_{\max}$ is subharmonic in $\mathbb{C}$. There exists a system Γ of analytic Jordan arcs such that $h_{\max}$ is harmonic in $\mathbb{C}\diagdown\Gamma$, but not harmonic in any neighborhood of a point $z \in \Gamma$. There exists a positive measure ν on Γ such that for any $R > 0$ we have*

$$h_{\max}(z) = h_R(z) + \int_{|t|<R} \log\left|\frac{R(z-t)}{R^2 - \bar{t}\,z}\right| d\nu(t) \quad \textit{for} \quad |z| \le R \tag{5.14}$$

with h_R a function that is harmonic in $\{|z| < R\}$ and satisfies

$$h_R(z) = h_{\max}(z) \qquad \textit{for} \quad |z| = R. \tag{5.15}$$

Since $R > 0$ is arbitrary, the measure ν in Lemma 5.3 is well defined throughout $\mathbb{C}$ and we have $\mathrm{supp}\,(\nu) = \Gamma$.

ETNA
Kent State University
etna@mcs.kent.edu

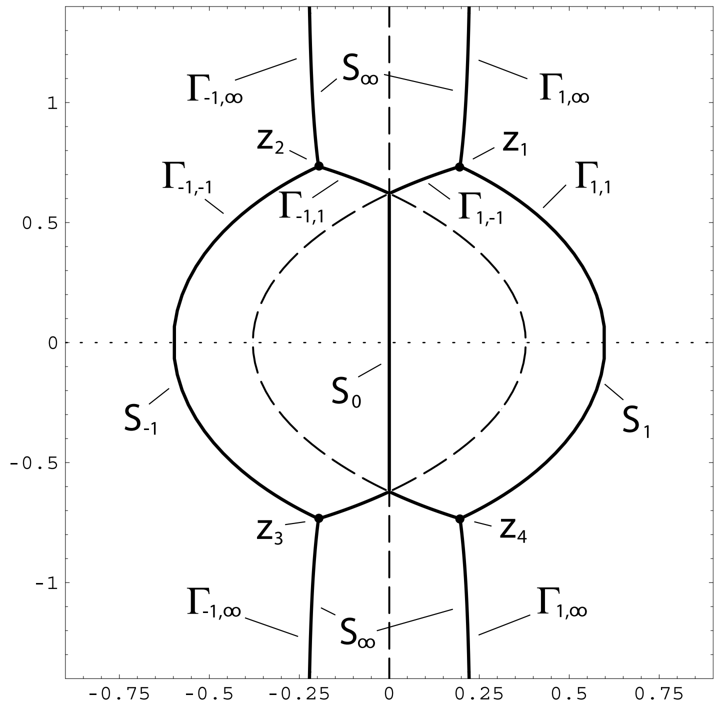

FIG. 5.1. *The picture contains all arcs of the set* $\widetilde{\Gamma}$ *from (5.16). The arcs of the set* Γ *fom Lemma 5.3 are represented by full lines, while those of* $\widetilde{\Gamma} \setminus \Gamma$ *are plotted by dashed ones. Explanations of the labels are given in the text after (5.16) and in Definition 5.4*

The main topic in the present subsection is the description of the system Γ of arcs. In Definition 5.4, Lemma 5.5, and Theorem 5.6, below, we will see that Γ consists of four subsets S_j, $j = -1, 0, 1, \infty$, that are the asymptotic cluster sets of the zeros of the rescaled Hermite-Padé polynomials P_n, Q_n, R_n and the remainder term E_n.

It is immediate from what has been said so far that the system Γ of analytic Jordan arcs is contained in the larger system of intersection arcs

$$\widetilde{\Gamma} := \left\{ z \in \overline{\mathbb{C}} \,\middle|\, \operatorname{card}\left\{ h(\zeta) \mid \zeta \in \pi^{-1}(\{z\}) \right\} < 3 \right\}. \tag{5.16}$$

Since the Riemann surface $\mathcal{R}$ has three sheets over $\mathbb{C}$, and since h is not identically constant on any open set, it follows that the set $\{ h(\zeta) \mid \zeta \in \pi^{-1}(\{z\}) \}$ contains three different points for almost all $z \in \overline{\mathbb{C}}$; the only exceptions are the analytic arcs in $\widetilde{\Gamma}$. These arcs are the totality of all intersection arcs of the multi-valued function $h \circ \pi^{-1}$.

Studying the behavior of the functions h and $h_{\max}$ in neighborhoods of the four branch points $\zeta_1, \ldots, \zeta_4 \in \mathcal{R}$ and $z_1, \ldots, z_4 \in \mathbb{C}$, respectively, shows that each branch point is the end point of exactly three different subarcs from $\widetilde{\Gamma}$. Three of these arcs connect the two points $\{z_1, z_4\}$, and three other ones connect the two points $\{z_2, z_3\}$. A formal proof of this last assertion is part of the proof of Lemma 2.6 in [37].

Let now $\Gamma_{1,j}$, $j = -1, 1, \infty$, denote the three arcs that connect z_1 with z_4, and $\Gamma_{-1,j}$, $j = -1, 1, \infty$, the three other arcs that connect z_2 with z_3. It turns out that two arcs, which are denoted by $\Gamma_{1,\infty}$ and $\Gamma_{-1,\infty}$, pass through infinity, while the other four arcs Γ_{ij}, $i, j = -1, 1$, are contained in $\mathbb{D}$. All six arcs are shown in Figure 5.1. The unbounded arcs $\Gamma_{1,\infty}$ and

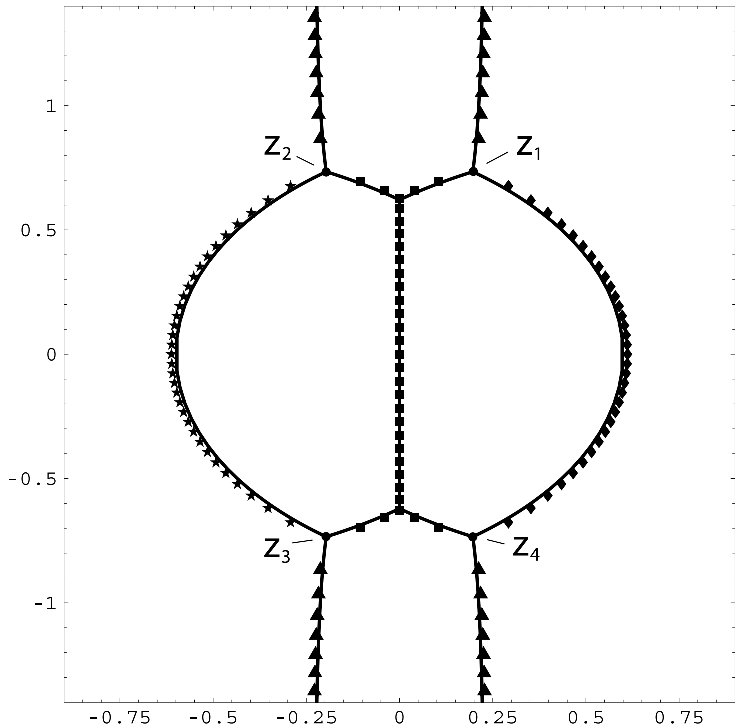

FIG. 5.2. *The zeros of the rescaled quadratic Hermite-Padé polynomials P_{33} (stars), Q_{33} (boxes), R_{33} (diamonds), and some of the zeros of the remainder term E_{33} (triangles) together with the arcs on which the zeros cluster. Note that the scales of the $x-$ and $y-$axes differ by a factor* 1.5.

$\Gamma_{-1,\infty}$ are represented only in part, and the two arcs $\Gamma_{1,-1}$ and $\Gamma_{-1,1}$ are partly represented by dashed lines; the reason for this is that along the dashed lines the values of the intersecting branches of $h \circ \pi^{-1}$ are no longer maximal, and therefore these dashed parts of $\Gamma_{1,-1}$ and $\Gamma_{-1,1}$ do not belong to the set Γ of Lemma 5.3, but they belong to $\widetilde{\Gamma}$.

The angles under which the six arcs $\Gamma_{i,j}$ $j = -1, 1, \infty$, $i = -1, 1$, end at the four branch points $z_1, \ldots, z_4$ can be calculated from local developments of the functions h and ψ. Having calculated these initial directions, the arcs themselves can then be calculated by using the property that along each of them two branches of the function $h \circ \pi^{-1}$ coincide.

Next, we come to the definition of the four sets S_j, $j = -1, 0, 1, \infty$, that will turn out to be the asymptotic cluster sets of the zeros of the rescaled Hermite-Padé polynomials P_n, Q_n, R_n and the remainder term E_n.

DEFINITION 5.4. *Let the point $y_1 \in i\,\mathbb{R}$ be defined by $y_1 := \Gamma_{1,-1} \cap \Gamma_{-1,1} \cap \{\, z \mid \operatorname{Im}(z) > 0 \,\}$. Then we set*

(5.17) $$S_1 := \Gamma_{1,1},$$

(5.18) $$S_{-1} := \Gamma_{-1,-1},$$

(5.19) $$S_0 := (\Gamma_{1,-1} \cap \{\, z \mid \operatorname{Re}(z) \geq 0 \,\}) \cup (\Gamma_{-1,1} \cap \{\, z \mid \operatorname{Re}(z) \leq 0 \,\}) \cup [-y_1, y_1]\,,$$

(5.20) $$S_\infty := \Gamma_{1,\infty} \cup \Gamma_{-1,\infty}.$$

The sets S_j, $j = -1, 0, 1, \infty$, are plotted in Figure 5.1. For the point y_1 one gets as

ETNA
Kent State University
etna@mcs.kent.edu

numerical value $y_1 \doteq 0.621391\, i$.

LEMMA 5.5. *The set Γ form Lemma 5.2 is given by*

$$\Gamma = S_{-1} \cup S_0 \cup S_1 \cup S_\infty. \tag{5.21}$$

If the four branch points $z_1, \ldots, z_4$ are removed from $\Gamma \subset \overline{\mathbb{C}}$, then $\Gamma \diagdown \{z_1, \ldots, z_4\}$ consists of four components and the closure of each of them is one of the four sets S_j, $j = -1, 0, 1, \infty$.

In many respects the next theorem is the analogue to Theorem 4.1 for diagonal quadratic Hermite-Padé polynomials. It is also the main result of the present subsection.

THEOREM 5.6. *For the zeros of the rescaled diagonal Hermite-Padé polynomials P_n, Q_n, R_n and the remainder term E_n introduced in (5.4) the following assertions hold true:*

(i) All zeros of the polynomial P_n cluster on S_{-1} for $n \to \infty$.

(ii) All zeros of the polynomial Q_n cluster on S_0 for $n \to \infty$.

(iii) All zeros of the polynomial R_n cluster on S_1 for $n \to \infty$.

(iv) All zeros of the remainder term E_n cluster on S_∞ for $n \to \infty$.

In Figure 5.2 all zeros of the rescaled Hermite-Padé polynomials P_{33}, Q_{33}, R_{33} and some zeros of the remainder term E_{33} are plotted together with the arcs of S_{-1}, S_0, S_1, and parts of the arcs of S_∞ on which the zeros cluster. The picture shows that already for the degree $n = 33$ there is a very good accordance between the zeros and their asymptotic cluster sets.

A proof of Theorem 5.6 follows rather immediately from a result about strong asymptotics for the P_n, Q_n, R_n and E_n in Theorem 2.9 in [37]. In the present review we will not concentrate on details of the proof, instead we shall discuss some other types of asymptotic relations in the next subsection.

5.3. Further Asymptotic Results. In [37] not only the asymptotic cluster sets of the zeros of the rescaled diagonal quadratic Hermite-Padé polynomials P_n, Q_n, R_n and the remainder term E_n have been studied. Also their asymptotic distributions, and besides of that also asymptotic relations for the polynomials and the remainder term themselves have been addressed. Again, the functions h and $h_{\max}$ are the basic input for the asymptotic relations.

We start our review with results concerning the asymptotic distributions of zeros, and for this purpose we introduce four measures μ_j, $j = -1, 0, 1, \infty$.

DEFINITION 5.7. *Let ν be the measure introduced in Lemma 5.3 and S_j, $j = -1, 0, 1, \infty$, the sets from Definition 5.2. Based on these objects we define*

$$\mu_j := \nu|_{S_j}, \quad j = -1, 0, 1, \infty. \tag{5.22}$$

Let $Z(p)$ be the (multi-) set of all zeros of a polynomial p with multiplicities taken account of by repetition, let further ν_p be the zero counting measure of the polynomial p, which puts weight 1 at each zero of p taking account of multiplicities, i.e., $\nu_p(B) = \operatorname{card}(Z(p) \cap B)$ for every Borel set $B \subset \mathbb{C}$. Let further $\overset{*}{\longrightarrow}$ denote the weak convergence of measures. The next result has been proved in [37].

THEOREM 5.8. *(cf. [37], Theorem 2.1) For the zeros of the rescaled diagonal Hermite-Padé polynomials P_n, Q_n, R_n and the remainder terms E_n we have the following asymptotic relations*

$$\frac{1}{n}\nu_{P_n} \overset{*}{\longrightarrow} \mu_{-1} \quad \text{as} \quad n \to \infty, \tag{5.23}$$

$$\frac{1}{n}\nu_{Q_n} \overset{*}{\longrightarrow} \mu_0 \quad \text{as} \quad n \to \infty, \tag{5.24}$$

$$\frac{1}{n}\nu_{R_n} \overset{*}{\longrightarrow} \mu_1 \quad \text{as} \quad n \to \infty, \tag{5.25}$$

$$\frac{1}{n}\nu_{E_n} \overset{*}{\longrightarrow} \mu_\infty + 3\,\delta_0 \quad \text{as} \quad n \to \infty, \tag{5.26}$$

ETNA
Kent State University
etna@mcs.kent.edu

where δ_0 is the Dirac measure at $z = 0$.

Theorem 5.8 complements Theorem 5.6 in the sense that we now not only know where the zeros of P_n, Q_n, R_n and of E_n cluster but also how these zeros are asymptotically distributed.

Via Lemma 5.3 and Definition 5.7 the asymptotic distributions μ_j, $j = -1, 0, 1, \infty$, can be traced back to the function $h_{\max}$, for which we have a constructive and a numerically efficient definition. More about this can be found in [37], Theorem 2.10 and .2.11. The next proposition is a rather immediate consequence of Lemma 5.3, and it sheds light on the analytic background of the definition of the asymptotic distributions μ_j, $j = -1, 0, 1, \infty$.

PROPOSITION 5.9. *(cf. [37], Theorem 2.8) The measures μ_j, $j = -1, 0, 1, \infty$, in (5.22) are absolutely continuous, their supports* $\operatorname{supp}(\mu_j) = S_j$, $j = -1, 0, 1, \infty$, *consist of analytic arcs, and for their densities we have the representation*

$$\frac{d\mu_j}{ds}(z) = \frac{1}{\pi}\left(\frac{\partial}{\partial n_+}h_{\max}(z) - \frac{\partial}{\partial n_-}h_{\max}(z)\right) \tag{5.27}$$

for $z \in S_j$, $j = -1, 0, 1, \infty$, where $\partial/\partial n_+$ and $\partial/\partial n_-$ are the normal derivatives to both sides of the arcs in S_j, ds is the line element on S_j, and orientations have to be chosen in (5.27) in such a way that all measures μ_j, $j = -1, 0, 1, \infty$, are positive.

In [38], Theorem 2.8, results about the asymptotic distributions of the zeros of P_n, Q_n, R_n and E_n have been proved that are a full degree more precise than those given in (5.23) - (5.26) of Theorem 5.8. These stronger relations in [38] are precise enough to distinguish asymptotically individual positions of zeros. Naturally, their formulation is more complicated and involved, and besides of the two functions h and $h_{\max}$ there are also branches of the function $\psi \circ \pi^{-1}$ needed.

In the second part of the present subsection, we come to the asymptotic relations for the polynomials P_n, Q_n, R_n and the remainder term E_n themselves. We started with some auxiliary definitions.

Near infinity the multi-valued function $h \circ \pi^{-1}$ has three branches, which we denote by h_{-1}, h_0, and h_1, and which have the local developments

$$h_{-1}(z) = -3\ \mathrm{Re}\,(z) + \log|z| + \mathrm{O}(1/z), \tag{5.28}$$

$$h_0(z) = \log|z| + \log(2) + \mathrm{O}(1/z), \tag{5.29}$$

$$h_1(z) = 3\ \mathrm{Re}\,(z) + \log|z| + \mathrm{O}(1/z), \tag{5.30}$$

as $z \to \infty$. The expressions in (5.28) - (5.30) follow rather directly from the defining relations (5.5) and (5.6) in Definition 5.1. In a neighborhood of the origin $z = 0$ the function $h \circ \pi^{-1}$ has again three branches; here we are only interested in the branch that has a logarithmic term. We denote this branch by h_∞, and remark that it corresponds to the function h near the point $\zeta_0 \in \mathcal{R}$ that has been introduced in (5.10). For h_∞ we have the local development

$$h_\infty(z) = 3\ \log|z| + \mathrm{O}(1) \quad \text{as} \quad z \to 0. \tag{5.31}$$

DEFINITION 5.10. *In the four domains*

$$D_j := \overline{\mathbb{C}} \setminus S_j, \qquad j = -1, 0, 1, \infty, \tag{5.32}$$

we define four functions h_j, $j = -1, 0, 1, \infty$, by harmonic continuations. In case of the first three functions h_{-1}, h_0, h_1 we start the continuation with the three function elements (5.28), (5.29), (5.30) at infinity and continue the process throughout the domains D_{-1}, D_0,

ETNA
Kent State University
etna@mcs.kent.edu

D_1, respectively. In the case of the function h_∞, we start from the origin and use (5.31) as initial function element.

Let us remark that the four branch points $z_1, \ldots, z_4$ belong to the two sets S_0 and S_∞, and hence the two functions h_0 and h_∞ can be continued throughout the two domains D_0 and D_∞ without any problem. In the cases of the two other functions h_{-1} and h_1, an inspection of the behavior of the function h on the Riemann surface $\mathcal{R}$ shows that also here the process of harmonic continuation can be carried out without hitting a branch point within one of the domains D_{-1} or D_1.

With the four functions h_j, $j = -1, 0, 1, \infty$, we can formulate the asymptotic relations for the polynomials P_n, Q_n, R_n and the remainder term E_n.

THEOREM 5.11. *(cf. [37], Theorem 2.2) For the rescaled diagonal Hermite-Padé polynomials P_n, Q_n, R_n and the remainder term E_n we have the following asymptotic relations:*

$$\lim_{n\to\infty} \frac{1}{n} \log|P_n(z)| = h_{-1}(z) + 3\ \mathrm{Re}\,(z) \quad \textit{for} \quad z \in D_{-1}, \tag{5.33}$$

$$\lim_{n\to\infty} \frac{1}{n} \log|Q_n(z)| = h_0(z) \qquad \textit{for} \quad z \in D_0, \tag{5.34}$$

$$\lim_{n\to\infty} \frac{1}{n} \log|R_n(z)| = h_1(z) - 3\ \mathrm{Re}\,(z) \quad \textit{for} \quad z \in D_1, \tag{5.35}$$

$$\lim_{n\to\infty} \frac{1}{n} \log|E_n(z)| = h_\infty(z) \qquad \textit{for} \quad z \in D_\infty. \tag{5.36}$$

In Theorem 2.2 of [37] a stronger version of asymptotic relations of the type (5.33) - (5.36) has been proved, which however will not be discussed here.

In the next Section we extend all results that have been reviewed in the present section to the non-diagonal case of quadratic Hermite-Padé polynomials.

6. Asymptotics for Non-Diagonal Quadratic Hermite-Padé Polynomials. The investigations by E.B. Saff and R.S. Varga in [35], which have been reviewed in Section 4, have strongly motivated the research in the present section. In [35] not only diagonal sequences, but the whole range of non-diagonal ray sequences of Padé polynomials has been studied, and among other very interesting results it has been shown how the asymptotic cluster sets of the zeros of non-diagonal Padé polynomials continuously change with the angle of the ray sequences in the Padé table. As a consequence one can see how the typical situation of diagonal Padé approximants transforms step by step into that of Taylor polynomials.

In the present section we study analogous questions for quadratic Hermite-Padé polynomials of type I, i.e., we extend the asymptotic relations that have been presented in the last section to non-diagonal ray sequences. Again, it is possible to see how the asymptotic relations of the diagonal case of quadratic Hermite-Padé polynomials transform step by step into those of Padé approximants.

It turns out that the basic structure of the asymptotic relations is very similar in a topological sense for all non-diagonal ray sequences. Therefore, we can use concepts and notations from the last section, only that several formulae have to be changed, they become, of course, somewhat more complicated and also dependent on parameters. On the other hand, the use of the earlier terminology and that of conceptional analogies allow us to give a rather short presentation of the new results.

In the next subsection we introduce general notations, which includes the introduction of the two parameters λ and κ, which specify ray sequences. In Subsection 6.2, a Riemann surface $\mathcal{R}$ and two functions ψ and h are introduced that generalize the corresponding objects from Subsection 5.1. In close analogy to Subsection 5.2, we define two systems Γ and $\widetilde{\Gamma}$ of Jordan arcs with the help of the function h from Subsection 6.1. As before, arcs from Γ will be

ETNA
Kent State University
etna@mcs.kent.edu

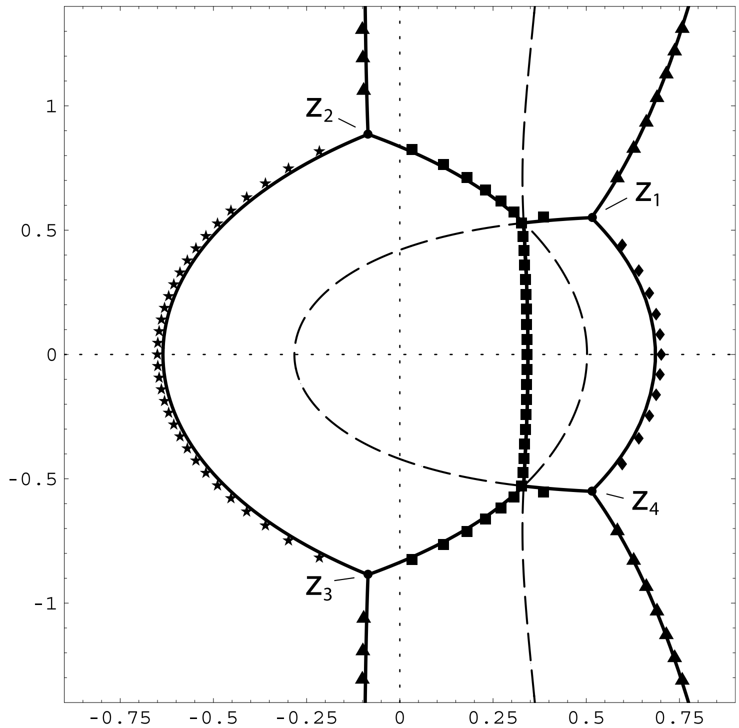

FIG. 6.1. *The zeros of the rescaled non-diagonal quadratic Hermite-Padé polynomials P_{33} (stars), Q_{33} (boxes), R_{11} (diamonds), and some of the zeros of the associated remainder term $E_{(34,34,12)}$ (triangles) together with the arcs of the set Γ (full lines) from Lemma 6.3 and the arc of $\widetilde{\Gamma} \setminus \Gamma$ (dashed lines) for $\lambda = 1$ and $\kappa = 1/3$. Note that the scales of the $x-$ and $y-$axes differ by a factor* 1.5.

the building blocks for the asymptotic cluster sets of the zeros of the non-diagonal Hermite-Padé polynomials and the associated remainder term. The description of these objects is done in Subsection 6.3. In Subsection 6.4, additional asymptotic results are given. Here, we use the analogy to results presented in Subsection 5.3 for the diagonal case of quadratic Hermite-Padé polynomials. At last, in Subsection 6.5 we describe and discuss the changes that are necessary in the proofs of [37] in order to adapt and transform them into proofs of the results stated here for the non-diagonal situation.

6.1. Non-Diagonal Ray Sequences of Quadratic Hermite-Padé Polynomials. By p_m, q_k, r_l we denote the three quadratic Hermite-Padé polynomials of type I in the same way as in Section 5 only that now their degrees $m, k, l \in \mathbb{N}$ are no longer assumed to be equal. The three polynomials $p_m \in \mathcal{P}_m$, $q_k \in \mathcal{P}_k$, $r_l \in \mathcal{P}_l$ correspond to the three polynomials $p_{1,n}$, $p_{2,n}$, $p_{3,n}$ in Definition 2.1 with multi-index $n := (m+1, k+1, l+1) \in \mathbb{N}^3$, $m = 2$, and the exponential system $\mathfrak{f} = (1, exp, exp^2)$. Note that each of the three polynomials p_m, q_k, r_l depends on all three indices m, k, l. The three polynomials satisfy the relation

$$p_m(w) + q_k(w)\, e^w + r_l(w)\, e^{2w} = \mathrm{O}(w^{|n|-1}) \quad \text{as} \quad w \to 0, \tag{6.1}$$

which is a rewriting of the defining relation (2.7) in Definition 2.1. We have $|n| = m + k + l + 3$.

ETNA
Kent State University
etna@mcs.kent.edu

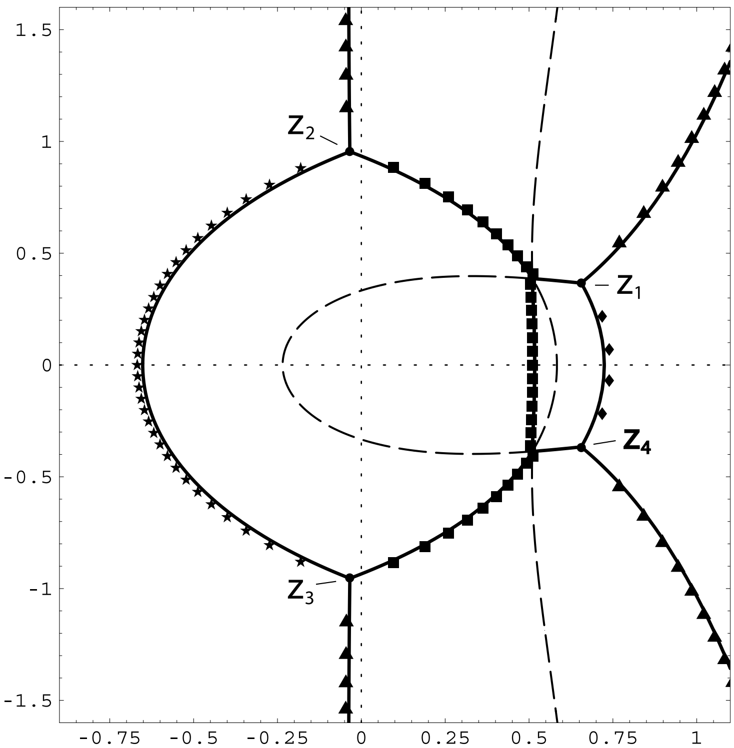

FIG. 6.2. *The zeros of the rescaled non-diagonal quadratic Hermite-Padé polynomials P_{33} (stars), Q_{33} (boxes), R_4 (diamonds), and some of the zeros of the associated remainder term $E_{(34,34,5)}$ (triangles) together with the arcs of the set Γ (full lines) from Lemma 6.3 and the arc of $\widetilde{\Gamma} \setminus \Gamma$ (dashed lines) for $\lambda = 1$, $\kappa = 4/33$. Note that the scales of the $x-$ and the $y-$axes differ by a factor 1.5.*

For the rescaling method we now use as new rescaled independent variable

$$z = \frac{w}{|n|}. \tag{6.2}$$

Note that the three polynomials p_m, q_k, r_l together possess $m + k + l + 3$ free coefficient. The transformation (6.2) leads to the rescaled Hermite-Padé polynomials

$$P_m(z) := p_m(|n|\, z),\, Q_k(z) := q_k(|n|\, z), \text{ and } R_l(z) := r_l(|n|\, z). \tag{6.3}$$

Together with a multiplication by $e^{-|n|\, z}$, relation (6.1) transform into

$$E_n(z) := e^{-|n|\, z} P_m(z) + Q_k(z) + R_l(z) e^{|n|\, z} = \mathrm{O}(z^{|n|-1}) \quad \text{as} \quad z \to 0 \tag{6.4}$$

in the new variable z. Since the two relations (6.1) and (6.4) are homogeneous, we can assume that P_m is monic, and since we also know that the exponential system $\mathfrak{f} = (1, exp, exp^2)$ is perfect (cf. [18]), we can make the further going assumption that

$$P_m(z) = z^m + \ldots \tag{6.5}$$

In the five Figures 6.1 - 6.5 the zeros of the rescaled polynomials P_m, Q_k, R_l, and some of the zeros of the remainder term E_n are plotted for different choices of degrees. Besides of

ETNA
Kent State University
etna@mcs.kent.edu

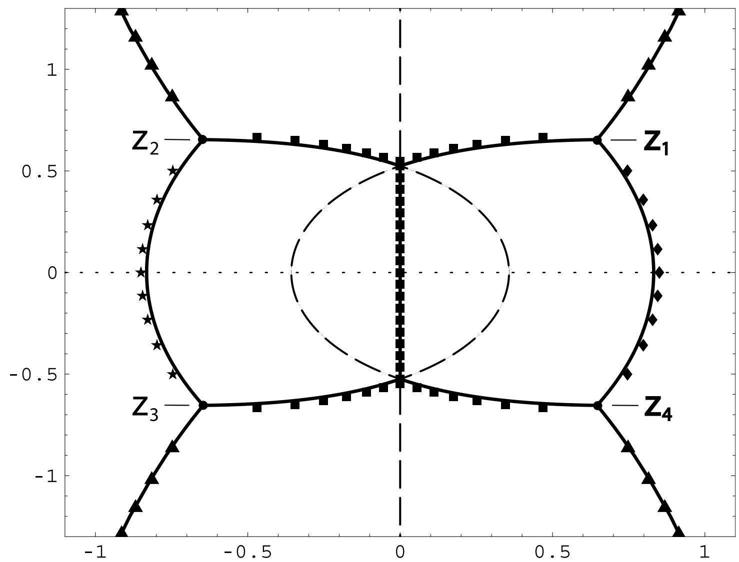

FIG. 6.3. *The zeros of the rescaled non-diagonal quadratic Hermite-Padé polynomials P_9 (stars), Q_{45} (boxes), R_9 (diamonds), and some of the zeros of the associated remainder term $E_{(10,46,10)}$ (triangles) together with the arcs of the set Γ (full lines) from Lemma 6.3 and the arc of $\widetilde{\Gamma} \setminus \Gamma$ (dashed lines) for $\lambda = 5$ and $\kappa = 1$. Note that the scales of the $x-$ and $y-$axes differ by a factor* 1.5.

the zeros, there are also lines, which, however, will only be explained further below. The two first Figures 6.1 and 6.2 should be seen together with the two earlier Figures 5.2 and 4.2. The whole sequence of the Figures 5.2, 6.1, 6.2, and 4.2 illustrates how the configuration of the diagonal case of Hermite-Padé polynomials in Figure 5.2 step by step transforms into that of the diagonal case of Padé polynomials in Figure 4.2.

The three Figures 6.3 - 6.5 illustrate how the asymptotic relations change when the degree k of the middle polynomial Q_k is varied, while the degrees m and l of the two outer polynomial P_m and R_l are kept fixed.

For the asymptotic analysis it is necessary that the three indices $m, k, l \in \mathbb{N}$ tend to infinity along ray sequences $\{(m_j, k_j, l_j)\}_{j\in\mathbb{N}}$. In a ray sequences $\{(m_j, k_j, l_j)\}_{j\in\mathbb{N}}$ we assume that by definition the following limits hold true:

$$m_j \to \infty, \quad \frac{k_j}{m_j} \to \lambda, \quad \frac{l_j}{m_j} \to \kappa \quad \text{as} \quad j \to \infty \quad \text{with} \quad \lambda, \kappa \in (0, \infty)\,. \tag{6.6}$$

In the case of the diagonal sequence we have $\lambda = \kappa = 1$. If the parameters λ and κ vary through $(0, \infty)^2$, we cover all possible ray sequences in the table of quadratic Hermite-Padé polynomials of type I.

6.2. The Riemann Surface $\mathcal{R}$ and two Functions ψ and h. In analogy to Subsection 5.1, a Riemann surface $\mathcal{R}$ and two functions ψ and h are introduced, which all three are

ETNA
Kent State University
etna@mcs.kent.edu

immediate generalizations of the corresponding objects in the Definitions 5.1 and 5.2.

DEFINITION 6.1. *For $\lambda, \kappa \in (0, \infty)$, the Riemann surface $\mathcal{R}$ and the bijective mapping $\psi : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$ are defined by the property that the two functions ψ and*

$$z(v) := \frac{1}{1+\lambda+\kappa}\left(\frac{1}{v+1} + \frac{\lambda}{v} + \frac{\kappa}{v-1}\right) = \frac{(1+\lambda+\kappa)\, v^2 + (\kappa - 1)\, v + \lambda}{(1+\lambda+\kappa)\, v(v^2-1)}, \quad v \in \overline{\mathbb{C}}, \tag{6.7}$$

satisfy the relation

$$z \circ \psi(\zeta) = \pi(\zeta) \quad \text{for} \quad \zeta \in \mathcal{R} \tag{6.8}$$

with $\pi : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$ the canonical projection of the Riemann surface $\mathcal{R}$.

The surface $\mathcal{R}$ as well as the function ψ depend on the two parameters λ, κ. It is immediate that Definition 5.1 is a special case of Definition 6.1 (one has only to choose $\lambda = \kappa = 1$). Again, $\psi \circ \pi^{-1}$ is an algebraic function of 3rd degree. For any combination of the two parameters $\lambda, \kappa \in (0, \infty)$, the Riemann surface $\mathcal{R}$ has four branch points over the base points $z_1, \ldots, z_4 \in \mathbb{C}$. The dependence of the points $z_1, \ldots, z_4$ on the two parameters λ, κ is too complicated for making it worthwhile to give an explicit formula for the points. This is a remarkable difference to formula (5.7). The four branch points $z_1, \ldots, z_4$ form two conjugated pairs $\{z_1, z_4\}$ and $\{z_2, z_3\}$, and they can be calculated in each concrete situation by solving the equation $z(v)' = 0$ in the $v-$plane and using then the mapping ψ. The procedure has been described after (5.7).

As before, the Riemann surface $\mathcal{R}$ can be broken down in three sheets $\widetilde{S}_{-1}$, $\widetilde{S}_0$, $\widetilde{S}_1$, which are glued together along two Jordan arcs $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_{-1}$ in the same way as this has been described after (5.10). As in (5.10) the point $\zeta_0 \in \mathcal{R}$ is defined by

$$\zeta_0 := \psi^{-1}(\infty), \tag{6.9}$$

and the defining relation (6.4) of the Hermite-Padé polynomials P_m, Q_k, R_l is lifted from $\mathbb{C}$ to a neighborhood of ζ_0 on $\mathcal{R}$.

After the Riemann surface $\mathcal{R}$ and the function ψ have been defined for the new situation, we come to the introduction of the function h on $\mathcal{R}$, which is done in the same way as in Definition 5.2.

DEFINITION 6.2. *For $\lambda, \kappa \in (0, \infty)$, the functions $h : \mathcal{R} \longrightarrow \overline{\mathbb{R}}$ is defined by*

$$h(\zeta) := \operatorname{Re}\, u \circ \psi(\zeta) \quad \text{for} \quad \zeta \in \mathcal{R} \tag{6.10}$$

and the function u in (6.10) is now given by

$$u(v) := \frac{(\lambda+\kappa)\, v^2 + (\kappa-1)\, v + (1-\lambda)}{v^2 - 1} + \log \frac{2^{\kappa}}{(1+\lambda+\kappa)(v+1)v^{\lambda}(v-1)^{\kappa}}. \tag{6.11}$$

Again, it is immediate that the earlier Definition 5.2 is a special case of Definition 6.2. As the Riemann surface $\mathcal{R}$ and the function ψ, so also the function h depends on the two parameters λ and κ in an essential way. As before in Subsection 5.1, it follows from (6.7), (6.8), (6.10), and (6.11) that for all $\lambda, \kappa \in (0, \infty)$ the function h is harmonic in $\mathcal{R} \setminus (\{\zeta_0\} \cup \pi^{-1}(\{\infty\}))$ and subharmonic at ζ_0.

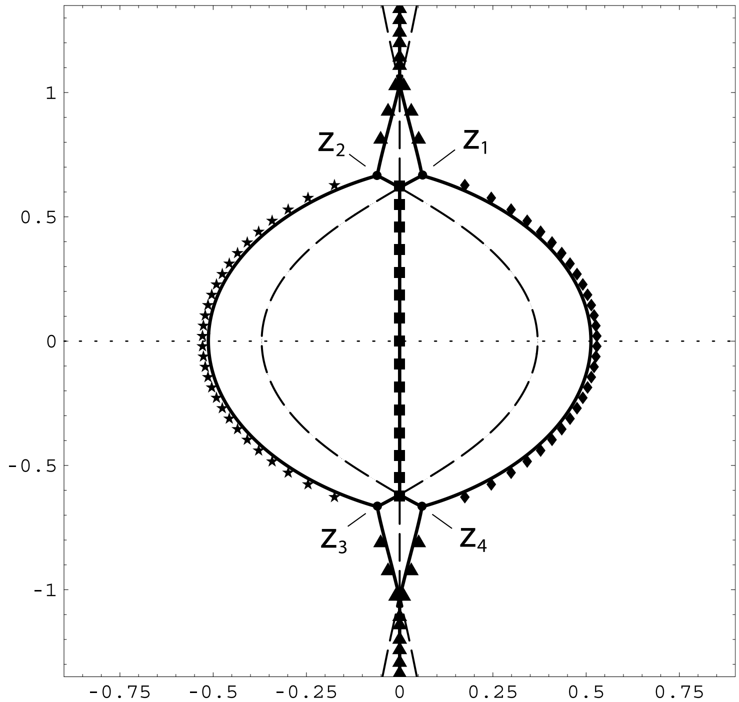

FIG. 6.4. *The zeros of the rescaled non-diagonal quadratic Hermite-Padé polynomials* P_{30} *(stars),* Q_{15} *(boxes),* R_{30} *(diamonds), and some of the zeros of the associated remainder term* $E_{(31,16,31)}$ *(triangles) together with the arcs of the set* Γ *(full lines) from Lemma 6.3 and the arc of* $\widetilde{\Gamma} \setminus \Gamma$ *(dashed lines) for* $\lambda = 1/2$ *and* $\kappa = 1$*. Note that the scales of the* $x-$ *and* $y-$*axes differ by a factor* 1.5*.*

6.3. The Definition of Jordan Arcs and a First Result. The definition of the asymptotic cluster sets S_{-1}, S_0, S_1, and S_∞ for the zeros of the polynomials P_m, Q_k, R_l, and the remainder term E_n, respectively, is central for all other asymptotic relations in the present investigation. Starting point for a definition is the function

$$h_{\max}(z) := \max\{\, h(\zeta) \mid \zeta \in \mathcal{R},\, \pi(\zeta) = z \,\} \quad \text{for} \quad z \in \overline{\mathbb{C}}, \tag{6.12}$$

which now depends on the two parameters λ and κ. With the same arguments from potential theory as those mentioned after (5.13) in Subsection 5.2, it follows that $h_{\max}$ is subharmonic in $\mathbb{C}$ and harmonic outside of a system Γ of analytic Jordan arcs. The consequences of this observation have been formulated in Lemma 5.3, and this lemma holds also true in the new situation.

LEMMA 6.3. *The conclusions of Lemma 5.3 hold true for any combination of parameters* $\lambda, \kappa \in (0, \infty)$*. The system* Γ *of analytic Jordan arcs in* $\mathbb{C}$ *and a measure* ν *with* $\operatorname{supp}(\nu) = \Gamma$ *depend now on the two parameters* λ *and* κ*.*

The analytic Jordan arcs in Γ belong to the larger set

$$\widetilde{\Gamma} := \left\{\, z \in \overline{\mathbb{C}} \,\middle|\, \operatorname{card}\{\, h(\zeta) \mid \zeta \in \pi^{-1}(\{z\})\} < 3 \right\} \tag{6.13}$$

of intersection arcs of the branches of the function $h \circ \pi^{-1}$. The arcs in Γ are the building blocks of the asymptotic cluster sets S_{-1}, S_0, S_1, and S_∞, and it follows from Lemma 6.3 and (6.13) that the set Γ consists of those subarcs of $\widetilde{\Gamma}$ on which the intersecting branches of

ETNA
Kent State University
etna@mcs.kent.edu

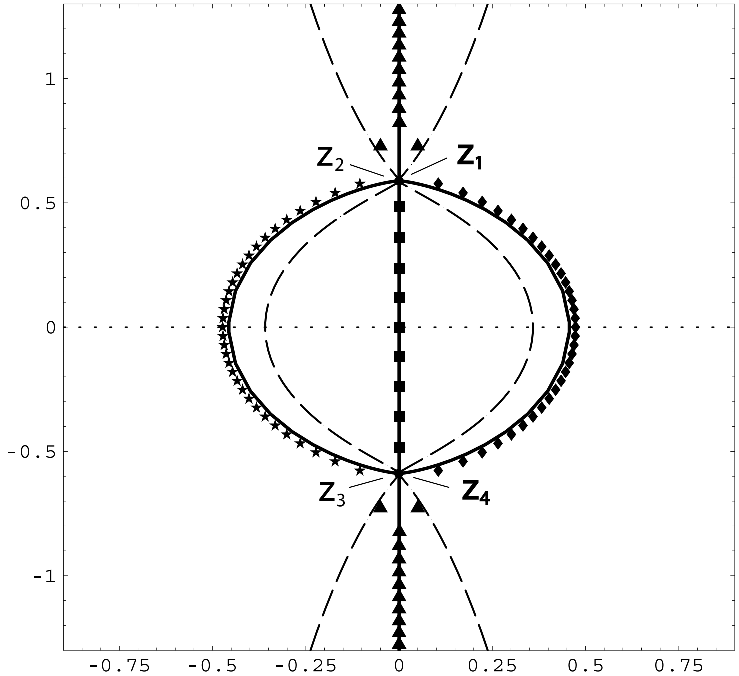

FIG. 6.5. *The zeros of the rescaled non-diagonal quadratic Hermite-Padé polynomials* P_{33} *(stars),* Q_9 *(boxes),* R_{33} *(diamonds), and some of the zeros of the associated remainder term* $E_{(34,10,34)}$ *(triangles) together with the arcs of the set* Γ *(full lines) from Lemma 6.3 and the arc of* $\widetilde{\Gamma} \setminus \Gamma$ *(dashed lines) for* $\lambda = 3/11$ *and* $\kappa = 1$. *Note that the scales of the* $x-$ *and* $y-$*axes differ by a factor* 1.5.

$h \circ \pi^{-1}$ have the same value as that of the function $h_{\max}$. Like the functions h and $h_{\max}$, so also the sets Γ and $\widetilde{\Gamma}$ depend on the two parameters λ and κ.

A good way of calculating the arcs of Γ is to start from the four branch points $z_1, \ldots, z_4$. There exist six Jordan arcs in $\widetilde{\Gamma}$ that end at these four points. They are denoted by $\Gamma_{i,j}$, $i = -1, 1$, $j = -1, 1, \infty$ as before. Three of them connect the two points z_1 and z_4, we denote them by $\Gamma_{1,j}$, $j = -1, 1, \infty$, and the other three connect the two points z_2 and z_3, and we denote them by $\Gamma_{-1,j}$, $j = -1, 1, \infty$. In each of the two groups, there exists one arc that passes through infinity, another one that has an intersection with $(-\infty, 0)$, and a third one that has an intersection with $(0, \infty)$. According to these three property we denote the three arcs by $\Gamma_{i,\infty}, \Gamma_{i,-1}, \Gamma_{i,1}$ in each group $i = -1, 1$, respectively. For a better orientation one may consult Figure 5.1.

The angle with which the six arcs $\Gamma_{i,j}$, $i = -1, 1$, $j = -1, 1, \infty$, end at the four points $z_1, \ldots, z_4$ can be calculated from local developments of the branches of the function $h \circ \pi^{-1}$ near the four points $z_1, \ldots, z_4$.

The two Jordan arcs $\Gamma_{1,-1}$ and $\Gamma_{-1,1}$ intersect in exactly two points $y_1, y_2 \in \mathbb{C}$, and we have $y_1 = \overline{y}_2$. In the symmetric situation of Section 5, these two intersection points y_1 and y_2 had to lie on the imaginary axis, this is no longer the case in the general situation. The intersection points y_1 and y_2 are important for the definition of the cluster set S_0 in Definition 6.4. The set S_0 is constructed in the following way: Starting from the four points $z_1, \ldots, z_4$, one follows the arcs $\Gamma_{1,-1}$ and $\Gamma_{-1,1}$ up to the two intersection points y_1 and y_2. Then one stops following the arcs $\Gamma_{1,-1}$ and $\Gamma_{-1,1}$, and instead one follows an intersection

ETNA
Kent State University
etna@mcs.kent.edu

arc from $\widetilde{\Gamma} \setminus (\Gamma_{1,-1} \cup \Gamma_{-1,1})$ that connects the two intersection points y_1 and y_2. Details are given in the next definition.

The two arcs $\Gamma_{1,\infty}$ and $\Gamma_{-1,\infty}$ are essential for the definition of the cluster set S_∞, but contrary to the symmetric situation in Subsection 5.2, now the two arcs may have two intersection points in $\mathbb{C}$, which has consequences for the construction of the set S_∞. If such intersection points exist, then the procedure is analogous to that applied in the definition of the set S_0.

After these preparations we can define the four asymptotic cluster sets $S_j = S_j(\lambda, \kappa)$, $j = -1, 0, 1, \infty$, for the zeros of non-diagonal sequences of the rescaled Hermite-Padé polynomials P_m, Q_l, R_k and the associated remainder term E_n. The definition is a generalization of Definition 5.4.

DEFINITION 6.4. *For $\lambda, \kappa \in (0, \infty)$, let the point $y_1 \in \mathbb{C}$ be defined by $y_1 := \Gamma_{1,-1} \cap \Gamma_{-1,1} \cap \{ z \mid \operatorname{Im}(z) > 0 \}$ and set $y_2 = \overline{y}_1$. Then we set*

$$S_1 = S_1(\lambda, \kappa) := \Gamma_{1,1}, \tag{6.14}$$

$$S_{-1} = S_{-1}(\lambda, \kappa) := \Gamma_{-1,-1}. \tag{6.15}$$

Let $[z_1, y_1]$ denote the section of the arc $\Gamma_{1,-1}$ that connects the two points z_1 and y_1, and let the three sections $[z_4, y_2] \subset \Gamma_{1,-1}$, $[z_2, y_1] \subset \Gamma_{-1,1}$, $[z_3, y_2] \subset \Gamma_{-1,1}$ be define in an analogous way. Let, further, $[y_1, y_2]$ be the shortest section of the arc or curve in $\widetilde{\Gamma} \setminus (\Gamma_{1,-1} \cup \Gamma_{-1,1})$ that connects the two points y_1 and y_2. With these definitions we set

$$S_0 = S_0(\lambda, \kappa) := [z_1, y_1] \cup [z_2, y_1] \cup [z_3, y_2] \cup [z_4, y_2] \cup [y_1, y_2]\,. \tag{6.16}$$

If the two arcs $\Gamma_{1,\infty}$ and $\Gamma_{-1,\infty}$ are disjoint in $\mathbb{C}$, then we set

$$S_\infty = S_\infty(\lambda, \kappa) := \Gamma_{1,\infty} \cup \Gamma_{-1,\infty}, \tag{6.17}$$

otherwise, the set S_∞ is put together from sections of the arcs $\Gamma_{1,\infty}$ and $\Gamma_{-1,\infty}$ that connect the points $z_1, \ldots, z_4$ with the intersection points, and in addition two sections from $\widetilde{\Gamma} \setminus (\Gamma_{1,\infty} \cup \Gamma_{-1,\infty})$ that connect the two intersection points with infinity. The situation is analogous to the definition of S_0, and Figure 6.4 is an illustrative example for this construction.

The next lemma can be helpful for the understanding of the definition of the sets S_{-1}, S_0, S_1, and S_∞.

LEMMA 6.5. *For every pair of parameters $(\lambda, \kappa) \in (0, \infty)$ the following three assertions hold true:*

(i) If from the set Γ of Lemma 6.2 the four branch points $z_1, \ldots, z_4$ are removed, then $\Gamma \diagdown \{z_1, \ldots, z_4\}$ consists of four components, and the closure of each of these components is one of the four sets $S_j = S_j(\lambda, \kappa)$, $j = -1, 0, 1, \infty$, from Definition 6.4.

(ii) The system Γ consists of those arcs in $\widetilde{\Gamma}$ on which the values of the intersecting branches of $h \circ \pi^{-1}$ are identical with the value of $h_{\max}$.

(iii) We have

$$\Gamma = S_{-1} \cup S_0 \cup S_1 \cup S_\infty. \tag{6.18}$$

Examples of the union $\Gamma = S_{-1} \cup S_0 \cup S_1 \cup S_\infty$ are shown (full lines) for different constellations of the parameters λ and κ in the five Figures 6.1 - 6.5. Since in these figures the whole set Γ has been represented by the same type of lines, the four cluster sets S_{-1}, S_0, S_1, S_∞ have to be identified and distinguished by the type of zeros that are lying close to them. The set Γ is contained in $\widetilde{\Gamma}$, and the arcs in $\widetilde{\Gamma} \setminus \Gamma$ are represented by dashed lines in the five Figures 6.1 - 6.5.

The next theorem generalizes Theorem 5.6, and it is the analogue to Theorem 4.1, which is concerned with non-diagonal Padé polynomials.

THEOREM 6.6. *Let the ray sequence of multi-indices* $\{(m_j+1, k_j+1, l_j+1)\}$ *satisfy conditions (6.6) with parameters* $\lambda, \kappa \in (0,\infty)$*, then the zeros of the rescaled quadratic Hermite-Padé polynomials* P_{m_j}*,* Q_{k_j}*,* R_{l_j} *and the associated remainder terms* E_{n_j}*,* $n_j = (m_j+1, k_j+1, l_j+1)$*, as introduced in (6.4) satisfy the following asymptotic relations:*

(i) All zeros of the polynomials P_{m_j} *cluster on* $S_{-1}(\lambda,\kappa)$ *for* $j \to \infty$*.*

(ii) All zeros of the polynomials Q_{k_j} *cluster on* $S_0(\lambda,\kappa)$ *for* $j \to \infty$*.*

(iii) All zeros of the polynomials R_{l_j} *cluster on* $S_1(\lambda,\kappa)$ *for* $j \to \infty$*.*

(iv) All zeros of the remainder terms E_{n_j} *cluster on* $S_\infty(\lambda,\kappa)$ *for* $j \to \infty$*.*

The sets S_{-1}*,* S_0*,* S_1*, and* S_∞ *have been introduced in Definition 6.4.*

In the five Figures 6.1 - 6.5, zeros of the rescaled non-diagonal Hermite-Padé polynomials and their associated remainder terms are plotted together with the union Γ of their asymptotic cluster sets S_{-1}, S_0, S_1, and parts of S_∞ for various constellations of degrees m, k, l, and consequently parameters $\lambda, \kappa \in (0,\infty)$. The pictures show that there is a very good accordance between the zeros and their asymptotic cluster sets already for the chosen moderate degrees. These pictures also show how the configuration of the asymptotic cluster sets S_{-1}, S_0, S_1, and S_∞ varies with the parameters $\lambda, \kappa \in (0,\infty)$.

Figure 6.5 deserves a special remark. In this figure the two branch points $z_1 = 0.00225 + 0.58856\,i$ and $z_2 = -0.00225 + 0.58856\,i$ as well as two branch points $z_3 = -0.00225 - 0.58856\,i$ and $z_4 = 0.00225 - 0.58856\,i$ are laying so close together that they cannot be distinguished in Figure 6.5.Near to these points lie also the intersection points y_1, y_2 of $\Gamma_{1,-1}$ with $\Gamma_{-1,1}$ and the two intersection points of $\Gamma_{1,\infty}$ and $\Gamma_{-1,\infty}$. These points can also not be distinguished from their neighbors. As a consequence, the sets S_0 and S_∞ appear in Figure 6.5 as straight lines, but in reality there are bifurcations at both ends.

6.4. Further Asymptotic Results. In the present Subsection the results from [37] that have been reviewed in Subsection 5.3 are extended to non-diagonal ray sequences. The structure of presentation is practically identical with that of Subsection 5.3, and remarks from there remain valid in the new circumstances. Changes and adaptations have become necessary only because of peculiarities connected with non-diagonal ray sequences, which are characterized, as before, by the two parameters $\lambda, \kappa \in (0,\infty)$ from (6.6).

Like in Definition 5.7, we introduce four measures μ_j, $j = -1, 0, 1, \infty$, which will be the asymptotic distributions of the zeros of the rescaled non-diagonal quadratic Hermite-Padé polynomials P_m, Q_l, R_k and the associated remainder terms E_n.

DEFINITION 6.7. *From Lemma 5.3 and Lemma 6.3 we know that there exists a measure* ν *for very parameter constellation* $\lambda, \kappa \in (0,\infty)$*. With the sets* $S_j = S_j(\lambda,\kappa)$*,* $j = -1, 0, 1, \infty$*, from Definition 6.2 we define*

$$\mu_j := \nu|_{S_j}, \quad j = -1, 0, 1, \infty. \tag{6.19}$$

The next theorem can be seen as a completion of Theorem 6.6 since now we have not only the cluster sets of the zeros of P_m, Q_l, R_k, and E_n, but also their asymptotic distributions.

THEOREM 6.8. *Let the ray sequence of multi-indices* $\{(m_j+1, k_j+1, l_j+1)\}$ *satisfy conditions (6.6) with parameters* $\lambda, \kappa \in (0,\infty)$*, let* μ_j*,* $j = -1, 0, 1, \infty$*, be the four measures introduced in (6.19), and let further* δ_0 *be the Dirac measure at* $z = 0$*. Then for the zeros of the rescaled non-diagonal Hermite-Padé polynomials* P_{m_j}*,* Q_{k_j}*,* R_{l_j} *and the associated remainder terms* E_{n_j}*,* $n_j = (m_j+1, k_j+1, l_j+1)$ *from (6.4) we have the following* *asymptotic relations:*

$$\frac{1}{m_j}\nu_{P_{m_j}} \xrightarrow{*} \mu_{-1} \quad as \quad j \to \infty, \tag{6.20}$$

$$\frac{1}{k_j}\nu_{Q_{k_j}} \xrightarrow{*} \frac{1}{\lambda}\mu_0 \quad as \quad j \to \infty, \tag{6.21}$$

$$\frac{1}{l_j}\nu_{R_{l_j}} \xrightarrow{*} \frac{1}{\kappa}\mu_1 \quad as \quad j \to \infty, \tag{6.22}$$

$$\frac{1}{m_j}\nu_{E_{n_j}} \xrightarrow{*} \mu_\infty + (1+\lambda+\kappa)\,\delta_0 \quad as \quad j \to \infty. \tag{6.23}$$

The measures μ_{-1}, $\frac{1}{\lambda}\mu_0$, and $\frac{1}{\kappa}\mu_1$ are probability measures, while μ_∞ is a measure with infinite mass.

Proposition 5.9 of Subsection 5.3 holds also in the new situation, and it provides means to calculate the density of the asymptotic distributions μ_j, $j = -1, 0, 1, \infty$.

In the second part of the present subsection, we present asymptotic relations for the rescaled non-diagonal polynomials P_{m_j}, Q_{k_j}, R_{l_j} and the associated remainder terms E_{n_j}. As in (5.28) - (5.29) we first consider the three branches of the multi-valued function $h \circ \pi^{-1}$ near infinity. They are denoted by h_{-1}, h_0, and h_1, and we have

$$h_{-1}(z) = -(1+\lambda+\kappa)\,\mathrm{Re}\,(z) + \log|z| + \mathrm{O}(1/z), \tag{6.24}$$

$$h_0(z) = \lambda\,\log|z| + (\lambda-1) + \log\Big(\frac{(1+\lambda+\kappa)^\lambda 2^\kappa}{(1+\lambda+\kappa)\,\lambda^\lambda}\Big) + \mathrm{O}(1/z), \tag{6.25}$$

$$\begin{aligned} h_1(z) = {} & (1+\lambda+\kappa)\,\mathrm{Re}\,(z) + \kappa\,\log|z| + (\kappa-1) \\ & + \log\Big(\frac{(1+\lambda+\kappa)^\kappa 2^{\kappa-1}}{(1+\lambda+\kappa)\,\kappa^\kappa}\Big) + \mathrm{O}(1/z), \end{aligned} \tag{6.26}$$

as $z \to \infty$. The expressions in (6.24 - 6.26) follow from the defining relations (6.7), (6.8), (6.10) and (6.11) in Definitions 6.1 and 6.2 in the same way as their forerunners (5.28 - 5.30) followed from the analogous relations. In a neighborhood of the origin $z = 0$ we need the branch of the function $h \circ \pi^{-1}$, which possesses a logarithmic term. It is denoted by h_∞, and we have

$$h_\infty(z) = (1+\lambda+\kappa)\log|z| + \mathrm{O}(1) \quad \text{as} \quad z \to 0. \tag{6.27}$$

DEFINITION 6.9. *In the four domains*

$$D_j := \overline{\mathbb{C}} \setminus S_j, \qquad j = -1, 0, 1, \infty, \tag{6.28}$$

we defined four functions h_j, $j = -1, 0, 1, \infty$, by harmonic continuations. In the cases of the first three functions h_{-1}, h_0, h_1 we start the continuation at infinity with the three function elements (6.24), (6.25), (6.26) and continue this process throughout the domains D_{-1}, D_0, D_1, respectively. In the case of the function h_∞, we start from the origin and use as initial function element (6.27).

With the four functions h_j, $j = -1, 0, 1, \infty$, we can formulate the asymptotic relations for the polynomials P_m, Q_k, R_l and the remainder term E_n.

THEOREM 6.10. *Let the ray sequence of multi-indices $\{(m_j+1, k_j+1, l_j+1)\}$ satisfy conditions (6.6) with parameters $\lambda, \kappa \in (0,\infty)$, then for the rescaled quadratic Hermite-Padé polynomials P_{m_j}, Q_{k_j}, R_{l_j} and the associated remainder terms E_{n_j}, $n_j = (m_j+1, k_j+1,$*

ETNA
Kent State University
etna@mcs.kent.edu

$l_j+1)$ *from (6.4) we have the following asymptotic relations:*

$$\lim_{j\to\infty}\frac{1}{m_j}\log|P_{m_j}(z)| = h_{-1}(z) + (1+\lambda+\kappa)\,\mathrm{Re}\,(z) \quad \textit{for} \quad z\in D_{-1}, \tag{6.29}$$

$$\lim_{j\to\infty}\frac{1}{m_j}\log|Q_{k_j}(z)| = h_0(z) \quad \textit{for} \quad z\in D_0, \tag{6.30}$$

$$\lim_{j\to\infty}\frac{1}{m_j}\log|R_{l_j}(z)| = h_1(z) - (1+\lambda+\kappa)\,\mathrm{Re}\,(z) \quad \textit{for} \quad z\in D_1, \tag{6.31}$$

$$\lim_{j\to\infty}\frac{1}{m_j}\log|E_{n_j}(z)| = h_\infty(z) \quad \textit{for} \quad z\in D_\infty. \tag{6.32}$$

The objects on the right-hand side of (6.29) - (6.32) can be calculated efficiently. The most difficult part is perhaps the selection of the right branch of the function $h\circ\pi^{-1}$ in each concrete situation.

6.5. Basic Ideas of the Proofs. The new results of the present subsection are natural extensions of those reviewed in Section 5. The basic structure has remained very much the same, and so the proofs of the earlier results from [37] can be transplanted without principle difficulties. Having this in mind and also the limitation and readability of the present contribution, it seems to be appropriate not to repeat too many details from [37], and instead to concentrate on the basic ideas of the proofs, which then will make plausible the necessary changes in procedure of [37].

The basic tool for the proofs in [37] is a saddle point method for the evaluation of certain integrals. The analysis starts from explicit representations for the rescaled Hermite-Padé polynomials P_m, Q_k, R_l and the associated remainder term E_n, $n=(m+1,k+1,l+1)$, which are given for the diagonal case in Section 1.3 of [37]. For arbitrarily degrees $m,k,l\in\mathbb{N}$ we have

$$P_m(z) \;=\; \frac{(-1)^{l+k}\,2^l\,m!}{i\pi\,(m+k+l)^m}e^{(m+k+l)z}\oint_{C_{-1}}\frac{e^{(m+k+l)z\,v}\,dv}{(v+1)^{m+1}v^{k+1}(v-1)^{l+1}}, \tag{6.33}$$

$$Q_k(z) \;=\; \frac{(-1)^{l+k}\,2^l\,m!}{i\pi\,(m+k+l)^m}\oint_{C_0}\frac{e^{(m+k+l)z\,v}\,dv}{(v+1)^{m+1}v^{k+1}(v-1)^{l+1}}, \tag{6.34}$$

$$R_l(z) \;=\; \frac{(-1)^{l+k}\,2^l\,m!}{i\pi\,(m+k+l)^m}e^{-(m+k+l)z}\oint_{C_{-1}}\frac{e^{(m+k+l)z\,v}\,dv}{(v+1)^{m+1}v^{k+1}(v-1)^{l+1}}, \tag{6.35}$$

$$E_n(z) \;=\; \frac{(-1)^{l+k}\,2^l\,m!}{i\pi\,(m+k+l)^m}\oint_{C_\infty}\frac{e^{(m+k+l)z\,v}\,dv}{(v+1)^{m+1}v^{k+1}(v-1)^{l+1}}, \tag{6.36}$$

where the integration paths $C_{-1}, C_0, C_1, C_\infty$ are closed curves encircling the point -1, 0, 1, or ∞, respectively. In (6.33) - (6.36) normalization (6.5) has been taken into account.

If we assume that the three degrees $m=m_j, k=k_j, l=l_j$ belong to a ray sequence $\{(m_j+1,k_j+1,l_j+1)\}_{j\in\mathbb{N}}$ that satisfies (6.6) with parameters $\lambda,\kappa\in(0,\infty)$, then it is rather immediate from (6.33) - (6.36) that the asymptotic analysis boils down to an asymptotic evaluation of integrals of the form

$$I_j(C;z) := \frac{2^{l_j}\,m_j!}{i\pi\,(m_j+k_j+l_j)^{m_j}}\oint_C\frac{e^{(m_j+k_j+l_j)z\,v}\,dv}{(v-1)^{m_j+1}v^{k_j+1}(v-1)^{l_j+1}},\quad j\in\mathbb{N}. \tag{6.37}$$

In order to apply the saddle point method, we have to rewrite (6.37) in the form

$$I_j(C;z) = \left(1+\mathrm{O}(\frac{1}{m_j})\right)\oint_C g(v)G(v,z)^{m_j}\,dv \tag{6.38}$$

with $\mathrm{O}(1/m_j)$ a Landau symbol for $j \to \infty$ that holds uniformly for z belonging to a compact set in $\mathbb{C}$. Comparing (6.38) with (6.37) and using the limits in (6.6) together with Stirling's formula one arrives after some calculations (for details see [37, Sections 4.1 and 4.2]) to the conclusion that the functions g and G in (6.38) are necessary of the form

$$G(v,z) = \exp((1+\lambda+\kappa)z\,v - \log(v+1) - \lambda\log(v) - \kappa\log(v-1) \\ - 1 - \log(1+\lambda+\kappa) + \kappa\log(2)), \tag{6.39}$$

$$g(v) = \frac{1}{i\pi\, v(v^2-1)}. \tag{6.40}$$

Let now $z \in \overline{\mathbb{C}}$ be fixed. From the saddle point method (cf. [20], [40], or [37], Section 4.1) it follows that for $m_j \in \mathbb{N}$ large the value of the integral (6.38) is dominated by the value $g(v)G(v,z)^{m_j}$ with $v \in \mathbb{C}$ one of the critical points that are defined by the condition

$$\frac{\partial}{\partial v} G(v,z) = 0. \tag{6.41}$$

Since $\frac{\partial}{\partial v}G(v,z) = G(v,z)\frac{\partial}{\partial v}\log G(v,z)$, it follows from (6.39) that equation (6.41) is equivalent to the equation

$$\frac{\partial}{\partial v}\log G(v,z) = (1+\lambda+\kappa)z - \frac{1}{v+1} - \frac{\lambda}{v} - \frac{\kappa}{v-1} = 0. \tag{6.42}$$

From (6.42) we deduce that for every $z \in \overline{\mathbb{C}}$ there exist exactly three critical points $v_j = v_{z,j} \in \mathbb{C}$, $j = 1,2,3$. As long as $z \notin \Gamma$, with Γ introduced in Lemma 6.3, the value of each of the four integrals in (6.33) - (6.36) depends only on one of the three critical points $v_{z,j}$, $j = 1,2,3$, and the interaction between the integration paths C_{-1}, C_0, C_1, C_∞ and the location of the critical points $v_{z,j}$, $j = 1,2,3$, decides which of the three critical points is relevant for each of the four integrals.

If the three critical points $v_{z,j}$, $j = 1,2,3$, in the $v-$plane are pulled back over the $z-$plane, then this defines the Riemann surface $\mathcal{R}$ of Definition 6.1. Indeed, the function $z(v)$ in (6.7) is an immediate consequence of (6.42), and its inverse defines the function ψ. The role of the critical points in the saddle point method is then also the reason for the specific definition of the function u in (6.11) of Definition 6.2. Indeed, the value of u is $\log|G(v_{z,j},z)|$ with $j \in \{1,2,3\}$, and the index j indicates on which sheet of the Riemann surface $\mathcal{R}$ the function h lives.

Zeros of the polynomials P_m, Q_k, R_l, and the remainder term E_n in (6.33) - (6.36) can only appear when more than one critical point v_j becomes effective in the evaluation of the integral in question. Because of this reason the zeros of the polynomials and the remainder term cluster on intersection arcs of branches of the multi-valued function $h \circ \pi^{-1}$.

In [37] the whole analysis is given in full detail, but of course limited to the diagonal case $\lambda = \kappa = 1$. In the review of results in Section5 a detailed account has been given with respect to the correspondence between the lemmas, theorems, and the proposition in Section 5 and the relevant proofs in [37]. Because of the structural identity between the Sections 5 and 6, this correspondence is also valid for the extended results in the present section, and we will not go into further details here.

REFERENCES

[1] A. I. APTEKAREV AND H. STAHL, *Asymptotics of Hermite-Padé polynomials*, in Progress in Approximation Theory, A. A. Gonchar and E. B. Saff, eds., Springer-Verlag, 1992, pp. 127–167.

[2] G. A. BAKER, JR. AND P. GRAVES-MORRIS, *Padé Approximants*, Cambridge University Press, Cambridge, 1996.

[3] P. B. BORWEIN, *Quadratic Hermite-Padé approximation to the exponential function*, Constr. Approx., 2 (1986), pp. 291–302.

[4] R. S. CARPENTER, R. S. VARGA, AND J. WALDVOGEL, *Asymptotics for the zeros of the partial sums of* e^z, Rocky Mountain J. Math., 21 (1991), pp. 99–120.

[5] J. COATES, *On the algebraic approximation of functions I–IV*, proc. Kon. Akad. v. Wet. A'd Ser. A 69 and 70, Indag. Math. 28 (1966), pp. 421–461, and 29 (1967), pp. 205–212.

[6] P. DEIFT AND X. ZHOU, *A deepest descent method for oscillatory Riemann-Hilbert problems: Asymptotics for the MKdV equation*, Ann. Math., 137 (1993), pp. 295–368.

[7] K. DRIVER, *Non-diagonal quadratic Hermite-Padé approximants to the exponential function*, J. Comp. Appl. Math., 65 (1995), pp. 125–134.

[8] K. DRIVER AND N. M. TEMME, *On polynomials related with Hermite-Padé approximants to the exponential function*, J. Approx. Theory, 95 (1998), pp. 101–122.

[9] A. EDREI, E. B. SAFF, AND R.S. VARGA, *Zeros of Sections of Power Series*, Lecture Notes in Mathematics, 1002, Springer-Verlag, Berlin-New York, 1983.

[10] G. FROBENIUS, *Über Relationen zwischen den Näherungsbrüchen von Potenzreihen*, J. für reine und angewandte Math., 90 (1881), pp. 1–17.

[11] C. HERMITE, *Sur la fonction exponentielle*, C. R. Acad. Sci. Paris, 77 (1873), pp. 18–24, pp. 74–79, pp. 226–233.

[12] H. JAGER, *A multidimensional generalisation of the Padé table*, proc. Kon. Akad. v. Wet. A'dam Ser. A 67, Indag. Math. 26 (1964), pp. 192–249.

[13] F. KLEIN, *Elementarmathematik vom höheren Standpunkt aus*, Springer-Verlag, Berlin, 1924.

[14] A. B. J. KUIJLAARS, W. VAN ASSCHE, AND F. WIELONSKY, *Quadratic Hermite-Padé approximants to the exponential function: a Riemann-Hilbert approach*, Constr. Approx., 21 (2005), pp. 351–412.

[15] A. B. J. KUIJLAARS, H. STAHL, W. VAN ASSCHE, AND F. WIELONSKY, *Asymptotique des approximants de Hermite-Padé quadratiques de la function exponentielle et problèmes de Riemann-Hilbert*, C. R. Acad. Sci. Paris, Ser. I, 336 (2003), pp. 893-896.

[16] K. MAHLER, *Zur Approximation der Exponentialfunktion und des Logarithmus I, II*, J. Reine Angew. Math., 166 (1931), pp. 118–137, pp. 138–150.

[17] K. MAHLER, *Application of some formulas by Hermite to the approximation of exponentials and logarithms*, Math. Ann., 168 (1967), pp. 200–227.

[18] K. MAHLER, *Perfect systems*, Comp. Math., 19 (1968), pp. 95–166.

[19] E.M. NIKISHIN AND V.N. SOROKIN, *Rational Approximation and Orthogonality*, Amer. Math. Soc., Providence, 1991.

[20] F. OLVER, *Aasymptotics and Special Functions*, Academic Press, San Diego, 1974.

[21] H. PADÉ, *Sur la répresentation approchée d'une function par des fractions rationelles*, Ann. de l'Ecole Normale Sup. 3iéme Série 9, 1892, Suppl., pp. 1-93.

[22] H. PADÉ, *Sur la généralisation des fractions continues algébraic*, C. R. Acad. Sci. Paris, 118 (1894), pp. 848-850.

[23] H. PADÉ, *Sur la généralisation des fractions continues algébraic*, J. Math. Pures et Appl., X (1894), pp. 291–329.

[24] H. PADÉ, *Memoire sur les dévelopments en fractions continues pouvent servir d'introduction à la théorie des fractions continues algébraic*, Ann. Sci. Ec. Norm. Sup., 16 (1899), pp. 395–426.

[25] I. E. PRITSKER AND R. S. VARGA, *The Szegö curve, zero distribution, and weighted approximation*, Trans. Amer. Math. Soc., 349 (1997), pp. 4085–4105.

[26] I. E. PRITSKER AND R. S. VARGA, *Zero distribution, the Szegö curve, and weighted approximation in the complex plane*, In: Numer. Math. Sci. Computation, Oxford Univ. Press, New York, 1998, pp. 167–188.

[27] T. RANSFORD, *Potential Theory in the Complex Plane*, Cambridge Universty Press, Cambridge, 1995.

[28] E. B. SAFF AND V. TOTIK, *Logarithmic Potentials with External Fields*, Springer-Verlag, Berlin, New York, 1997.

[29] E. B. SAFF AND R. S. VARGA, *On the zeros and poles of Padé approximants to* e^z, Numer. Math., 25 (1975/76), pp. 1–14.

[30] E. B. SAFF AND R. S. VARGA, *Convergence of of Padé approximants to* e^{-z} *on unbounded sets*, J. Approx. Theory, 13 (1975), pp. 470–488.

[31] E. B. SAFF AND R. S. VARGA, *Zeros-free parabolic regions for sequences of polynomials*, SIAM J. Math. Anal., 7 (1976), pp. 344–357.

[32] E. B. SAFF AND R. S. VARGA, *The behavior of the Padé table for the exponential*, in Approximation Theory II, Proc. Int. Sympos. Texas, Austin, TX, 1976, Academic Press, New York, 1976, pp. 519–531.

[33] E. B. SAFF, R. S. VARGA, AND W. C. NI, *Geometric convergence of rational approximants to* e^{-z} *in infinite sectors*, Numer. Math., 26 (1976), pp. 211–225.

[34] E. B. SAFF AND R. S. VARGA, *On the zeros and poles of Padé approximants to* e^z *II*, in Padé and Rational

Approximation, Proc. Int. Sympos. Univ. South Florida, Tampa, FL, 1976, Academic Press, New York, 1977, pp. 195–213.

[35] E. B. SAFF AND R. S. VARGA, *On the zeros and poles of Padé approximants to e^z III*, Numer. Math., 30 (1978), pp. 241–266.

[36] H. STAHL, *Spurious poles in Padé approximation*, J. Comp. Appl. Math., 99 (1998), pp. 511–527.

[37] H. STAHL, *Quadratic Hermite-Padé polynomials associated with the exponential function*, J. Approx. Theory, 125 (2003), pp. 238-294.

[38] H. STAHL, *Asymptotic distributions of zeros of quadratic Hermite-Padé polynomials associated with the exponential function*, Constr. Approx., 23 (2006), pp. 121-164.

[39] G. SZEGÖ, *Über einige Eigenschaften der Exponentialreihe*, Sitzunggsberichte Berliner Math. Ges., 23 (1924), pp. 50–64.

[40] N. M. TEMME, *Special Functions*, John Wiley, New York 1996.

[41] F. WIELONSKY, *Asymptotics of diagonal Hermite-Padé approximants to e^z*, J. Approx. Theory, 90 (1997), pp. 283–298.

[42] F. WIELONSKY, *Hermite-Padé approximants to exponential functions and an inequality of Mahler*, J. Number. Theory, 74 (1999), pp. 230–249.