

## *Full Length Research Paper*

# **Design of feedback controller for functional projective synchronization of chaotic systems with disturbances**

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This paper considers the synchronization problem of chaotic systems with disturbance using a new feedback control scheme. Functional projective synchronization (FPS), which is a generalized synchronization concept recently developed, is investigated. Based on Lyapunov stability theory, a novel stability criterion for the synchronization between master and slave chaotic systems is derived. The proposed method is applied to unified chaotic systems and Chen-Lee chaotic systems in order to show the effectiveness of our method.

**Key words:** Chaotic system, functional projective synchronization, Lyapunov method.

## **INTRODUCTION**

Synchronization of one system with another is a very important process in the control of complex physical, chemical and biological systems. Therefore, many researchers have focused on this topic and have developed several efficient synchronization techniques for various systems, including chaotic systems, which are very sensitive to variations in the parameters and initial conditions. Since Pecora and Carroll (1990) introduced the concept of synchronization, the study of chaos synchronization has been of increasing interest to scientists and engineers. Up to date, various applications of chaos synchronization have been introduced (Ji et al., 2009; Lü et al., 2004; Yassen, 2006; Park et al., 2009; Li et al., 2005; Vincent, 2005; Park, 2006). Originally, chaos synchronization refers to the state in which the master (or drive) and the slave (or response) systems have precisely identical trajectories as time goes to infinity. We usually regard such a synchronization as complete synchronization or identical synchronization. Over the last decade, various types of chaos synchronization have been proposed (Rosenblum et al., 1997; Boccaletti et al., 2000; Hramov et al., 2005; Park, 2007), such as anti-synchronization, phase synchronization, projective synchronization (PS), generalized synchronization, lag

synchronization and functional projective synchronization (FPS). Amongst all kinds of chaos synchronization, FPS is the state of the art and generalized concept of synchronization schemes. As compared PS, FPS means that the master and slave systems could be synchronized up to a scaling function, but not a constant. Recently, various control methods which include adaptive control (Park, 2008; Wang et al., 2008; Park et al., 2009), active control (Lee, 2009), backstepping design (Vincent et al., 2007) and nonlinear control (Chen, 2005), have been applied to handle with chaos synchronization. Most of the works done on FPS of chaotic systems have used active control method since it is easy to design a control input and to deal with equations including scaling function. The controller based on the active control method is complex and contains various variables, so it may not be suitable for real practical purpose. This reason makes that many engineers try to make a controller with a simple structure, which consists of a few variables. In real-world control problem, the external disturbance should be considered in the system model. However, the disturbance is not considered in most of research on chaotic synchronization.

In this paper, a new feedback controller including adaptive control scheme is utilized to achieve functional projective synchronization of a general class of chaotic systems with external disturbances. Based on Lyapunov method, a new stability criterion for the synchronization is derived. In order to verify our synchronization scheme, unified chaotic systems and Chen-Lee chaotic systems are considered and new feed-back controllers for the

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systems are designed. Finally, numerical simulations are given to demonstrate the effectiveness of the proposed method.

## MAIN RESULTS

Consider the following master (drive) and slave (response) chaotic systems:

$$\dot{x}(t) = Ax(t) + f(x) + d(t), \quad (1)$$

$$\dot{y}(t) = Ay(t) + f(y) + u(t), \quad (2)$$

Where  $x(t) = (x_1, x_2, \dots, x_n)^T \in \mathfrak{R}^n$  and  $y(t) = (y_1, y_2, \dots, y_n)^T \in \mathfrak{R}^n$  are state vectors of master and slave systems, respectively,  $f(x): \mathfrak{R}^n \rightarrow \mathfrak{R}^n$  is a continuous nonlinear vector function,  $A \in \mathfrak{R}^{n \times n}$  is a constant matrix,  $d(t)$  is the disturbance signal bounded in magnitude  $\|d(t)\| \leq \bar{d}$  and  $u(t) = (u_1, u_2, \dots, u_n)^T \in \mathfrak{R}^n$  is the control input for synchronization between master system (1) and slave system (2).

Let us define the error vector as

$$e(t) = y(t) - \alpha(t)x(t) \quad (3)$$

Where  $\alpha(t)$  is a continuously differentiable bounded function satisfying  $\max |\alpha(t)| = h$  and  $e(t) = (e_1, e_2, \dots, e_n)^T \in \mathfrak{R}^n$  with  $e_i = y_i - \alpha(t)x_i$ .

### Definition 1

It is said that FPS occurs between master system (1) and response system (2) if the condition  $\lim_{t \rightarrow \infty} \|y(t) - \alpha(t)x(t)\| = 0$  holds for a given scaling function  $\alpha(t)$ .

### Remark 1

Chaos synchronization schemes such as complete synchronization, anti-synchronization and projective synchronization are special case of FPS. When,  $\alpha = -1$  and  $\alpha = \text{constant}$ , then FPS becomes complete synchronization, anti-synchronization and projective synchronization, respectively.

From the definition of error signal, the time derivative of (3) is

$$\dot{e}(t) = Ae(t) + f(y) - \alpha(t)f(x) - \dot{\alpha}(t)x(t) - \alpha(t)d(t) + u(t) \quad (4)$$

Here, the non-linear term  $f(y) - \alpha(t)f(x)$  can be decomposed as follows:

$$f(y) - \alpha(t)f(x) = L(e, x)e(t) + \zeta(x), \quad (5)$$

Where  $L \in \mathfrak{R}^{n \times n}$  and  $\zeta: \mathfrak{R}^n \rightarrow \mathfrak{R}^n$  are continuous nonlinear function.

Then, by applying Equation (5) to error dynamics (4), we have

$$\dot{e}(t) = Ae(t) + L(e, x)e(t) + \zeta(x) - \dot{\alpha}(t)x - \alpha(t)d(t) + u(t) \quad (6)$$

Now, we have the following main result.

### Theorem 1

Master system (1) and slave system (2) can be synchronized up to a scaling function  $\alpha(t)$  via the controller:

$$u(t) = Ke(t) + \dot{\alpha}(t)x(t) - \zeta(x) - \frac{h^2 \bar{d}^2}{h\bar{d}\|e(t)\| + \varepsilon\|e(t)\|^2} e(t), \quad (7)$$

Where  $K = \text{diag}\{k_1, k_2, \dots, k_n\} \in \mathfrak{R}^{n \times n}$  is an adaptive control law with update rule  $\dot{k}_i = -e_i^2, (i = 1, \dots, n)$  and  $\varepsilon$  is positive scalar.

### Proof

By substituting the control input (7) into Equation (6), we have

$$\dot{e}(t) = (A + L(e, x) + K)e(t) - \alpha(t)d(t) - \frac{h^2 \bar{d}^2}{h\bar{d}\|e(t)\| + \varepsilon\|e(t)\|^2} e(t). \quad (8)$$

For stability analysis, let us consider the following Lyapunov function

$$V = \frac{1}{2} e^T(t) e(t) + \frac{1}{2} \sum_{i=1}^n (k_i + p_i)^2, \quad (9)$$

Where  $p_i (i = 1, 2, \dots, n)$  are real numbers.

Its time derivative along the solution of system (8) is

$$\begin{aligned}
\dot{V} &= e^T(t) \dot{e}(t) + \sum_{i=1}^n (k_i + p_i) \dot{k}_i \\
&= e^T(t) (A + L(e, x) + K) e(t) - e^T(t) (K + P) e(t) - \alpha(t) e^T(t) d(t) - e^T(t) \frac{h^2 \bar{d}^2}{h \bar{d} \|e(t)\| + \varepsilon \|e(t)\|^2} e(t) \\
&\leq e^T(t) (A + L(e, x) - P) e(t) + h \bar{d} \|e(t)\| - \frac{h^2 \bar{d}^2 \|e(t)\|^2}{h \bar{d} \|e(t)\| + \varepsilon \|e(t)\|^2} \\
&= e^T(t) (A + L(e, x) - P) e(t) + \frac{h \bar{d} \|e(t)\| \cdot \varepsilon \|e(t)\|^2}{h \bar{d} \|e(t)\| + \varepsilon \|e(t)\|^2} \\
&\leq e^T(t) (A + L(e, x) - P) e(t) + \varepsilon \|e(t)\|^2
\end{aligned} \quad (10)$$

Where  $P = \text{diag}\{p_1, p_2, \dots, p_n\} \in \mathbb{R}^{n \times n}$  is a constant matrix and the well-known inequality  $0 \leq ab/(a+b) \leq a \forall a, b > 0$  is used.

Note that when dynamic system behaves chaotically, the state variable  $x_i$  and error signal  $e_i$  are bounded in magnitude, i.e., the chaotic attractor is bounded. Also, we have

$$\begin{aligned}
e^T L(e, x) e &\leq |e|^T \bar{L} |e|, \\
e^T A e &\leq |e|^T \bar{A} |e|
\end{aligned} \quad (11)$$

Where  $\bar{L} \in \mathbb{R}^{n \times n}$  is the constant matrix obtained by taking absolute values on all the entries of  $L(e, x)$ , but  $\bar{A} \in \mathbb{R}^{n \times n}$ , is the constant matrix obtained by taking absolute values on off-diagonal entries of  $A$ .

Then, we have

$$\dot{V} < |e|^T (\bar{A} + \bar{L} - P + \varepsilon I) |e| \equiv -|e|^T Q |e| \quad (12)$$

Where  $Q = -(\bar{A} + \bar{L} - P + \varepsilon I)$ .

Obviously, the error system (6) is asymptotically stable if we can choose the matrix  $P$ , which makes matrix  $Q$  be positive definite. This completes the proof.

## Remark 2

In order to analyze the stability in Theorem 1, the matrix  $P = \text{diag}\{p_1, p_2, \dots, p_n\}$  is utilized in the Lyapunov function (9). However, the information about  $P$  is not used to construct the control law (7).

## Remark 3

If the original active control method widely used in the

literature (Lü et al., 2002; Park et al., 2006; Park et al., 2007; Li et al., 2005; Vincent, 2005; Chen et al., 2005) is applied to systems (1) and (2) for functional synchronization, all the state variables of master and slave systems are needed for the synchronization. Whereas, our controller (7) only needs the state variables of master system. That is, we use fewer variables for control. Note that the purpose of the last term of control input (7) is for compensation of disturbance signals, which are not considered in the literature.

## Application 1: Unified chaotic systems

Consider the following unified chaotic master and slave systems (Lü et al., 2002):

$$\begin{aligned}
\text{Master system: } \dot{x}_1(t) &= (25+10)(x_2(t)-x_1(t))+d_1(t) \\
\dot{x}_2(t) &= (28-35)x_1(t)-x_1(t)x_3(t)+(29-1)x_2(t)+d_2(t) \\
\dot{x}_3(t) &= x_1(t)x_2(t)-\frac{s+8}{3}x_3(t)+d_3(t) \\
\text{Slave system: } \dot{y}_1(t) &= (25+10)(y_2(t)-y_1(t))+u_1(t) \\
\dot{y}_2(t) &= (28-35)y_1(t)-y_1(t)y_3(t)+(29-1)y_2(t)+u_2(t) \\
\dot{y}_3(t) &= y_1(t)y_2(t)-\frac{s+8}{3}y_3(t)+u_3(t)
\end{aligned} \quad (13)$$

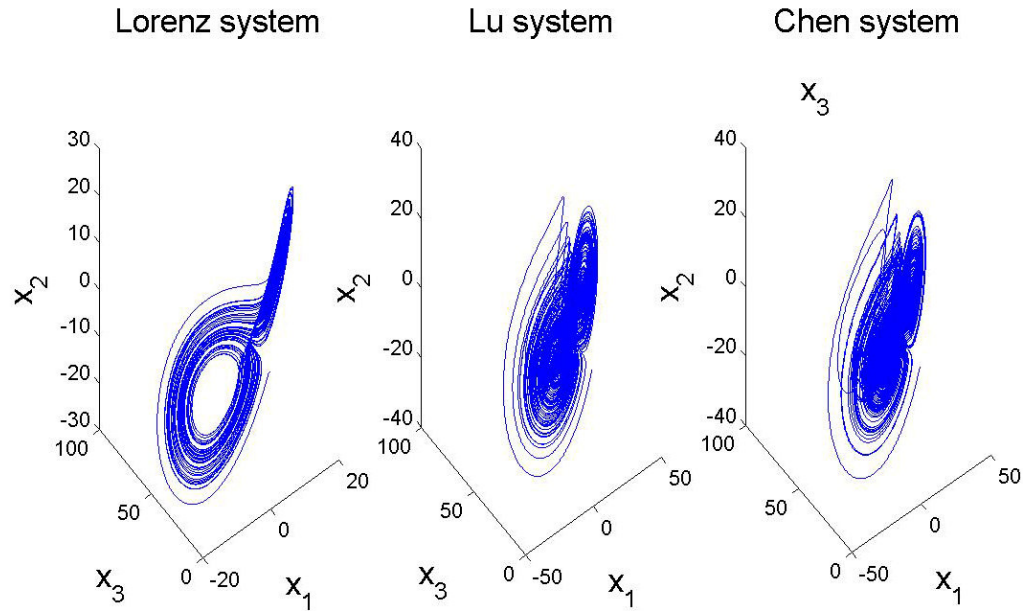
The system (13) is chaotic for any  $s \in [0, 1]$ . It is called the general Lorenz, Lü and Chen system when  $s \in [0, 0.8]$ ,  $s = 0.8$  and  $s \in (0.8, 1]$ , respectively. In order to see chaotic motion of the system (13), let us take initial condition  $x(0) = (0, 1, 1)^T$ . Figure 1 shows three chaotic motions of unified system at  $s = 0, s = 0.8$  and  $s = 1$ , respectively.

We can rewrite the system (13) in the form of Equation (1):

$$A = \begin{bmatrix} -(25s+10) & 25s+10 & 0 \\ 28-35s & 29s-1 & 0 \\ 0 & 0 & -\frac{s+8}{3} \end{bmatrix}, f(x) = \begin{bmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{bmatrix}, f(y) = \begin{bmatrix} 0 \\ -y_1y_3 \\ y_1y_2 \end{bmatrix} \quad (14)$$

The nonlinear term  $f(y) - \alpha(t)f(x)$  is partitioned into

$$\begin{aligned}
f(y) - \alpha(t)f(x) &= \begin{bmatrix} 0 \\ -y_1y_3 + \alpha(t)x_1x_3 \\ y_1y_2 - \alpha(t)x_1x_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -e_1e_3 - \alpha(t)x_1e_3 - \alpha(t)x_3e_1 - \alpha(t)^2x_1x_3 + \alpha(t)x_1x_3 \\ e_1e_2 + \alpha(t)x_1e_2 + \alpha(t)x_2e_1 + \alpha(t)^2x_1x_2 - \alpha(t)x_1x_2 \end{bmatrix} \\
&= L(e, x)e + \zeta(x)
\end{aligned} \quad (15)$$



**Figure 1.** Chaotic motion of unified system.

Where:

$$L(e, x) = \begin{bmatrix} 0 & 0 & 0 \\ -e_3 - \alpha(t)x_3 & 0 & -\alpha(t)x_1 \\ e_2 + \alpha(t)x_2 & \alpha(t)x_1 & 0 \end{bmatrix},$$

$$\zeta(x) = \begin{bmatrix} 0 \\ -\alpha(t)^2 x_1 x_3 + \alpha(t)x_1 x_3 \\ \alpha(t)^2 x_1 x_2 - \alpha(t)x_1 x_2 \end{bmatrix}. \quad (16)$$

### Theorem 2

The FPS for the unified chaotic system (13) is achieved by the following controller.

$$u_1 = k_1 e_1 + \dot{\alpha}(t)x_1 - \frac{h^2 \bar{d}^2}{h\bar{d}\|e\| + \varepsilon\|e\|^2} e_1$$

$$u_2 = k_2 e_2 + \dot{\alpha}(t)x_2 + \alpha(t)^2 x_1 x_3 - \alpha(t)x_1 x_3 - \frac{h^2 \bar{d}^2}{h\bar{d}\|e\| + \varepsilon\|e\|^2} e_2$$

$$u_3 = k_3 e_3 + \dot{\alpha}(t)x_3 - \alpha(t)^2 x_1 x_2 + \alpha(t)x_1 x_2 - \frac{h^2 \bar{d}^2}{h\bar{d}\|e\| + \varepsilon\|e\|^2} e_3. \quad (17)$$

### Proof

From (16),  $e^T L e$  is

$$e^T L e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ -\alpha(t)x_3 & 0 & -\alpha(t)x_1 \\ \alpha(t)x_2 & \alpha(t)x_1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \quad (18)$$

Note that the constant  $m$  satisfying  $|x_i| < m, (i=1,2,3)$  exists since the unified chaos system has bounded orbits. Also since the function  $\alpha(t)$  is bounded, so we have

$$e^T L e < \begin{bmatrix} |e_1| \\ |e_2| \\ |e_3| \end{bmatrix}^T \begin{bmatrix} 0 & \frac{1}{2}hm & \frac{1}{2}hm \\ \frac{1}{2}hm & 0 & hm \\ \frac{1}{2}hm & hm & 0 \end{bmatrix} \begin{bmatrix} |e_1| \\ |e_2| \\ |e_3| \end{bmatrix}$$

$$\equiv |e|^T \bar{L} |e|. \quad (19)$$

So, by using the control (17) and (19) and further computation,  $\dot{V}$  in Equation (12) is given by

$$\dot{V} < \begin{bmatrix} |e_1| \\ |e_2| \\ |e_3| \end{bmatrix}^T \begin{bmatrix} -\delta_1 - p_1 & \frac{1}{2}(|\delta_1| + \delta_2 + hm) & \frac{1}{2}hm \\ \frac{1}{2}(|\delta_1| + \delta_2 + hm) & \delta_3 - p_2 & hm \\ \frac{1}{2}hm & hm & -\delta_4 - p_3 \end{bmatrix} \begin{bmatrix} |e_1| \\ |e_2| \\ |e_3| \end{bmatrix}$$

$$= -|e|^T Q e. \quad (20)$$

Chen-Lee system

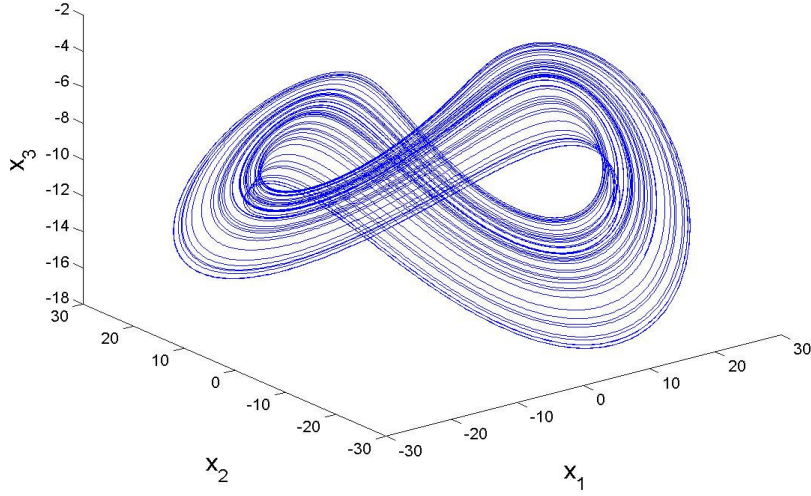


Figure 2. Chaotic motion of Chen-Lee system.

Where

$$\delta_1 = 25s + 10 + \varepsilon, \delta_2 = |28 - 35s|, \delta_3 = 29s - 1 + \varepsilon, \delta_4 = \frac{s+8}{3} + \varepsilon.$$

If there exist  $p_1, p_2, p_3$  such that

$$\begin{cases} p_1 > -\delta_1 \\ p_2 > \frac{(\delta_1 + p_1)\delta_3 - \delta_5^2}{\delta_1 + p_1} \\ p_3 > \begin{cases} \frac{-\delta_4(-\delta_1\delta_3 + \delta_1p_2 - \delta_3p_1 + p_1p_2) + h^2m^2(\delta_5 - \frac{1}{4}\delta_3 + \frac{1}{4}p_2 + \delta_1 + p_1) + \delta_4\delta_5^2}{-\delta_1\delta_3 + \delta_1p_2 - \delta_3p_1 + p_1p_2 - \delta_5^2} & \text{in case of positive denominator} \\ \frac{-\delta_4(-\delta_1\delta_3 + \delta_1p_2 - \delta_3p_1 + p_1p_2) + h^2m^2(\delta_5 - \frac{1}{4}\delta_3 + \frac{1}{4}p_2 + \delta_1 + p_1) + \delta_4\delta_5^2}{-\delta_1\delta_3 + \delta_1p_2 - \delta_3p_1 + p_1p_2 - \delta_5^2} & \text{in case of negative denominator} \end{cases} \end{cases} \quad (21)$$

Where  $\delta_5 = \frac{1}{2}(|\delta_1| + \delta_2 + hm)$ , then the matrix  $Q$  given in (12) is positive-definite. This means that error signals between master and slave systems (13) approach to zero as time goes by. This completes the proof.

### Application 2: Chen-Lee chaotic systems

Consider the following Chen-Lee chaotic master and slave systems (Chen and Lee, 2004):

$$\text{Master system} : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -x_2x_3 \\ x_1x_3 \\ \frac{1}{3}x_1x_2 \end{bmatrix} + \begin{bmatrix} d_1(t) \\ d_2(t) \\ d_3(t) \end{bmatrix},$$

$$\text{Slave system: } \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} -y_2y_3 \\ y_1y_3 \\ \frac{1}{3}y_1y_2 \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}, \quad (22)$$

Where  $a = 5, b = 10, c = 3.8$  and Figure 2 shows the chaotic motion of the Chen-Lee systems.

Then we have the following system parameters:

$$\begin{aligned} L(e, x) &= \begin{bmatrix} 0 & -e_3 - \alpha(t)x_3 & -\alpha(t)x_2 \\ e_3 + \alpha(t)x_3 & 0 & \alpha(t)x_1 \\ \frac{1}{3}(e_2 + \alpha(t)x_2) & \frac{1}{3}\alpha(t)x_1 & 0 \end{bmatrix}, \\ \zeta(x) &= \begin{bmatrix} -\alpha(t)^2x_2x_3 + \alpha(t)x_2x_3 \\ \alpha(t)^2x_1x_3 - \alpha(t)x_1x_3 \\ \frac{1}{3}(\alpha(t)^2x_1x_2 - \alpha(t)x_1x_2) \end{bmatrix}. \end{aligned} \quad (23)$$

### Theorem 3

The Chen-Lee master and slave chaotic systems (22) are synchronized up to a scaling function  $\alpha(t)$  via the controller

$$\begin{aligned} u_1 &= k_1e_1 + \dot{\alpha}(t)x_1 + \alpha(t)^2x_2x_3 - \alpha(t)x_2x_3 - \frac{h^2\bar{d}^2}{h\bar{d}\|e\| + \varepsilon\|e\|^2}e_1 \\ u_2 &= k_2e_2 + \dot{\alpha}(t)x_2 - \alpha(t)^2x_1x_3 + \alpha(t)x_1x_3 - \frac{h^2\bar{d}^2}{h\bar{d}\|e\| + \varepsilon\|e\|^2}e_2 \end{aligned}$$

$$u_3 = k_3 e_3 + \dot{\alpha}(t) x_3 \frac{1}{3} (-\alpha(t)^2 x_1 x_2 + \alpha(t) x_1 x_2) - \frac{h^2 \bar{d}^2}{h \bar{d} \|e\| + \varepsilon \|e\|^2} e_3. \quad (24)$$

### Proof

For this case, we have

$$\begin{aligned} e^T L e &= -e_1 e_2 e_3 - e_1 e_2 \alpha(t) x_3 - e_1 e_3 \alpha(t) x_2 + e_1 e_2 e_3 + e_1 e_2 \alpha(t) x_3 + e_1 e_3 \alpha(t) x_1 \\ &\quad + \frac{1}{3} (e_1 e_2 e_3 + e_1 e_3 \alpha(t) x_2 + e_2 e_3 \alpha(t) x_1) \\ &= \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^T \begin{bmatrix} 0 & -\alpha(t) x_3 & -\alpha(t) x_2 \\ \alpha(t) x_3 & 0 & \alpha(t) x_1 \\ \frac{1}{3} (e_2 + \alpha(t) x_2) & \frac{1}{3} \alpha(t) x_1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \end{aligned} \quad (25)$$

Further, we obtain

$$\begin{aligned} e^T L e &< \begin{bmatrix} |e_1| \\ |e_2| \\ |e_3| \end{bmatrix}^T \begin{bmatrix} 0 & hm & \frac{1}{6} (w + 4hm) \\ hm & 0 & \frac{2}{3} hm \\ \frac{1}{6} (w + 4hm) & \frac{2}{3} hm & 0 \end{bmatrix} \begin{bmatrix} |e_1| \\ |e_2| \\ |e_3| \end{bmatrix} \\ &\equiv |e|^T \bar{L} |e|. \end{aligned} \quad (26)$$

Where  $|e_i| \leq w, i=1,2,3$ ,  $|\alpha(t)| \leq h$  and  $|x_i| \leq m, i=1,2,3$  are used.

Therefore  $\dot{V}$  in Equation (12) is

$$\begin{aligned} \dot{V} &< \begin{bmatrix} |e_1| \\ |e_2| \\ |e_3| \end{bmatrix}^T \begin{bmatrix} \delta_1 - p_1 & hm & \frac{1}{6} (w + 4hm) \\ hm & \delta_2 - p_2 & \frac{2}{3} hm \\ \frac{1}{6} (w + 4hm) & \frac{2}{3} hm & \delta_3 - p_3 \end{bmatrix} \begin{bmatrix} |e_1| \\ |e_2| \\ |e_3| \end{bmatrix} \\ &= -|e|^T Q |e|, \end{aligned} \quad (27)$$

Where  $\delta_1 = a + \varepsilon$ ,  $\delta_2 = -b + \varepsilon$ ,  $\delta_3 = -c + \varepsilon$ .

If there exist  $p_1, p_2, p_3$  such that

$$\begin{aligned} p_1 &> \delta_1 \\ p_2 &> \frac{-\delta_2(\delta_1 - p_1) + h^2 m^2}{-\delta_1 + p_1} \\ p_3 &\begin{cases} > \frac{\delta_3(\delta_1 \delta_2 - \delta_1 p_2 - \delta_2 p_1 + p_1 p_2) + h^2 m^2 (-\frac{4}{9} \delta_1 + \frac{4}{9} p_1 - \delta_3) + \delta_3^2 (-\delta_2 + p_2)}{(\delta_1 - p_1)(\delta_2 - p_2) - h^2 m^2} \\ &\text{incase of positive denominator} \\ < \frac{\delta_3(\delta_1 \delta_2 - \delta_1 p_2 - \delta_2 p_1 + p_1 p_2) + h^2 m^2 (-\frac{4}{9} \delta_1 + \frac{4}{9} p_1 - \delta_3) + \delta_3^2 (-\delta_2 + p_2)}{(\delta_1 - p_1)(\delta_2 - p_2) - h^2 m^2} \\ &\text{incase of negative denominator} \end{cases} \end{aligned} \quad (28)$$

Where  $\delta_5 = \frac{1}{6} (w + 4hm)$ , then the matrix  $Q$  is positive-definite and  $\dot{V} < 0$ . This guarantees that FPS occurs between Chen-Lee master and slave chaotic systems.

### NUMERICAL SIMULATIONS

In this section, two numerical examples are presented to show the effectiveness of our synchronization scheme.

For numerical simulation, a forth-order Runge-Kutta method with sampling time 0.00001 s is used to solve differential equations in this paper. Initial conditions of master and slave systems are chosen as  $x(0) = (-5, -1, 2)^T$  and  $y(0) = (0, 2, -1)^T$ , respectively. Initial conditions of adaptive feedback gain are taken by  $k(0) = (1, 1, 1)^T$  and the scaling function for FPS is chosen as  $\alpha(t) = 1.5 + \sin 2\pi t$ . For simulation, the random noises of  $|d_1(t)| \leq 0.1$ ,  $|d_2(t)| \leq 0.2$ ,  $d_3(t) = 0$  are used and the control parameter  $\varepsilon = 0.00001$  is chosen.

#### Example 1

This example is for showing chaotic synchronization of Lorenz system addressed in Section 3. By applying the controller designed in Equation (17), Figure 3 shows that error signals go to zero rapidly, so FPS is achieved. Also, Figure 4 displays adaptive control gains  $k_i (i=1,2,3)$ . Note

that for  $s=0$ ,  $\delta_1=10$ ,  $\delta_2=28$ ,  $\delta_3=1$ ,  $\delta_4=\frac{8}{3}$ ,  $h=5$ ,  $m=50$ ,  $\bar{d}=0.25$  in Equation (21), we can obtain  $p_1=4.062$ ,  $p_2=3510.1$ ,  $p_3=3471.7$  for guaranteeing Equation (21).

#### Example 2

This example deals with synchronization of Chen-Lee system. By applying the controller in (24), Figure 5 illustrates that the error dynamics of Chen-Lee system is asymptotically stable. It means FPS has occurred by the controller obtained from theorem 3. Figure 6 shows adaptive control gain  $k_i (i=1,2,3)$ . In this example, we took the parameters  $p_1=27.5$ ,  $p_2=7552.9$ ,  $p_3=7559.1$  satisfying Equation (28) for  $h=5$ ,  $m=50$ ,  $w=500$ ,  $\bar{d}=0.25$ .

### Conclusion

In this paper, we have investigated the functional projective synchronization problem for general class of chaos systems. A novel stability criterion for FPS is derived based on Lyapunov method and adaptive control

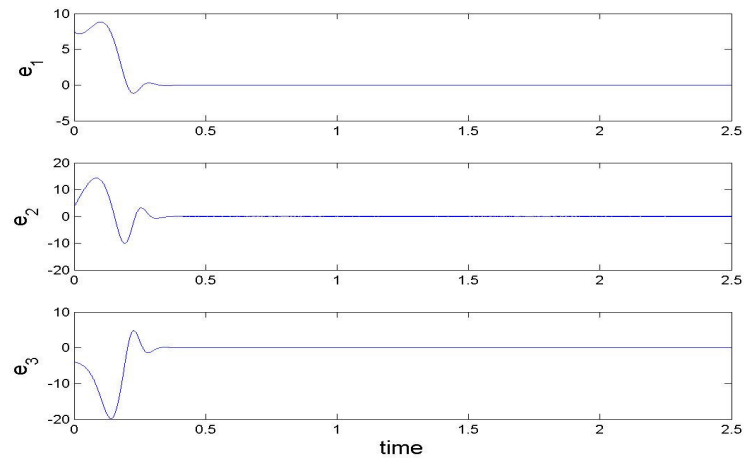


Figure 3. Error signals in Example 1.

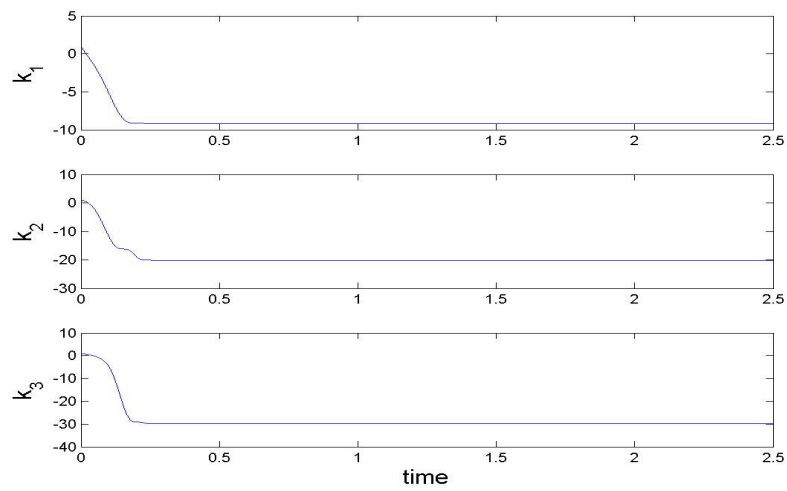


Figure 4. Adaptive control gain in example 1.

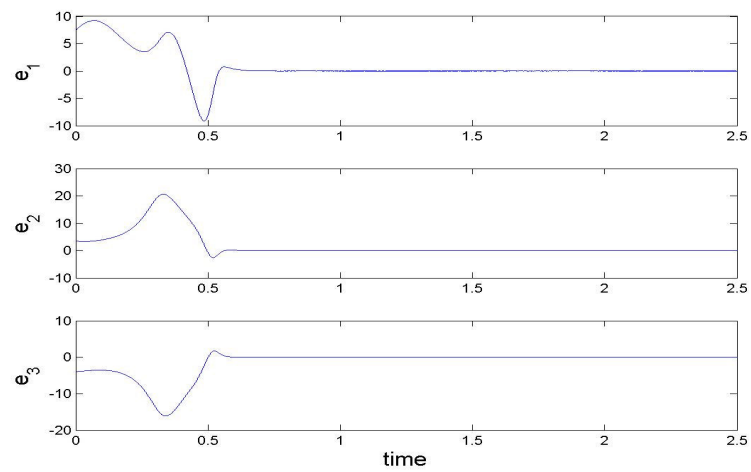


Figure 5. Error signals in Example 2.

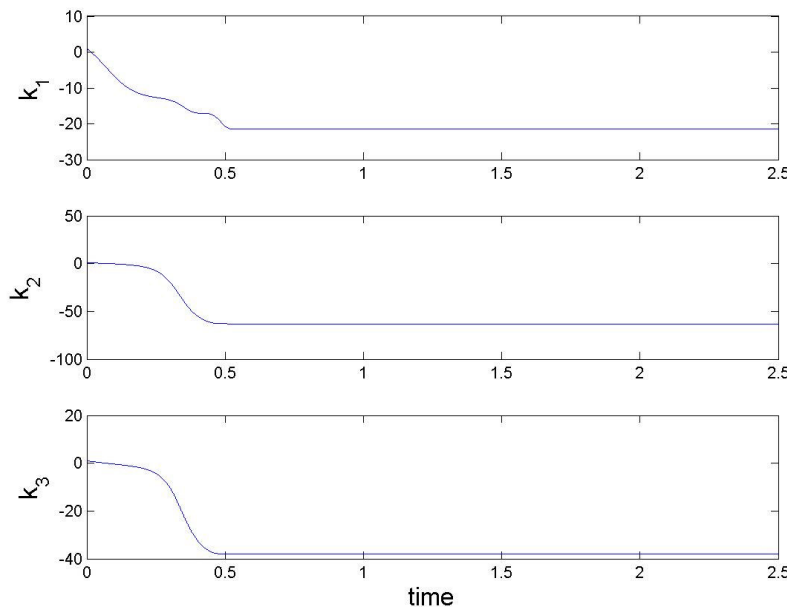


Figure 6. Adaptive control gain in Example 2.

scheme. Numerical simulations show that our method is effective for FPS of various types of chaos systems.

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