

*Full Length Research Paper***Geometric phases for two-mode squeezed vacuum state****Da-Bao Yang^{1*} and Ji-Xuan Hou²**¹Department of Fundamental Physics, School of Science, Tianjin Polytechnic University, Tianjin 300387, People's Republic of China.²Department of Physics, Southeast University, Nanjing 211189, People's Republic of China.

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Although the geometric phase for one-mode squeezed state had been studied in detail, the counterpart for two-mode squeezed state is vacant. In this paper, we aim at the special case, namely, two-mode squeezed vacuum state. Furthermore, the total phase factor is to be written in an elegant form, which is just identical to one term of product of two squeezed operators. In addition, when this system undergoes cyclic evolutions, the corresponding geometric phase is obtained, which is the sum of the counterparts of two isolated one-mode squeezed vacuum state. Finally, the relationship between the cyclic geometric phase and entanglement of two-mode squeezed vacuum state is established.

Key words: Quantum optics, Geometric phases, Entanglement

INTRODUCTION

Squeezed light plays an important role in the development of quantum optics (Walls, 1983). It preserves the minimum uncertainty and exhibits non-classical nature of light, such as sub-Poissonian statistics which can be observed as photon antibunching effect. It also has many applications in optical communications and detection of gravitational radiation. It was generalized to nonlinear case (Kwek and Kiang, 2003). But their studies were just confined to one mode case. Moreover, two mode squeezed state was systematically studied (Caves and Schumaker, 1985; Schumaker and Caves, 1985).

Since geometric phase had been discovered (Berry, 1984) in the quantum system which underwent adiabatic and unitary evolution, its research exploded. Subsequently, it was extended to non-Abelian case (Wilczek and Zee,

1984). Its non-adiabatic and cyclic counterpart was studied (Aharonov and Anandan, 1987; Anandan, 1988). Soon, by getting rid of the condition of cyclic evolution, it was generalized to a more general case (Samuel and Bhandari, 1988), which depended on the earlier study (Pancharatnam, 1956). Subsequently, using kinematic approach, geometric phase was derived as well (Mukunda and Simon, 1993).

Moreover, the geometric phases also had other generalizations, such as off-diagonal ones (Kult, 2007; Manini and Pistolesi, 2000; Mukunda et al., 2001) and mixed state counterparts (Singh et al., 2003; Sjoqvist et al., 2000; Tong et al., 2004).

In addition, geometric phases also have many applications, which range from quantum information and

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computation science (Duan et al., 2001; Jones et al., 2000) to condensed matter (Xiao et al., 2010). Also, this subject is introduced elaborately by many monographs (Bohm et al., 2003; Chruściński and Jamiołkowski, 2004; Wilczek and Shapere, 1989).

Meanwhile, the interdisciplinary study between quantum optics and geometric phase has also emerged. Berry phase for coherent and squeezed states was researched (Chaturvedi et al., 1987); the non-adiabatic geometric phase for squeezed state was studied by Liu et al. (1998); the geometric phase for nonlinear coherent and squeezed state in kinematic approach was discussed (Yang et al., 2011). However, the above studies are all confined to one-mode case. In seeking for theoretical progress, the two-mode case will be researched in this paper. Moreover, the degree of entanglement between the two-mode state is to be evaluated.

This paper is organized as follows. First is a presentation of the features of two-modes squeezed states followed by a review of the kinematic approach to geometric phase. Next, the geometric phase for two-mode squeezed state was calculated. From the above outcome, when the system undergoes cyclic evolution, the corresponding result was also obtained. Moreover, the Von Neumann entropy was calculated, and its relation with geometric phase established. Finally, the research was concluded.

REVIEW OF TWO-MODE SQUEEZED VACUUM STATES AND GEOMETRIC PHASES

The Hamiltonian for two-mode of electromagnetic field (Schumaker and Caves, 1985) takes the form

$$H_0 = \Omega(a_+^\dagger a_+ + a_-^\dagger a_-) + \varepsilon(a_+^\dagger a_+ - a_-^\dagger a_-), \quad (1)$$

where $\Omega \pm \varepsilon$ are the frequencies for the two-mode; also, we take $\hbar=1$ for simplicity. Furthermore, Ω and ε can be regarded as a carrier frequency and a modulation frequency respectively. And the electromagnetic field are quantized by the following commutation relations

$$[a_+, a_-] = [a_+^\dagger, a_-^\dagger] = 0$$

$$[a_+, a_+^\dagger] = [a_-, a_-^\dagger] = 1.$$

The squeezed operator (Schumaker and Caves, 1985) is generalized as

$$S(r, \phi) \equiv \exp[r(a_+ a_- e^{-2i\phi} - a_+^\dagger a_-^\dagger e^{2i\phi})], \quad (2)$$

where the real number r is called the squeeze factor and ϕ is a real phase angle. Moreover, the above operator (2) is unitary,

$$S^{-1}(r, \phi) = S^\dagger(r, \phi) = S(-r, \phi).$$

Hence, the squeezed vacuum state is

$$S(r, \phi)|0\rangle \quad (3)$$

Under the Hamiltonian (1), it evolves as

$$\begin{aligned} e^{-iH_0 t} S(r, \phi)|0\rangle &= e^{-iH_0 t} S(r, \phi) e^{iH_0 t} |0\rangle \\ &= S(r, \phi - \Omega t)|0\rangle \end{aligned} \quad (4)$$

which uses the following formulas (Schumaker and Caves, 1985)

$$\exp[-i\varepsilon t(a_+^\dagger a_+ - a_-^\dagger a_-)] S(r, \phi) \exp[i\varepsilon t(a_+^\dagger a_+ - a_-^\dagger a_-)] = S(r, \phi)$$

and

$$\exp[-i\theta(a_+^\dagger a_+ + a_-^\dagger a_-)] S(r, \phi) \exp[i\theta(a_+^\dagger a_+ + a_-^\dagger a_-)] = S(r, \phi - \theta).$$

The geometric phases γ (Mukunda and Simon, 1993) for arbitrary time t takes the form

$$\gamma = \arg\langle\psi(0)|\psi(t)\rangle + \int_0^t \langle\psi(\tau)|H|\psi(\tau)\rangle d\tau. \quad (5)$$

It is a physical reality, due to it is invariant under gauge transformation. And it can be explained as an outcome of parallel transportation in the framework of fiber bundle, that is, holonomy, hence it is fittingly called geometric phase.

EVALUATIONS OF THE GEOMETRIC PHASE FACTOR

For convenience, instead of calculating the geometric phase, we evaluate the geometric phase factor,

$$e^{i\gamma} = \frac{\langle\psi(0)|\psi(t)\rangle}{\|\langle\psi(0)|\psi(t)\rangle\|} e^{i\delta}, \quad (6)$$

where

$$\delta = \int_0^t \langle\psi(\tau)|H|\psi(\tau)\rangle d\tau \quad (7)$$

which is identical to negative the dynamical phase.

At first, let us calculate the inner product

$$\begin{aligned} \langle\psi(0)|\psi(t)\rangle &= \langle 0|S^\dagger(r, \phi) e^{-iH_0 t} S(r, \phi)|0\rangle \\ &= \langle 0|S^\dagger(r, \phi) S(r, \phi - \Omega t)|0\rangle, \end{aligned} \quad (8)$$

which uses Equation (4). In order to work out the total

phase, the following formula (Schumaker and Caves, 1985) is very useful

$$S^\dagger(r', \phi') S(r'', \phi'') = e^{-i\Theta} S(R, \Phi - \Theta) U(\Theta), \quad (9)$$

where the above parameters satisfy the matrix equation

$$C_{R, \Phi} e^{i\Theta \sigma_3} = C_{r'', \phi''} C_{-r', \phi'}, \quad (10)$$

where the matrix $C_{r, \phi}$ is defined by

$$C_{r, \phi} = \begin{pmatrix} \cosh r & e^{2i\phi} \sinh r \\ e^{-2i\phi} \sinh r & \cosh r \end{pmatrix} \quad (11)$$

and σ_3 is the famous Pauli matrix in the z direction. By substituting Equation 9 into Equation 8, one obtains

$$\langle \psi(0) | \psi(t) \rangle = \langle 0 | e^{-i\Theta} S(R, \Phi - \Theta) U(\Theta) | 0 \rangle,$$

where

$$U(\Theta) = \exp[-i\Theta(a_+^\dagger a_+ + a_-^\dagger a_-)].$$

By use of the explicit decomposition of squeezed operator (Schumaker and Caves, 1985):

$$S(R, \Psi) = (\cosh R)^{-1} e^{-a_+^\dagger a_+ e^{2i\Psi} \tanh R} e^{-(a_+^\dagger a_+ + a_-^\dagger a_-) \ln(\cosh R)} e^{a_+ a_- e^{-2i\phi} \tanh R}, \quad (12)$$

the total phase can be transformed to be an elegant manner

$$\langle \psi(0) | \psi(t) \rangle = (\cosh R)^{-1} e^{-i\Theta}, \quad (13)$$

of which parameters are determined by Equation (10). Its explicit form is

$$\begin{pmatrix} e^{i\Theta} \cosh R & e^{i(2\Phi - \Theta) \sinh R} \\ e^{i(\Theta - 2\Phi) \sinh R} & e^{-i\Theta} \cosh R \end{pmatrix} = \begin{pmatrix} e^{-i\Omega t (\cosh 2r \sin \Omega t + \cos \Omega t)} & e^{i(2\phi - \Omega t - \pi/2) \sinh 2r \sin \Omega t} \\ e^{-i(2\phi - \Omega t - \pi/2) \sinh 2r \sin \Omega t} & e^{i\Omega t (\cos \Omega t - i \sin \Omega t \cosh 2r)} \end{pmatrix}.$$

Therefore, the element (2,2) can tell us the total phase factor $e^{-i\Theta}$, which take the form

$$\frac{e^{i\Omega t} (\cos \Omega t - i \sin \Omega t \cosh 2r)}{(\cos^2 \Omega t + \sin^2 \Omega t \cosh^2 2r)^{1/2}}. \quad (14)$$

Moreover, let us calculate another term δ (7) in the expression of geometric phase (5). By substituting Equation 4 into Equation 7, one can obtain

$$\delta = \int_0^t \langle 0 | S^\dagger(r, \phi - \Omega \tau) H S(r, \phi - \Omega \tau) | 0 \rangle d\tau.$$

By use of the following formulas (Schumaker and Caves, 1985)

$$\begin{aligned} S^\dagger(r, \eta) a_+ S(r, \eta) &= a_+ \cosh r - a_-^\dagger e^{2i\eta} \sinh r \\ S^\dagger(r, \eta) a_- S(r, \eta) &= a_- \cosh r - a_+^\dagger e^{2i\eta} \sinh r, \end{aligned}$$

the formula for δ can be simplified as

$$\delta = 2\Omega t \sinh^2 r. \quad (15)$$

Finally, by inserting Equations 14 and 15 into Equation 6, the geometric phase is achieved as

$$e^{i\gamma} = \frac{e^{i\Omega t \cosh 2r} (\cos \Omega t - i \sin \Omega t \cosh 2r)}{(\cos^2 \Omega t + \sin^2 \Omega t \cosh^2 2r)^{1/2}}$$

Now, the cyclic geometric phase will be discussed. From the total phase factor (8), it is not hard to see that when $\Omega t = 2\pi$, the state (4) will undergo a genuine cyclic evolution of which the final state is exactly the initial state. In other words, the total phase can be regarded as zero. Hence the geometric phase can be explicitly expressed

$$\gamma_c = 4\pi \sinh^2 r \mod 2\pi, \quad (16)$$

which is exactly negative the dynamical phase. Because total phase vanish and the geometric phase is equal to the difference between the total phase and dynamical phase along with Yang et al. (2011), the geometric phase for isolated one-mode squeezed state is

$$\gamma_{ic} = 2\pi \sinh^2 r \mod 2\pi,$$

where the subscript index i implies mode. So if we define cyclic geometric phase to the simplest form, in combination with Equation 16, $\gamma_c = \gamma_{1c} + \gamma_{2c}$, which reveals the additional relationship between the two-mode system and the isolated one mode system.

Moreover, the cyclic geometric phase γ_c is related to the Von Neumann entropy which can measure the entanglement between the two modes in the squeezed state. In order to establish the relationship. Let us calculate the entropy first. By use of Equation 12,

$$S(r, \phi - \Omega t) | 0 \rangle = \frac{1}{\cosh r} \sum_{n=0}^{\infty} (-e^{2i(\phi - \Omega t)} \tanh r)^n | n \rangle_+ | n \rangle_-.$$

Fortunately, it is already in the form of Schmidt decomposition. By a brute force calculation, the Von Neumann entropy reads

$$E = \cosh^2 r \ln(\cosh^2 r) - \sinh^2 r \ln \sinh^2 r, \quad (17)$$

which is identical to the result in Van Enk (1999). Finally,

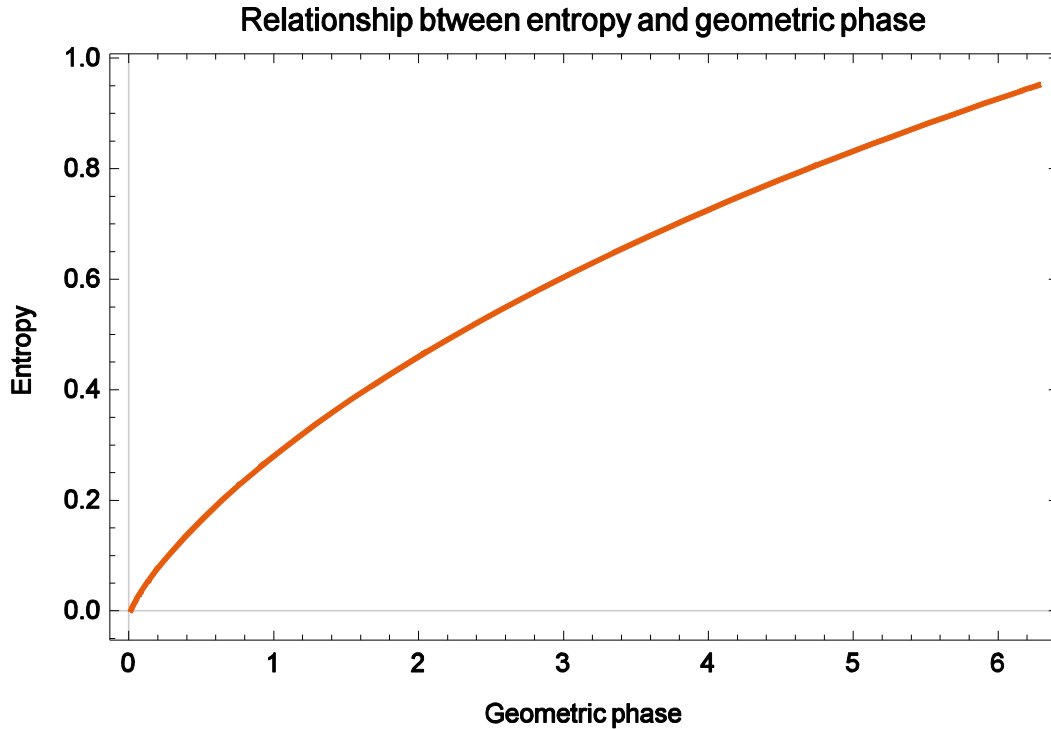


Figure 1. γ_c is set to vary from 0 to 2π .

substituting Equation 16 into Equation 17, we obtain

$$E = \left(1 + \frac{\gamma_c}{4\pi}\right) \ln\left(1 + \frac{\gamma_c}{4\pi}\right) - \frac{\gamma_c}{4\pi} \ln \frac{\gamma_c}{4\pi},$$

which shows the relationship between the entanglement and the cyclic geometric phase. And the corresponding graph is as shown in Figure 1.

Conclusions

In this article, the geometric phase factor for two-mode squeezed vacuum state is evaluated explicitly. The total phase factor (13) is turned to be an elegant outcome, which is just one term of the product of the initial squeezed operator and final squeezed operator (9). When this system undergoes cyclic evolutions, the corresponding geometric phase is obtained, which is just the sum of the counterparts of two isolated one-mode squeezed state. Furthermore, the relationship between the cyclic geometric phase and entanglement of two-mode squeezed state is established.

Conflict of Interests

The authors have not declared any conflict of interest.

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