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Efficient implementation of a modified and relaxed hybrid steepest-descent method for a type of variational inequality

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Abstract

To reduce the difficulty and complexity in computing the projection from a real Hilbert space onto a nonempty closed convex subset, researchers have provided a hybrid steepest-descent method for solving VI(F, K) and a subsequent three-step relaxed version of this method. In a previous study, the latter was used to develop a modified and relaxed hybrid steepest-descent (MRHSD) method. However, choosing an efficient and implementable nonexpansive mapping is still a difficult problem. We first establish the strong convergence of the MRHSD method for variational inequalities under different conditions that simplify the proof, which differs from previous studies. Second, we design an efficient implementation of the MRHSD method for a type of variational inequality problem based on the approximate projection contraction method. Finally, we design a set of practical numerical experiments. The results demonstrate that this is an efficient implementation of the MRHSD method.

Keywords: hybrid steepest-descent method, variational inequalities, approximate projection contraction method, strong convergence, nonexpansive mapping

1 Introduction

Let H be a real Hilbert space with inner product $< \bullet, \bullet >$ and norm $|| \bullet ||$, let K be a nonempty closed convex subset of H, and let F: $H \to H$ be an operator. Then the variational inequality problem $VI(F, K)$ involves finding $x^* \in K$ such that

$$x^* \in K, < x - x^*, F(x^*) > \geq 0, \forall x \in K. \tag{1}$$

Variational inequality problems were introduced by Hartman and Stampacchia and were subsequently expanded in several classic articles [1,2]. Variational inequality theory provides a method for unifying the treatment of equilibrium problems encountered in areas as diverse as economics, optimal control, game theory, transportation science, and mechanics. Variational inequality problems have many applications, such as in mathematical optimization problems, complementarity problems and fixed point problems [3-7]. Thus, it is important to solve variational inequality problems and much research has been devoted to this topic [8-12].

It is known that

$$x^* \text{ is the solution of } VI(F, K) \Leftrightarrow x^* = P_K[x^* - \beta F(x^*)], \quad \beta > 0.$$

where P_K is the projection from H onto K, i.e.,

$$P_K(x) = \operatorname{argmin}_{y \in K} \|x - y\|, \ \forall x \in H.$$

Thus, we can solve a variational inequality problem using a fixed-point problem with some appropriate conditions. For example, if F is a strongly monotone and Lipschitzian mapping on K and $\beta > 0$ is small enough, then P_K is a contraction. Hence, Banach's fixed point theorem guarantees convergence of the Picard iterates generated by $P_K[x - \beta F(x)]$. Such a method is called a projection method, as described elsewhere [13-17].

To reduce the complexity of computing the projection P_K, Yamada and Deutsch developed a hybrid steepest-descent method for solving $VI(F, K)$ [7,8], but choosing an efficient and implementable nonexpansive mapping is still a difficult problem. Subsequently, Xu and Kim [9] and Zeng et al. [10] proved the convergence of hybrid steepest-descent method. Noor introduced iterations after analysis of several three-step iterative methods [18]. Ding et al. provided a three-step relaxed hybrid steepest-descent method for variational inequalities [11] and Yao et al. [19] provided a simple proof of the method under different conditions. Our group has described a modified and relaxed hybrid steepest descent (MRHSD) method that makes greater use of historical information and minimizes information loss [20].

This article makes three new contributions compared to previous results. First, we prove a strong convergence of the MRHSD method under different and suitable restrictions imposed on the parameters (Condition 3.2). The proof of strong convergence is different from the previous proof [20]. Second, based on the approximate projection contraction method, we design an efficient implementation of the MRHSD method for a type of variational inequality problem. Third, we design some practical numerical experiments and the results verify that it is efficient implementation. Furthermore, the MRHSD method under Condition 3.2 is more efficient than under Condition 3.1.

The remainder of the article is organized as follows. In Section 2, we review several lemmas and preliminaries. In Section 3, we prove the convergence theorem. We discuss an implementation of the MRHSD method for a type of variational inequality problem in Section 4. Section 5 presents numerical experiments and results applicable to finance and statistics. Section 6 concludes.

2 Preliminaries

To prove the convergence theorem, we first introduce several lemmas and the main results reported by others [10,11,21].

Lemma 1 *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space X and let* $\{\zeta_n\}$ *be a sequence in [0, 1] with* $0 < \liminf_{n\to\infty} \zeta_n \le \limsup_{n\to\infty} \zeta_n < 1$.

Suppose

$$x_{n+1} = (1 - \zeta_n) y_n + \zeta_n x_n \quad \forall n \ge 0$$

and

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0 \quad \forall n\ge 0.$$

Then $\lim_{n\to\infty}\|y_n-x_n\|=0$.

Lemma 2 *Let* $\{s_n\}$ *be a sequence of non-negative real numbers satisfying the inequality:*

$$s_{n+1}\le(1-\alpha_n)s_n+\alpha_n\tau_n+\gamma_n, \forall n\ge 0,$$

where $\{\alpha_n\}$, $\{\tau_n\}$, *and* $\{\gamma_n\}$ *satisfy the following conditions:*

(1) $\alpha_n\subset[0,1]$, $\sum_{n=0}^{\infty}\alpha_n=\infty$, *or* $\prod_{n=0}^{\infty}(1-\alpha_n)=0$;
(2) $\limsup_{n\to\infty}\tau_n\le 0$;
(3) $\gamma_n\subset[0,\infty)$, $\sum_{n=0}^{\infty}\gamma_n<\infty$.

Then $\lim_{n\to\infty}s_n=0$.

Lemma 3 *(Demiclosedness principle) Assume that T is a nonexpansive self-mapping on a nonempty closed convex subset K of a Hilbert space H. If T has a fixed point, then (I - T) is demiclosed; that is, whenever* $\{x_n\}$ *is a sequence in K weakly converging to some* $x\in K$ *and the sequence* $\{(I-T)x_n\}$ *strongly converges to some* $y\in H$, *it follows that* $(I-T)x=y$, *where I is the identity operator of H.*

The following lemma is an immediate result of the inner product of a Hilbert space.

Lemma 4 *In a real Hilbert space H, the following inequality holds:*

$$\|x+y\|^2\le\|x\|^2+2<y,x+y>, \forall x, y\in H.$$

Lemma 5 *Let* $\{\alpha_n\}$ *be a sequence nonnegative numbers with and be sequence of real numbers with* $\limsup_{n\to\infty}\alpha_n<\infty$ *and* $\{\beta_n\}$ *be sequence of real numbers with* $\limsup_{n\to\infty}\beta_n\le 0$. *Then*

$$\limsup_{n\to\infty}\alpha_n\beta_n\le 0.$$

A basic property of the projection mapping onto a closed convex subset of Hilbert space will be given out in the following lemma.

Lemma 6 *Let K be a nonempty closed convex subset of H.* $\forall x, y\in H$ *and* $z\in K$,

(1) $<P_K(x)-x, z-P_K(x)>\ \ge 0$,
(2) $\|P_K(x)-P_K(y)\|^2\le\|x-y\|^2-\|P_K(x)-x+y-P_K(y)\|^2$.

We now introduce some basic assumptions. Let F: $H\to H$ be an operator with F: κ-Lipschtz and η-strongly monotone; that is, F satisfies the following conditions:

$$\|F(x)-F(y)\|\le\kappa\|x-y\|$$

and

$$< F(x) - F(y),\ x - y > \geq \eta \|x - y\|^2,\ \forall x,\ y \in K.$$

Assuming the solution set of $VI(F, K)$ is nonempty, naturally $VI(F, K)$ has a unique solution $x^* \in K$ under these conditions. Following Yamada [8] and to reduce the complexity of computing the projection P_K, we replace the projection P_K with a nonexpansive mapping T: $H \to H$ with the property that the fixed point set is $Fix(T) = K$. Now we introduce some notation. For any given numbers $\lambda \in (0, 1)$ and $\mu \in (0, 2\eta/\kappa^2)$, we define the mapping $T_\mu^\lambda : H \to H$ as

$$T_\mu^\lambda x : Tx - \lambda\mu F(Tx),\ \forall x \in H,$$

where T_μ^λ satisfies the following property under some conditions.

Lemma 7 *If* $0 < \mu < 2\eta/\kappa^2$ *and* $0 < \lambda < 1$, *then* T_μ^λ *is a contraction. In fact,*

$$\|T_\mu^\lambda x - T_\mu^\lambda y\| \leq (1 - \lambda\delta)\, \|x - y\|\,,\ \forall x,\ y \in H,$$

where $\delta = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$.

3 Convergence theorem

Before analysis and proof, we first review the MRHSD method and related results [20].

Algorithm [20]

Take three fixed numbers $t, \rho, \gamma \in (0, 2\eta/\kappa^2)$, and let $\{\alpha_n\} \subset [0, 1)$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ and $\{\lambda_n\}$, $\{\lambda'_n\}$, $\{\lambda''_n\} \subset (0,\ 1)$. Starting with arbitrarily chosen initial points $x_0 \in H$, compute the sequences $\{x_n\}$, $\{\bar{x}_n\}$, $\{\tilde{x}_n\}$ such that

Step 1: $\bar{x}_n = \gamma_n x_n + (1 - \gamma_n)[Tx_n - \lambda''_{n+1}\gamma F(Tx_n)]$,
Step 2: $\tilde{x}_n = \beta_n x_n + (1 - \beta_n)[T\bar{x}_n - \lambda'_{n+1}\rho F(T\bar{x}_n)]$,
Step 3: $x_{n+1} = \alpha_n \bar{x}_n + (1 - \alpha_n)[T\tilde{x}_n - \lambda_{n+1} t F(T\tilde{x}_n)]$,

where T: $H \to H$ is a nonexpansive mapping. However, choosing an efficient and implementable nonexpansive mapping T is a difficult problem, and previous studies did not design numerical experiments or describe an efficient and implementable nonexpansive mapping T [8-11,19,20]. In Section 4, we design an efficient and implementable nonexpansive mapping T for a type of variational problem based on the approximate projection contraction method. We then review the conditions and theorem presented by Xu et al. [20].

Condition 3.1

(1) $\sum_1^\infty |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_1^\infty |\beta_n - \beta_{n-1}| < \infty$, $\sum_1^\infty |\gamma_n - \gamma_{n-1}| < \infty$;
(2) $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 1$, $\lim_{n\to\infty} \gamma_n = 1$;
(3) $\lim_{n\to\infty} \lambda_n = 0$, $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$, $\sum_1^\infty \lambda_n = \infty$;
(4) $\lambda_n \geq \max\{\lambda'_n,\ \ \lambda''_n\}$, $\forall\ n \geq 1$.

Theorem 1 [20]*Under the Condition 3.1, the sequence* $\{x_n\}$ *generated by algorithm* [20]*converges strongly to* $x^* \in K$*, and* x^* *is the unique solution of the VI*(F, K).

We provide different conditions and establish a strong convergence theorem for the MRHSD method for variational inequalities under these conditions. Note that Condition 3.2 and a strong convergence theorem (Theorem 2) are the first contributions of the article.

Condition 3.2

(1) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, $\lim_{n\to\infty} \beta_n = 1$, $\lim_{n\to\infty} \gamma_n = 1$;

(2) $\lim_{n\to\infty} \lambda_n = 0$, $\sum_1^{\infty} \lambda_n = \infty$;

(3) $\lambda_n \geq \max\{\lambda'_n, \ \lambda''_n\}$, $\forall n \geq 1$.

Theorem 2 *The sequence* $\{x_n\}$ *generated by algorithm* [20]*converges strongly to* $x^* \in K$*, and* x^* *is the unique solution of the VI*(F, K)*; assume that* α_n*,* β_n*,* γ_n *and* λ_n*,* λ'_n*,* λ''_n*satisfy the Condition 3.2.*

Proof. We divide the proof into several steps.

Step 1. [20] The sequences $\{x_n\}$, $\{\bar{x}_n\}$, $\{\tilde{x}_n\}$ are bounded.

According to Step 1, we have that

$$\{Tx_n\}, \{T\bar{x}_n\}, \{T\tilde{x}_n\}, \{F(Tx_n)\}, \{F(T\bar{x}_n)\}, \{F(T\tilde{x}_n)\}\}$$

are also bounded and

$$\left\|x_n - x^*\right\| \leq M_0, \forall n \geq 0,$$

where $M_0 = \max\{3\|x_0 - x^*\|, 3(\rho + \gamma + t)\|F(x^*)\|/\tau\}$.
and

$$\begin{aligned}
\left\|\tilde{x}_n - x^*\right\| &\leq \beta_n \left\|x_n - x^*\right\| + (1-\beta_n)\lambda_{n+1}(\gamma + \rho)\left\|F(x^*)\right\| \leq (1+\tau)M_{0,}\\
\left\|\bar{x}_n - x^*\right\| &\leq \left\|x_n - x^*\right\| + (1-\gamma_n)\lambda''_{n+1}\gamma\left\|F(x^*)\right\| \leq (1+\tau)M_0.
\end{aligned}$$

Step 2. $\|x_{n+1} - x_n\| \to 0$.

Indeed, a series of computations yields:

$$\begin{aligned}
\|\bar{x}_n - \bar{x}_{n-1}\| &= \left\|\gamma_n x_n - \gamma_{n-1}x_{n-1} + (1-\gamma_n)T_\gamma^{\lambda''_{n+1}}x_n - (1-\gamma_{n-1})T_\gamma^{\lambda''_n}x_{n-1}\right\|\\
&\leq \|\gamma_n x_n - \gamma_{n-1}x_{n-1}\| + \left\|(1-\gamma_n)T_\gamma^{\lambda''_{n+1}}x_n - (1-\gamma_{n-1})T_\gamma^{\lambda''_n}x_{n-1}\right\|\\
&\leq \|x_n - x_{n-1}\| + \left|(1-\gamma_n)\lambda''_{n+1} - (1-\gamma_{n-1})\lambda''_n\right|\gamma\left\|F(Tx_{n-1})\right\|\\
&\quad + |\gamma_n - \gamma_{n-1}|\left(\|x_{n-1}\| + \|Tx_{n-1}\|\right).
\end{aligned} \tag{2}$$

By $T_\rho^{\lambda'_{n+1}}\bar{x}_n = T\bar{x}_n - \lambda'_{n+1}\rho F(T\bar{x}_n)$, $T_\rho^{\lambda'_n}\bar{x}_{n-1} = T\bar{x}_{n-1} - \lambda'_n\rho F(T\bar{x}_{n-1})$ and (2), we can obtain

$$\begin{aligned}
\left\|\tilde{x}_n - \tilde{x}_{n-1}\right\| &= \left\|\beta_n x_n - \beta_{n-1}x_{n-1} + (1-\beta_n)T_\rho^{\lambda'_{n+1}}\bar{x}_n - (1-\beta_{n-1})T_\rho^{\lambda'_n}\bar{x}_{n-1}\right\|\\
&\leq \|\beta_n x_n - \beta_{n-1}x_{n-1}\| + \left\|(1-\beta_n)T_\rho^{\lambda'_{n+1}}\bar{x}_n - (1-\beta_{n-1})T_\rho^{\lambda'_n}\bar{x}_{n-1}\right\|\\
&\leq \|x_n - x_{n-1}\| + \left|(1-\beta_n)\lambda'_{n+1} - (1-\beta_{n-1})\lambda'_n\right|\rho\left\|F(T\bar{x}_{n-1})\right\|\\
&\quad + (1-\beta_n)(1-\lambda'_{n+1}\tau')\,|\gamma_n - \gamma_{n-1}|\left(\|x_{n-1}\| + \|Tx_{n-1}\|\right)\\
&\quad + (1-\beta_n)(1-\lambda'_{n+1}\tau')\left|(1-\gamma_n)\lambda''_{n+1} - \gamma_{n-1}\lambda''_n\right|\gamma\left\|F(Tx_{n-1})\right\|\\
&\quad + |\beta_n - \beta_{n-1}|\left(\|x_{n-1}\| + \|T\bar{x}_{n-1}\| + \|T\bar{x}_{n-1}\|\right).
\end{aligned} \tag{3}$$

Let

$$\tilde{y}_n = T_t^{\lambda_{n+1}} \tilde{x}_n = T\tilde{x}_n - \lambda_{n+1} t F(T\tilde{x}_n),$$

so we obtain

$$x_{n+1} = \alpha_n \bar{x}_n + (1-\alpha_n)\tilde{y}_n.$$

Furthermore,

$$\begin{aligned} \left\|\tilde{y}_n - \tilde{y}_{n-1}\right\| &= \left\|T\tilde{x}_n - T\tilde{x}_{n-1} + \lambda_n t F(T\tilde{x}_{n-1}) - \lambda_{n+1} t F(T\tilde{x}_n)\right\| \\ &\leq \left\|T\tilde{x}_n - T\tilde{x}_{n-1}\right\| + \lambda_n t \left\|F(T\tilde{x}_{n-1})\right\| + \lambda_{n+1} t \left\|F(T\tilde{x}_n)\right\| \\ &\leq \left\|\tilde{x}_n - \tilde{x}_{n-1}\right\| + \lambda_n t \left\|F(T\tilde{x}_{n-1})\right\| + \lambda_{n+1} t \left\|F(T\tilde{x}_n)\right\|. \end{aligned} \tag{4}$$

By $\lim_{n\to\infty} \beta_n = 1$, $\lim_{n\to\infty} \lambda_n = 0$ and (3), (4), we obtain:

$$\begin{aligned} &\left\|\tilde{y}_n - \tilde{y}_{n-1}\right\| - \|x_n - x_{n-1}\| \\ &\leq \left|(1-\beta_n)\lambda'_{n+1} - (1-\beta_{n-1})\lambda'_n\right| \rho \left\|F(T\tilde{x}_{n-1})\right\| \\ &\quad + (1-\beta_n)(1-\lambda'_{n+1}\tau') \left|\gamma_n - \gamma_{n-1}\right| \left(\|x_{n-1}\| + \|Tx_{n-1}\|\right) \\ &\quad + (1-\beta_n)(1-\lambda'_{n+1}\tau') \left|(1-\gamma_n)\lambda''_{n+1} - \gamma_{n-1}\lambda''_n\right| \gamma \left\|F(Tx_{n-1})\right\| \\ &\quad + \left|\beta_n - \beta_{n-1}\right| \left(\|x_{n-1}\| + \|T\bar{x}_{n-1}\| + \|T\bar{x}_{n-1}\|\right) \\ &\quad + \lambda_n t \left\|F(T\tilde{x}_{n-1})\right\| + \lambda_{n+1} t \left\|F(T\tilde{x}_n)\right\| \to 0. \end{aligned} \tag{5}$$

According to Lemma 1, we can obtain

$$\lim_{n\to\infty} \left\|\tilde{y}_{n-1} - x_{n-1}\right\| = 0.$$

Furthermore, using the conditions $\lim_{n\to\infty} \gamma_n = 1$, $\max\left\{\lambda'_n, \lambda''_n\right\} \leq \lambda_n \to 0$, we obtain

$$\begin{aligned} \|\bar{x}_n - x_n\| &= \left\|-(1-\gamma_n)x_n + (1-\gamma_n)\left(Tx_n - \lambda'_{n+1}\gamma F(Tx_n)\right)\right\| \\ &\leq (1-\gamma_n)\|x_n\| + (1-\gamma_n)\|Tx_n\| + \lambda'_{n+1}\gamma \left\|F(Tx_n)\right\| \to 0. \end{aligned} \tag{6}$$

According to (5) and (6), we conclude that

$$\begin{aligned} \|x_n - x_{n-1}\| &= \left\|\alpha_{n-1}\bar{x}_{n-1} + (1-\alpha_{n-1})\tilde{y}_{n-1} - x_{n-1}\right\| \\ &\leq \alpha_{n-1}\|\bar{x}_{n-1} - x_{n-1}\| + (1-\alpha_{n-1})\left\|\tilde{y}_{n-1} - x_{n-1}\right\| \to 0, \end{aligned}$$

so we immediately obtain

$$\|x_{n+1} - x_n\| \to 0.$$

Step 3. $\|x_{n+1} - Tx_n\| \to 0$.
In fact,

$$\begin{aligned} \left\|\tilde{x}_n - x_n\right\| &= \left\|-(1-\beta_n)x_n + (1-\beta_n)\left(T\bar{x}_n - \lambda'_{n+1}\rho F(T\bar{x}_n)\right)\right\| \\ &\leq (1-\beta_n)\|x_n\| + (1-\beta_n)\|T\bar{x}_n\| + \lambda'_{n+1}\rho \|F(T\bar{x}_n)\|. \end{aligned} \tag{7}$$

According to the assumptions $\lim_{n\to\infty} \beta_n = 1$ and $\lim_{n\to\infty} \lambda_n = 0$, then

$$\left\|\tilde{x}_n - x_n\right\| \to 0.$$

A series of computations yields:

$$\begin{aligned}\|x_{n+1}-Tx_n\| &= \left\|\alpha_n(\bar{x}_n - Tx_n) + (1-\alpha_n)\left(T_t^{\lambda_{n+1}}\tilde{x} - Tx_n\right)\right\| \\ &\le \alpha_n\|\bar{x}_n - Tx_n\| + (1-\alpha_n)\left\|T\tilde{x}_n - Tx_n\right\| \\ &\quad + (1-\alpha_n)\lambda_{n+1}t\left\|F\left(T\tilde{x}_n\right)\right\| \\ &\le \alpha_n\|\bar{x}_n - Tx_n\| + \left\|\tilde{x}_n - x_n\right\| + \lambda_{n+1}t\left\|F\left(T\tilde{x}\right)\right\| \\ &\le \alpha_n\|x_{n+1} - Tx_n\| + \alpha_n\|\bar{x}_n - x_{n+1}\| + \left\|\tilde{x}_n - x_n\right\| \\ &\quad + \lambda_{n+1}t\left\|F\left(T\tilde{x}\right)\right\|.\end{aligned} \tag{8}$$

Hence, by (6), (7), (8) and Conditions 3.2, we obtain:

$$\|x_{n+1}-Tx_n\| \le \frac{\alpha_n}{1-\alpha_n}\|\bar{x}_n - x_{n+1}\| + \frac{\left\|\tilde{x}_n - x_n\right\|}{1-\alpha_n} + \frac{\lambda_{n+1}t\left\|F\left(T\tilde{x}\right)\right\|}{1-\alpha_n} \to 0. \tag{9}$$

Corollary 1 $\|x_n - Tx_n\| \to 0$.
Applying Steps 2 and 3, we get

$$\|x_{n+1} - Tx_n\| \to 0$$

and

$$\|x_{n+1} - x_n\| \to 0,$$

So then

$$\|x_n - Tx_n\| \le \|x_{n+1} - Tx_n\| + \|x_{n+1} - x_n\| \to 0.$$

Step 4. $\limsup_{n\to\infty} < -F(x^*), T\tilde{x}_n - x^* > \le 0$.
For some $\tilde{x} \in H$, here exits $\left\{Tx_{n_i}\right\} \to \tilde{x}$ weakly, and such that

$$\limsup_{n\to\infty} < -F\left(x^*\right), Tx_n - x^* > = \limsup_{n\to\infty} < -F\left(x^*\right), Tx_{n_i} - x^* > .$$

According to $\left\{Tx_{n_i}\right\} \to \tilde{x}$, we have

$$\tilde{x} \in Fix(T) = K.$$

Moreover, we have x^* is the unique solution of $VI(F, K)$, so we can obtain:

$$\begin{aligned}&\limsup_{n\to\infty} < -F\left(x^*\right), Tx_n - x^* > \\ &= \limsup_{n\to\infty} < -F\left(x^*\right), \tilde{x} - x^* > \\ &\le 0.\end{aligned}$$

Since $\left\|T\tilde{x}_n - Tx_n\right\| \le \left\|\tilde{x}_n - x_n\right\| \to 0$, we immediately conclude that

$$\begin{aligned}&\limsup_{n\to\infty} < -F\left(x^*\right), T\tilde{x}_n - x^* > \\ &\le \limsup_{n\to\infty} < -F\left(x^*\right), T\tilde{x}_n - Tx_n > \\ &\quad + \limsup_{n\to\infty} < -F\left(x^*\right), Tx_n - x^* > \\ &\le \limsup_{n\to\infty} < -F\left(x^*\right), Tx_n - x^* > \\ &\le 0.\end{aligned}$$

Step 5. $\|x_n - x^*\| \to 0$. To prove this conclusion, we have to apply the Lemma 2 several times.

By Step 1 and Lemma 4, we have:

$$\begin{aligned}
\left\|x_{n+1} - x^*\right\|^2 &= \left\|\alpha_n \left(\bar{x}_n - x^*\right) + (1-\alpha_n)\left(T_t^{\lambda_{n+1}}\tilde{x}_n - x^*\right)\right\|^2 \\
&\le \left\|\alpha_n\left(\bar{x}_n - x^*\right)\right\|^2 \\
&\quad + (1-\alpha_n)\left\|\left(T_t^{\lambda_{n+1}}\tilde{x}_n - T_t^{\lambda_{n+1}}x^* + T_t^{\lambda_{n+1}}x^* - x^*\right)\right\|^2 \\
&\le \left\|\alpha_n\left(\bar{x}_n - x^*\right)\right\|^2 + (1-\alpha_n)\left[\left\|\left(T_t^{\lambda_{n+1}}\tilde{x}_n - T_t^{\lambda_{n+1}}x^*\right)\right\|^2\right. \\
&\quad \left. +2 < T_t^{\lambda_{n+1}}x^* - x^*, T_t^{\lambda_{n+1}}\tilde{x}_n - x^* >\right] \\
&\le \alpha_n\left[\left\|x_n - x^*\right\| + (1-\gamma_n)\lambda''_{n+1}\gamma\left\|F\left(x^*\right)\right\|\right]^2 \\
&\quad + (1-\alpha_n)(1-\lambda_{n+1}\tau)^2\left[\left\|x_n - x^*\right\| + (1-\beta_n)\lambda_{n+1}(\gamma+\rho)\left\|F\left(x^*\right)\right\|\right]^2 \\
&\quad + 2t\lambda_{n+1} < -F\left(x^*\right), T\tilde{x}_n - x^* - t\lambda_{n+1}F\left(T\tilde{x}_n\right) > \\
&\le \alpha_n\left\|x_n - x^*\right\|^2 + (1-\gamma_n)\lambda_{n+1}\gamma M + (1-\alpha_n)(1-\lambda_{n+1}\tau)^2\left\|x_n - x^*\right\|^2 \\
&\quad + (1-\alpha_n)(1-\lambda_{n+1}\tau)^2(1-\beta_n)\lambda_{n+1}M \\
&\quad + 2t\lambda_{n+1} < -F\left(x^*\right), T\tilde{x}_n - x^* - t\lambda_{n+1}F\left(T\tilde{x}_n\right) > \\
&\le \left(1-(1-\alpha_n)\lambda_{n+1}\tau\right)\left\|x_n - x^*\right\|^2 + (1-\alpha_n)\lambda_{n+1}\tau w'_{n+1},
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
w'_{n+1} &= \frac{2t < -F\left(x^*\right), T\tilde{x}_n - x^* - t\lambda_{n+1}F\left(T\tilde{x}_n\right) >}{\tau(1-\alpha_n)} \\
&\quad + \frac{\varphi_n}{\tau(1-\alpha_n)} + \frac{\xi_n}{\tau(1-\alpha_n)}, \\
\varphi_n &= (1-\gamma_n)\gamma M, \\
\xi_n &= (1-\alpha_n)(1-\lambda_{n+1}\tau)^2(1-\beta_n)M
\end{aligned}$$

and $M_0 \ll M < \infty$.

If we denote

$$s'_{n+1} = \left\|x_{n+1} - x^*\right\|, u_n = (1-\alpha_n)\lambda_{n+1}\tau,$$

we can rewrite (10):

$$s'_{n+1} \le (1-u_n)s'_n + u_n w'_n + 0.$$

In fact, u_n, w'_n satisfies Lemma 2, according to

$$\lim_{n\to\infty}\beta_n = 1, \lim_{n\to\infty}\gamma_n = 1, \lim_{n\to\infty}\lambda_n = 0$$

and step 4, we obtain

$$\frac{\varphi_n}{\tau(1-\alpha_n)} \to 0$$

and

$$\frac{\xi_n}{\tau(1-\alpha_n)} \to 0.$$

Furthermore, $\limsup_{n\to\infty} < -F\left(x^*\right), T\tilde{x}_n - x^* > \le 0$, so we have:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{2t < -F\left(x^*\right), T\tilde{x}_n - x^* - t\lambda_{n+1} F\left(T\tilde{x}_n\right) >}{\tau\left(1-\alpha_n\right)} \\
&\le \frac{2t}{\tau} \limsup_{n\to\infty} \left\{< -F\left(x^*\right), T\tilde{x}_n - x^* > \right. \\
&\quad \left. + \lambda_{n+1} < -F\left(x^*\right), -tF\left(T\tilde{x}_n\right) > \right\} \\
&\le \frac{2t}{\tau} \limsup_{n\to\infty} \left\{< -F\left(x^*\right), T\tilde{x}_n - x^* > \right\} \\
&\quad + \limsup_{n\to\infty} \left\{\lambda_{n+1} < -F\left(x^*\right), -tF\left(T\tilde{x}_n\right) > \right\} \\
&\le 0 + 0 = 0.
\end{aligned}$$

Consequently we obtain

$$\limsup_{n\to\infty} w'_n \le 0,$$

and then from Lemma 2, we have

$$\left\|x_n - x^*\right\| \to 0.$$

which completes the proof.

The following section is our second contribution in this article.

4 Implementation of the MRHSD method for a kind of variational inequalities

Now we consider the variational inequality problem $VI(F, K_1 \cap K_2)$, which involves finding $x^* \in K_1 \cap K_2$ such that

$$x^* \in K_1 \cap K_2, < x - x^*, F\left(x^*\right) > \ge 0, \forall x \in K_1 \cap K_2, \tag{11}$$

where K_1 and K_2 are nonempty and closed convex subsets of H.

To reduce the difficulty and complexity in computing the projection P_K, we solve $VI(F, K_1 \cap K_2)$ by the MRHSD method. Then we have to choose an efficient and implementable nonexpansive mapping T. Based on the spirit of the approximate projection contraction method, we define Tx as:

$$Tx = H\left(G(x)\right) \approx P_K[x], \tag{12}$$

where

$$G(x) = P_{K_2}(x), \; H(x) = P_{K_1}(x).$$

Assuming that $P_{K_2}(x)$, $P_{K_1}(x)$ can be computed without much difficulty, we can efficiently compute Tx. According to $Tx \approx P_K[x]$, we can partly retain the efficiency of the projection contraction method. Obviously, the fixed point set is $\mathrm{Fix}(T) = K$ and T satisfies the property of nonexpansive mapping.

5 Numerical experiments

To show the effects of the MRHSD method for $VI(F, K_1 \cap K_2)$, we test a set of problems that arise in finance and statistics [12,22]. Let H_L, H_U be given $n \times n$ symmetric matrices, let C be asymmetric, which differs from previous approaches [12,22], and H_L

$\leq H_U$ in terms of elements. The problem considered in this section is:

$$\min\left\{\frac{1}{2}\,\|X-C\|_F^2\,|X\in K=S_+^n\cap ß\right\},\tag{13}$$

where $\|\bullet\|_F$ is the matrix Fröbenis norm, i.e.,

$$\|C\|_F=\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|C_{ij}\right|^2\right)^{\frac{1}{2}}.$$

Furthermore,

$$S_+^n=\left\{H\in IR^{n\times n}|H^T=H,H\succeq 0\right\}$$

and

$$ß=\left\{H\in IR^{n\times n}|H^T=H,H_L\leq H\leq H_U\right\}.$$

Note that the matrix Fröbenis norm is induced by the inner product

$$\langle A,B\rangle=\text{Trace}\left(A^TB\right).$$

It is known that optimization problem (13) is equivalent to the following variational inequality problem:

$$\left\langle X'-X,\nabla\left(\frac{1}{2}\|X-C\|^2\right)\right\rangle\geq 0,\forall X'\in K,$$

so we obtain

$$\left\langle X'-X,X-C\right\rangle\geq 0,\forall X'\in K.\tag{14}$$

To solve variational inequality problem (14) by the MRHSD method, we take one set of parameter sequences satisfying Condition 3.1.

Condition 3.1.

$$\alpha_n=\lambda_n=\lambda_n'=\lambda_n''=\frac{1}{n},$$
$$\beta_n=\gamma_n=1-\frac{1}{n},$$
$$\gamma=\rho=t=c_0>0.$$

Furthermore, we take two different parameter sequences satisfying Condition 3.2 to demonstrate the different effects for different α_n.

Condition 3.2a.

$$\begin{cases}\alpha_n=0.3-1/\left(100*n\right);\ n=2k\\ \alpha_n=0.1-1/(100*n);\ n=2k-1;\end{cases}$$
$$\lambda_n=\lambda_n'=\lambda_n''=1/\left(n+1\right);$$
$$\begin{cases}\beta_n=1-1/n;\gamma_n=1-1/n;\ n=2k\\ \beta_n=1-1/n;\ \gamma_n=1-1/(2n);\ n=2k-1;\end{cases}$$
$$\gamma=\rho=t=c_0>0.$$

Condition 3.2b.

$$\begin{cases} \alpha_n = 0.8 - 1/(100 * n);\ n = 2k \\ \alpha_n = 0.3 - 1/(100 * n);\ n = 2k - 1; \end{cases}$$

$$\lambda_n = \lambda'_n = \lambda''_n = 1/(n+1);$$

$$\begin{cases} \beta_n = 1 - 1/n; \gamma_n = 1 - 1/n;\ n = 2k \\ \beta_n = 1 - 1/n;\ \gamma_n = 1 - 1/(2n);\ n = 2k - 1; \end{cases}$$

$$\gamma = \rho = t = c_0 > 0.$$

According to Tx(12), we define TX as

$$TX = H(G(X)), \tag{15}$$

where

$$G(X) = \min(H_U, \max(X, H_L)), H(X) = P_{S_+^n}(X),$$

which can easily be computed and the fixed point set to $Fix(T) = K$. Moreover, according to Theorems 1 and 2, the sequences generated by algorithm [20] under Conditions 3.1 and 3.2 are convergent.

The computation started with ones(n, n) in MATLAB and stopped when $\|x_{k+1} - x_k\| \le 10^{-4}$ or 10^{-5}. All codes were implemented in MATLAB 7.0 and were run using a Pentium R 1.70 G processor on a 768 M ASUS notebook computer.

We tested the problem using n = 100, 200, 300, 400, 500. The test results for the MRHSD method under different conditions and tolerances are reported in Tables 1 and 2.

Test examples

In this example we generate the data in a similar manner as in [12]. Note that it is very difficult to compute the examples using the extended contraction method [12] when C is asymmetric. However, the MRHSD method can efficiently compute the examples when C is asymmetric.

The diagonal elements of C are randomly generated in the interval (0, 2) and the off-diagonal elements are randomly generated in the interval (-1, 1):

$$(H_U)_{jj} = (H_L)_{jj} = 1,$$
$$(H_U)_{ij} = -(H_L)_{ij} = 0.1, \forall i \neq j, i, j = 1, 2, \ldots, n.$$

Table 1 Numerical results for tolerance of 10^{-4}

Matrix	$\gamma = \rho = t = c_0 = 0.01$					
	Condition 3.1		Condition 3.2a		Condition 3.2b	
n	It	cpu	It	cpu	It	cpu
100	24	1.04	22	0.98	20	0.84
200	34	8.35	30	7.48	22	5.25
300	41	29.93	36	27.48	26	19.10
400	46	72.46	42	67.63	30	51.84
500	52	149.64	48	142.07	34	98.93

Table 2 Numerical results for tolerance of 10^{-5}

Matrix	$\gamma = \rho = t = c_0 = 0.01$					
	Condition 3.1		Condition 3.2a		Condition 3.2b	
n	It	cpu	It	cpu	It	cpu
100	75	3.59	68	2.83	46	2.37
200	106	25.94	96	22.71	66	15.53
300	128	94.37	116	89.52	78	58.30
400	147	236.59	132	205.70	88	138.28
500	165	529.44	148	420.73	100	285.24

Matlab code:

```
C = zeros(n, n); HU = ones(n, n)*0.1; HL = -HU;
for i = 1:n
    for j = 1:n
        t = mod(t*42108+13846,46273);
        C(i, j) = t*2/46273-1;
    end;
end;
for i = 1:n
    C(i, i) = abs(C(i, i))*2; HU(i, i) = 1; HL(i, i) = 1;
end;
```

The numerical results demonstrate that this implementation of the MRHSD method is efficient. Furthermore, the MRHSD Method under Condition 3.2 is more efficient than under Condition 3.1. These numerical experiments and results are the third contribution of the article.

6 Conclusions and discussions

We have proved strong convergence of the MRHSD method under Condition 3.2, which differs from Condition 3.1. The proof can be simplified using Condition 3.2, which imposes suitable restrictions on the parameters. The result can be considered an improvement and refinement of previous results [20]. In particular, we designed an efficient implementation of the MRHSD method based on the approximate projection contraction method. Numerical experiments demonstrated that this is an efficient implementation and that the MRHSD method under Condition 3.2 is more efficient than under Condition 3.1. However, choosing an efficient and implementable nonexpansive mapping for a general $VI(F, K)$ is still a difficult problem.

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Competing interests
The author declares that they have no competing interests.

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