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# Embedding theorem on RD-spaces

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## Abstract

An RD-space  $(X, d, \mu)$  is a space of homogeneous type in the sense of Coifman and Weiss with the additional property that a reverse doubling property holds. An important class of RD-spaces is provided by Carnot-Carathéodory spaces with a doubling measure. In this article, the author establishes the embedding theorem for Besov and Triebel-Lizorkin spaces on RD-spaces.

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## 1 Introduction and statement of main results

Spaces of homogeneous type, particularly including metric measure spaces, play a prominent role in many fields of mathematics. These spaces constitute natural generalizations of manifolds admitting all kinds of singularities and still providing rich geometric structure; see [1, 2]. Analysis on spaces of homogenous type has been performed quite intensively; see, for example, [3–5]. Recently, a theory of Besov and Triebel-Lizorkin spaces on RD-spaces was developed in [6, 7], which includes  $n$ -regular measure spaces.

Let us now recall some notations and definitions. Spaces of homogeneous type were introduced by Coifman and Weiss in the early 1970s, in [4]. A *quasi-metric*  $d$  on a set  $X$  is a function  $d : X \times X \rightarrow [0, \infty)$  satisfying (i)  $d(x, y) = d(y, x) \geq 0$  for all  $x, y \in X$ ; (ii)  $d(x, y) = 0$  if and only if  $x = y$ ; and (iii) the *quasi-triangle inequality*: there is a constant  $A_0 \in [1, \infty)$  such that for all  $x, y, z \in X$ ,

$$d(x, y) \leq A_0 [d(x, z) + d(z, y)]. \quad (1.1)$$

We define the quasi-metric ball by  $B(x, r) := \{y \in X : d(x, y) < r\}$  for  $x \in X$  and  $r > 0$ . Note that the quasi-metric, in contrast to a metric, may not be Hölder regular and quasi-metric balls may not be open. We say that  $(X, d, \mu)$  is a space of homogeneous type in sense of Coifman and Weiss if  $d$  is a quasi-metric and  $\mu$  is a nonnegative Borel regular measure on  $X$  satisfying the *doubling condition*, that is, for all  $x \in X$ ,  $r > 0$ , then  $0 < \mu(B(x, r)) < \infty$  and

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)), \quad (1.2)$$

where  $\mu$  is assumed to be defined on a  $\sigma$ -algebra which contains all Borel sets and all balls  $B(x, r)$  and the constant  $0 < C < \infty$  is independent of  $x \in X$  and  $r > 0$ .

We point out that the doubling condition (1.2) implies that there exists a positive constant  $\omega := \log_2 C$  (the *upper dimension* of  $\mu$ ) such that for all  $x \in X$ ,  $\lambda \geq 1$  and  $r > 0$ ,

$$\mu(B(x, \lambda r)) \leq C \lambda^\omega \mu(B(x, r)). \quad (1.3)$$

Macías and Segovia [8] showed that the quasi-metric  $d$  can be replaced by another quasi-metric  $\tilde{d}$  such that the topologies induced on  $X$  by  $d$  and  $\tilde{d}$  coincide. Moreover,  $\tilde{d}$  has the following regularity property: there exist constants  $C > 0$  and  $0 < \theta < 1$  such that for all  $0 < r < \infty$  and all  $x, x', y \in X$ ,

$$|\tilde{d}(x, y) - \tilde{d}(x', y)| \leq C \tilde{d}(x, x')^\theta [\tilde{d}(x, y) + \tilde{d}(x', y)]^{1-\theta}. \quad (1.4)$$

Analysis on spaces of homogeneous type has been performed quite intensively in recent years in [3, 9] and [10]. For example, Coifman and Weiss introduced atomic Hardy space  $H_{at}^p$  for  $p \in (0, 1]$  in [5] and proved that if  $T$  is a Calderón-Zygmund singular integral operator and is bounded on  $L^2$ , then  $T$  extends a bounded operator from  $H^p$  to  $L^p$  for suitable  $p \leq 1$ . In many applications, the additional assumptions on the measure  $\mu$  are required. For instance, Macías and Segovia in [8] provided the maximal function characterization of the Hardy spaces  $H_{at}^p$  on spaces of homogeneous type with additional assumption that the quasi-metric  $d$  satisfies the regularity condition in (1.4) and the measure  $\mu$  satisfies the following property:

$$\mu(B(x, r)) \sim r. \quad (1.5)$$

Note that property (1.5) is much stronger than the doubling condition. More precisely, Macías and Segovia provided the maximal function characterization for Hardy spaces  $H^p(X)$  with  $(1 + \theta)^{-1} < p \leq 1$ , on spaces of homogeneous type  $(X, d, \mu)$  that satisfy the property (1.4) on the quasi-metric  $d$  and the property (1.5) on the measure  $\mu$ .

In [11], Nagel and Stein developed the product  $L^p$  ( $1 < p < \infty$ ) theory in the setting of the Carnot-Carathéodory spaces formed by vector fields satisfying Hörmander's finite rank condition. The particular Carnot-Carathéodory spaces studied in [11] are metric spaces with a measure  $\mu$  satisfying the conditions  $\mu(B(x, sr)) \sim s^{m+2} \mu(B(x, r))$  for  $s \geq 1$  and  $\mu(B(x, sr)) \sim s^4 \mu(B(x, r))$  for  $s \leq 1$ . These conditions on the measure are weaker than property (1.5) but are still stronger than the original doubling condition (1.2). In [6], motivated by the work of Nagel and Stein, Besov and Triebel-Lizorkin spaces were developed on spaces of homogeneous type with a regular quasi-metric and a measure satisfying the reverse doubling condition, that is, there are constants  $\kappa \in (0, \omega]$  and  $c \in (0, 1]$  such that

$$c \lambda^\kappa \mu(B(x, r)) \leq \mu(B(x, \lambda r)) \quad (1.6)$$

for all  $x \in X$ ,  $0 < r < \sup_{x, y \in X} d(x, y)/2$  and  $1 \leq \lambda < \sup_{x, y \in X} d(x, y)/2r$ .

We would like to mention that spaces of homogeneous type encompass several important examples in harmonic analysis, such as Euclidean spaces with  $A_\infty$ -weights (of the Muckenhoupt class), Ahlfors  $n$ -regular metric measure spaces, Lie groups of polynomial growth and Carnot-Carathéodory spaces with doubling measures. All these examples fall under the scope of the study of RD-spaces introduced in [6]. An RD-space  $(X, d, \mu)$  is a

space of homogeneous type in sense of Coifman and Weiss which has a ‘dimension’  $\omega$  and satisfies the quasi-metric  $d$  satisfying (1.4) and the ‘reverse’ doubling property (1.6). For further developments, including analogous theories of function spaces on RD-spaces, we refer to [6, 7, 11, 12] and [13].

On the other hand, embedding theorems are essential tools in many fields for function spaces, especially partial differential equations. For embedding theorems on  $\mathbb{R}^n$ , see [14–17] and [18]. Han, Lin and Yang in [19] and [20] have proved embedding theorems for Besov and Triebel-Lizorkin spaces on spaces of homogeneous type  $(X, d, \mu)$ , where the quasi-metric  $d$  satisfies (1.4) and, however, measure  $\mu$  satisfies (1.5).

The main purpose in this paper is to establish the embedding theorem for Besov and Triebel-Lizorkin spaces on RD-spaces. We would like to point out that the reverse doubling property on the measure played an important role. More precisely, this reverse doubling property ensures that

$$\sum_{k \in \mathbb{Z}: \delta^k \geq r} \frac{1}{\mu(B(x, \delta^k))} \leq \frac{C}{\mu(B(x, r))},$$

which is the key to developing the theory of Besov and Triebel-Lizorkin spaces on spaces of homogeneous type. See [6, 7] for more details. But this reverse doubling property on the measure does not play any role in the proof of the embedding theorem in this paper. However, to achieve the embedding theorem, one needs the density condition on the measure, namely

$$\mu(B(x, r)) \geq Cr^\omega \tag{1.7}$$

for any  $x \in X$  and  $r > 0$ .

Throughout this paper, we use  $C$  to denote positive constants, whose value may vary from line to line. Constants with subscripts, such as  $C_1$ , do not change in different occurrences. By  $V_r(x)$  we denote the measure of  $B(x, r)$ , the ball centered at  $x$  with radius  $r > 0$ ; and  $V(x, y)$  denotes the measure of  $B(x, y)$ , the ball centered at  $x$  with radius  $d(x, y) > 0$ . In addition, we use the notation  $a \lesssim b$  to mean that there is a constant  $C > 0$  such that  $a \leq Cb$ , and the notation  $a \sim b$  to mean that  $a \lesssim b \lesssim a$ . The implicit constants  $C$  are meant to be independent of other relevant quantities. Also, for two topological spaces  $\mathfrak{A}$  and  $\mathfrak{B}$ ,  $\mathfrak{A} \hookrightarrow \mathfrak{B}$  means a linear and continuous embedding. For  $p > 1$ , let  $p'$  be its conjugate index.

Before stating the embedding theorem, we now recall test functions and distributions on RD-spaces  $(X, d, \mu)$ .

**Definition 1.1** (Test functions, [6]) Fix  $x_0 \in X$ ,  $r > 0$ ,  $\gamma > 0$  and  $\beta \in (0, \theta)$ . A function  $f$  defined on  $X$  is said to be a *test function of type  $(x_0, r, \beta, \gamma)$  centered at  $x_0 \in X$*  if  $f$  satisfies the following three conditions.

- (i) (Size condition) For all  $x \in X$ ,

$$|f(x)| \leq C \frac{1}{V_r(x_0) + V(x, x_0)} \left( \frac{r}{r + d(x, x_0)} \right)^\gamma.$$

(ii) (Regularity condition) For all  $x, y \in X$  with  $d(x, y) < (2A_0)^{-1}(r + d(x, x_0))$ ,

$$|f(x) - f(y)| \leq C \left( \frac{d(x, y)}{r + d(x, x_0)} \right)^\beta \frac{1}{V_r(x_0) + V(x, x_0)} \left( \frac{r}{r + d(x, x_0)} \right)^\gamma.$$

(iii) (Cancellation condition)

$$\int f(x) d\mu(x) = 0.$$

We denote by  $G(x_0, r, \beta, \gamma)$  the set of all test functions of type  $(x_0, r, \beta, \gamma)$ . The norm of  $f$  in  $G(x_0, r, \beta, \gamma)$  is defined by

$$\|f\|_{G(x_0, r, \beta, \gamma)} := \inf \{C > 0 : \text{(i) and (ii) hold}\}.$$

For each fixed  $x_0$ , let  $G(\beta, \gamma) := G(x_0, 1, \beta, \gamma)$ . It is easy to check that for each fixed  $x_1 \in X$  and  $r > 0$ , we have  $G(x_1, r, \beta, \gamma) = G(\beta, \gamma)$  with equivalent norms. Furthermore, it is also easy to see that  $G(\beta, \gamma)$  is a Banach space with respect to the norm on  $G(\beta, \gamma)$ .

For  $0 < \beta < \theta$  and  $\gamma > 0$ , let  $\mathring{G}(\beta, \gamma)$  be the completion of the space  $G(\theta, \gamma)$  in the norm of  $G(\beta, \gamma)$ . For  $f \in \mathring{G}(\beta, \gamma)$ , define  $\|f\|_{\mathring{G}(\beta, \gamma)} := \|f\|_{G(\beta, \gamma)}$ .

**Definition 1.2** (Distributions) The *distribution space*  $(\mathring{G}(\beta, \gamma))'$  is defined to be the set of all linear functionals  $\mathcal{L}$  from  $\mathring{G}(\beta, \gamma)$  to  $\mathbb{C}$  with the property that there exists  $C > 0$  such that for all  $f \in \mathring{G}(\beta, \gamma)$ ,

$$|\mathcal{L}(f)| \leq C \|f\|_{\mathring{G}(\beta, \gamma)}.$$

We begin with recalling the definition of approximation to the identity, which plays the same role as the heat kernel  $H(s, x, y)$  does in [11].

**Definition 1.3** ([6]) Let  $\theta$  be the regularity exponent of  $X$ . A sequence  $\{S_k\}_{k \in \mathbb{Z}}$  of linear operators is said to be an approximation to the identity (in short, ATI) if there exists a constant  $C, C_1 > 0$  such that for all  $k \in \mathbb{Z}$  and all  $x, x', y, y' \in X$ ,  $S_k(x, y)$ , the kernel of  $S_k$  is a function from  $X \times X$  into  $\mathbb{C}$  satisfying

- (i)  $S_k(x, y) = 0$  if  $d(x, y) \geq C2^{-k}$  and  $|S_k(x, y)| \leq C \frac{1}{V_{2^{-k}(x)+V_{2^{-k}}(y)}}$ ;
- (ii)  $|S_k(x, y) - S_k(x', y)| \leq C2^{k\theta} d(x, x')^\theta \frac{1}{V_{2^{-k}(x)+V_{2^{-k}}(y)}}$  for  $\rho(x, x') \leq \max\{C_1, 2\}2^{-k}$ ;
- (iii)  $|S_k(x, y) - S_k(x, y')| \leq C2^{k\theta} d(y, y')^\theta \frac{1}{V_{2^{-k}(x)+V_{2^{-k}}(y)}}$  for  $\rho(y, y') \leq \max\{C_1, 2\}2^{-k}$ ;
- (iv)  $||[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]| \leq C2^{2k\theta} d(x, x')^\theta d(y, y')^\theta \frac{1}{V_{2^{-k}(x)+V_{2^{-k}}(y)}}$  for  $\rho(x, x') \leq \max\{C_1, 2\}2^{-k}$  and  $\rho(y, y') \leq \max\{C_1, 2\}2^{-k}$ ;
- (v)  $\int_X S_k(x, y) d\mu(y) = 1$ ;
- (vi)  $\int_X S_k(x, y) d\mu(x) = 1$ .

The Besov and Triebel-Lizorkin spaces on RD-spaces are defined as follows.

**Definition 1.4** ([6]) Suppose that  $|s| < \theta$  and  $\omega$  is the upper dimension of  $(X, d, \mu)$ . Let  $\{S_k\}_{k \in \mathbb{Z}}$  be an ATI and  $D_k = S_k - S_{k-1}$  for  $k \in \mathbb{Z}$ .

The Besov space  $\dot{B}_p^{s,q}(X)$  is the collection of all  $f \in (\dot{G}(\beta, \gamma))'$  with  $\beta, \gamma \in (0, \theta)$  and  $\max(\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s}) < p \leq \infty$  and  $0 < q \leq \infty$  such that

$$\|f\|_{\dot{B}_p^{s,q}(X)} := \left\{ \sum_{k \in \mathbb{Z}} 2^{ksq} \|D_k(f)\|_{L^p(X)}^q \right\}^{1/q} < \infty.$$

The Triebel-Lizorkin space  $\dot{F}_p^{s,q}(X)$  is the collection of all  $f \in (\dot{G}(\beta, \gamma))'$  with  $\beta, \gamma \in (0, \theta)$  and  $\max(\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s}) < p < \infty$ ,  $\max(\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s}) < q \leq \infty$  such that

$$\|f\|_{\dot{F}_p^{s,q}(X)} := \left\| \left\{ \sum_{k \in \mathbb{Z}} 2^{ksq} |D_k(f)|^q \right\}^{1/q} \right\|_{L^p(X)} < \infty.$$

The main result of this paper is the following.

**Theorem 1.5** *Suppose that  $\mu(B(x, r)) \geq Cr^\omega$  for any  $x \in X$  and  $r > 0$  and  $-\theta < s_1 < s_2 < \theta$ .*

- (i) *Let  $\max\{\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s_i}\} < p_i \leq \infty$ ,  $0 < q \leq \infty$ ,  $i = 1, 2$  and  $-\theta < s_1 - \omega/p_1 = s_2 - \omega/p_2 < \theta$ . Then*

$$\dot{B}_{p_2}^{s_2,q} \hookrightarrow \dot{B}_{p_1}^{s_1,q}. \quad (1.8)$$

- (ii) *Let  $\max\{\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s_i}\} < p_i < \infty$  and  $\max\{\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s_i}\} < q_i \leq \infty$  for  $i = 1, 2$ , and  $-\theta < s_1 - \omega/p_1 = s_2 - \omega/p_2 < \theta$ . Then*

$$\dot{F}_{p_2}^{s_2,q_2} \hookrightarrow \dot{F}_{p_1}^{s_1,q_1}. \quad (1.9)$$

## 2 The proof of Theorem 1.5

In this section, we will prove Theorem 1.5. Since there is no Fourier transform on spaces of homogeneous type, the proof of Theorem 1.5 is quite different from the proof on  $\mathbb{R}^n$  as given on p.131 in [16]. The density property (1.7) on the measure  $\mu$  plays a crucial role in the proof of Theorem 1.5 in this paper.

We first recall the following lemmas, namely the construction provided as an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type by Christ in [3], the discrete Calderón reproducing formulae and the frame characterizations of Besov and Triebel-Lizorkin spaces on RD-spaces established in [6].

**Lemma 2.1** *Let  $X$  be a space of homogeneous type in the sense of Coifman and Weiss. Then there exist a collection  $\{Q_\alpha^k \subset X : k \in \mathbb{Z}, \alpha \in I_k\}$  of open subsets, where  $I_k$  is some (possible finite) index set, and constants  $\delta \in (0, 1)$  and  $C_2, C_3 > 0$  such that*

- (i)  $\mu(X \setminus \bigcup_\alpha Q_\alpha^k) = 0$  for each fixed  $k$  and  $Q_\alpha^k \cap Q_\beta^k = \emptyset$  if  $\alpha \neq \beta$ ;
- (ii) for any  $\alpha, \beta, k, l$  with  $l \geq k$ , either  $Q_\beta^l \subset Q_\alpha^k$  or  $Q_\beta^l \cap Q_\alpha^k = \emptyset$ ;
- (iii) for each  $(k, \alpha)$  and each  $l < k$ , there is a unique  $\beta$  such that  $Q_\alpha^k \subset Q_\beta^l$ ;
- (iv)  $\text{diam}(Q_\alpha^k) \leq C_2 \delta^k$ ;
- (v) each  $Q_\alpha^k$  contains some ball  $B(z_\alpha^k, C_3 \delta^k)$ , where  $z_\alpha^k \in X$ .

In fact, we can think of  $Q_\alpha^k$  as being a dyadic cube with a diameter roughly  $\delta^k$  and centered at  $z_\alpha^k$ . In what follows, we always suppose  $\delta = 1/2$ . See [6] for how to remove this restriction. Also, in the following, for  $k \in \mathbb{Z}$ ,  $\tau \in I_k$ , we will denote by  $Q_\tau^{k,v}$ ,  $v = 1, \dots, N(k, \tau)$ , the set of all cubes  $Q_\tau^{k+M} \subset Q_\tau^k$ , where  $M$  is a fixed large positive integer.

**Lemma 2.2** Suppose that  $\{S_k\}_{k \in \mathbb{Z}}$  is an approximation to the identity as in Definition 1.3. Set  $D_k = S_k - S_{k-1}$  for  $k \in \mathbb{Z}$ . Then, for any fixed  $M \in \mathbb{N}$  large enough, there exists a family of functions  $\{\tilde{D}_k(x, y)\}_{k \in \mathbb{Z}}$  such that for any fixed  $y_\tau^{k,v} \in Q_\tau^{k,v}$ ,  $k \in \mathbb{Z}$ ,  $\tau \in I_k$  and  $v \in \{1, \dots, N(k, \tau)\}$  and all  $f \in (\dot{G}(\beta, \gamma))'$  with  $0 < \beta, \gamma < \theta$  and  $x \in X$ ,

$$f(x) = \sum_{k \in \mathbb{Z}} \sum_{\tau \in I_k} \sum_{v=1}^{N(k, \tau)} \mu(Q_\tau^{k,v}) \tilde{D}_k(x, y_\tau^{k,v}) D_k(f)(y_\tau^{k,v}), \quad (2.1)$$

where the series converges in the norm of  $(\dot{G}(\beta', \gamma'))'$  with  $\theta > \beta' > \beta$  and  $\theta > \gamma' > \gamma$ . Moreover,  $\tilde{D}_k(x, y)$ ,  $k \in \mathbb{Z}$ , the kernel of  $\tilde{D}_k$ , satisfy the following estimates: for  $0 < \epsilon < \theta$ ,

$$|\tilde{D}_k(x, y)| \leq C \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)} \frac{2^{-k\epsilon}}{(2^{-k} + d(x, y))^\epsilon}; \quad (2.2)$$

$$\begin{aligned} & |\tilde{D}_k(x, y) - \tilde{D}_k(x', y)| \\ & \leq C \left( \frac{d(x, x')}{2^{-k} + d(x, y)} \right)^\epsilon \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)} \frac{2^{-k\epsilon}}{(2^{-k} + d(x, y))^\epsilon} \end{aligned} \quad (2.3)$$

for  $d(x, x') \leq (2^{-k} + d(x, y))/2A$ ;

$$\int_X \tilde{D}_k(x, y) d\mu(y) = \int_X \tilde{D}_k(x, y) d\mu(x) = 0 \quad (2.4)$$

for all  $k \in \mathbb{Z}$ .

**Lemma 2.3** Let all the other notation be as in Lemma 2.2. Suppose that  $|s| < \theta$ ,  $\omega$  is the upper dimension of  $(X, d, \mu)$ .

For all  $f \in (\dot{G}(\beta, \gamma))'$  with  $\beta, \gamma \in (0, \theta)$  and  $\max(\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s}) < p \leq \infty$  and  $0 < q \leq \infty$ , then

$$\|f\|_{\dot{B}_p^{s,q}(X)} \sim \left\{ \sum_{k \in \mathbb{Z}} 2^{ksq} \left[ \sum_{\tau \in I_k} \sum_{v=1}^{N(k, \tau)} \mu(Q_\tau^{k,v}) |D_k(f)(y_\tau^{k,v})|^p \right]^{q/p} \right\}^{1/q}.$$

For all  $f \in (\dot{G}(\beta, \gamma))'$  with  $\beta, \gamma \in (0, \theta)$  and  $\max(\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s}) < p < \infty$ ,  $\max(\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s}) < q \leq \infty$ , then

$$\|f\|_{\dot{B}_p^{s,q}(X)} \sim \left\| \left\{ \sum_{k \in \mathbb{Z}} \sum_{\tau \in I_k} \sum_{v=1}^{N(k, \tau)} 2^{ksq} |D_k(f)(y_\tau^{k,v})|^q \chi_{Q_\tau^{k,v}} \right\}^{1/q} \right\|_{L^p(X)}.$$

We now prove Theorem 1.5.

*Proof of Theorem 1.5* To prove (1.8), let  $f \in \dot{B}_{p_2}^{s_2, q}(X)$  with  $|s_2| < \theta$ ,  $\max(\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s_2}) < p_2 \leq \infty$  and  $0 < q_2 \leq \infty$ . Since  $s_1 < s_2$  and  $s_1 - \omega/p_1 = s_2 - \omega/p_2$ , it follows that  $p_2 < p_1$ . First, we recall the following known estimates of Lemma 3.2 in [6]: for  $k, k' \in \mathbb{Z}$ ,  $\tau \in I_k$ ,  $\tau' \in I_{k'}$  and  $v = 1, \dots, N(k, \tau)$ ,  $v' = 1, \dots, N(k', \tau')$ ,

$$\begin{aligned} & |D_k \tilde{D}_{k'}(y_\tau^{k,v}, y_{\tau'}^{k',v'})| \\ & \leq C 2^{-|k-k'|\epsilon} \frac{1}{V_{2^{-(k \wedge k')}}(y_\tau^{k,v}) + V(y_\tau^{k,v}, y_{\tau'}^{k',v'})} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(y_\tau^{k,v}, y_{\tau'}^{k',v'}))^\epsilon}. \end{aligned} \quad (2.5)$$

By the discrete Calderón reproducing formula (2.1) in Lemma 2.2, we can write

$$\begin{aligned} |D_k(f)(y_{\tau}^{k,v})| &\leq \sum_{k' \in \mathbb{Z}} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',v'}) |D_k \tilde{D}_{k'}(y_{\tau}^{k,v}, y_{\tau'}^{k',v'}) D_{k'}(f)(y_{\tau'}^{k',v'})| \\ &\leq C \sum_{k' \in \mathbb{Z}} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',v'}) |D_{k'}(f)(y_{\tau'}^{k',v'})| 2^{-|k-k'|\epsilon} \\ &\quad \times \frac{1}{V_{2-(k \wedge k')} (y_{\tau}^{k,v}) + V(y_{\tau}^{k,v}, y_{\tau'}^{k',v'})} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k,v}, y_{\tau'}^{k',v'}))^\epsilon}. \end{aligned} \quad (2.6)$$

Combining Lemma 2.3 and (2.6), we obtain

$$\begin{aligned} \|f\|_{\dot{B}_{p_1}^{s_1,q}(X)} &\lesssim \left\{ \sum_{k \in \mathbb{Z}} 2^{ks_1q} \left[ \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \left( \mu(Q_{\tau}^{k,v})^{1/p_1} \sum_{k' \in \mathbb{Z}} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',v'}) |D_{k'}(f)(y_{\tau'}^{k',v'})| \right. \right. \\ &\quad \times \left. \left. 2^{-|k-k'|\epsilon} \frac{1}{V_{2-(k \wedge k')} (y_{\tau}^{k,v}) + V(y_{\tau}^{k,v}, y_{\tau'}^{k',v'})} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k,v}, y_{\tau'}^{k',v'}))^\epsilon} \right)^{p_1} \right]^{q/p_1} \right\}^{1/q}. \end{aligned} \quad (2.7)$$

We need to consider two cases.

Case 1:  $p_1 > 1$ .

We choose  $\epsilon_1 > 0$  and  $\epsilon_2 > 0$  such that  $\epsilon = \epsilon_1 + \epsilon_2$ ,  $\epsilon_1$  can be taken arbitrarily close to  $\epsilon$ , and using the Hölder inequality, we get

$$\begin{aligned} &\sum_{k' \in \mathbb{Z}} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',v'}) |D_{k'}(f)(y_{\tau'}^{k',v'})| \\ &\quad \times 2^{-|k-k'|\epsilon} \frac{1}{V_{2-(k \wedge k')} (y_{\tau}^{k,v}) + V(y_{\tau}^{k,v}, y_{\tau'}^{k',v'})} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k,v}, y_{\tau'}^{k',v'}))^\epsilon} \\ &\lesssim \left( \sum_{k' \in \mathbb{Z}} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',v'}) |D_{k'}(f)(y_{\tau'}^{k',v'})|^{p_1} 2^{-|k-k'|p_1\epsilon_1} \frac{1}{V_{2-(k \wedge k')} (y_{\tau}^{k,v}) + V(y_{\tau}^{k,v}, y_{\tau'}^{k',v'})} \right. \\ &\quad \times \left. \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k,v}, y_{\tau'}^{k',v'}))^\epsilon} \right)^{1/p_1} \left( \sum_{k' \in \mathbb{Z}} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',v'}) \right. \\ &\quad \times \left. \frac{2^{-|k-k'|p_1\epsilon_2}}{V_{2-(k \wedge k')} (y_{\tau}^{k,v}) + V(y_{\tau}^{k,v}, y_{\tau'}^{k',v'})} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k,v}, y_{\tau'}^{k',v'}))^\epsilon} \right)^{1/p_1'} \\ &\lesssim \left( \sum_{k' \in \mathbb{Z}} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',v'}) |D_{k'}(f)(y_{\tau'}^{k',v'})|^{p_2} \right. \\ &\quad \times \left. 2^{-|k-k'|p_1\epsilon_1} \frac{1}{V_{2-(k \wedge k')} (y_{\tau}^{k,v}) + V(y_{\tau}^{k,v}, y_{\tau'}^{k',v'})} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k,v}, y_{\tau'}^{k',v'}))^\epsilon} \right)^{1/p_1}, \end{aligned}$$

where, since Christ's construction in [3], the dyadic cubes on spaces of homogeneous type are disjoint, the last inequalities follow from the facts

$$\begin{aligned} & \frac{1}{V_{2^{-(k \wedge k')}}(y_{\tau}^{k,v}) + V(y_{\tau}^{k,v}, y_{\tau'}^{k',v'})} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k,v}, y_{\tau'}^{k',v'}))^\epsilon} \\ & \sim \frac{1}{V_{2^{-(k \wedge k')}}(y_{\tau}^{k,v}) + V(y_{\tau}^{k,v}, y)} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k,v}, y))^\epsilon} \end{aligned} \quad (2.8)$$

for any  $y \in Q_{\tau'}^{k',v'}$  and

$$\begin{aligned} & \sum_{k' \in \mathbb{Z}} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',v'}) \frac{2^{-|k-k'|p_1\epsilon_2}}{V_{2^{-(k \wedge k')}}(y_{\tau}^{k,v}) + V(y_{\tau}^{k,v}, y_{\tau'}^{k',v'})} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k,v}, y_{\tau'}^{k',v'}))^\epsilon} \\ & \lesssim \sum_{k' \in \mathbb{Z}} 2^{-|k-k'|p_1\epsilon_2} \int_X \frac{1}{V_{2^{-(k \wedge k')}}(y_{\tau}^{k,v}) + V(y_{\tau}^{k,v}, y)} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k,v}, y))^\epsilon} d\mu(y) \\ & \leq C. \end{aligned}$$

From this, it follows that

$$\begin{aligned} & \|f\|_{\dot{B}_{p_1}^{s_1,q}(X)} \\ & \lesssim \left\{ \sum_{k \in \mathbb{Z}} 2^{ks_1q} \left[ \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \mu(Q_{\tau}^{k,v}) \sum_{k' \in \mathbb{Z}} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',v'}) |D_{k'}(f)(y_{\tau}^{k,v})|^{p_1} \right. \right. \\ & \quad \times \left. \left. 2^{-|k-k'|p_1\epsilon_1} \frac{1}{V_{2^{-(k \wedge k')}}(y_{\tau}^{k,v}) + V(y_{\tau}^{k,v}, y_{\tau'}^{k',v'})} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k,v}, y_{\tau'}^{k',v'}))^\epsilon} \right]^{q/p_1} \right\}^{1/q} \\ & \lesssim \left\{ \sum_{k \in \mathbb{Z}} 2^{ks_1q} \left[ \sum_{k' \in \mathbb{Z}} 2^{-|k-k'|p_1\epsilon_1} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',v'}) |D_{k'}(f)(y_{\tau'}^{k',v'})|^{p_1} \right]^{q/p_1} \right\}^{1/q}. \end{aligned}$$

Applying the  $p_2/p_1$ -inequality for  $\frac{p_2}{p_1} \leq 1$

$$\left( \sum_{j \in \mathbb{Z}} |a_j| \right)^{\frac{p_2}{p_1}} \leq \sum_{j \in \mathbb{Z}} |a_j|^{\frac{p_2}{p_1}} \quad (2.9)$$

with  $a_j \in \mathbb{C}$  for all  $j \in \mathbb{Z}$  implies that the last term above is dominated by

$$\left\{ \sum_{k \in \mathbb{Z}} \left[ \sum_{k' \in \mathbb{Z}} 2^{-|k-k'|p_2\epsilon_1} 2^{ks_1p_2} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',v'})^{p_2/p_1} |D_{k'}(f)(y_{\tau'}^{k',v'})|^{p_2} \right]^{q/p_2} \right\}^{1/q}.$$

Applying the density condition (1.7), it immediately follows that  $\mu(Q_{\tau}^{k',v'}) \geq C2^{-k'\omega}$  for any  $k' \in \mathbb{Z}$ ,  $\tau' \in I_{k'}$  and hence we have  $\mu(Q_{\tau'}^{k',v'})^{p_2/p_1} = \mu(Q_{\tau'}^{k',v'})^{p_2/p_1-1} \mu(Q_{\tau'}^{k',v'}) \leq C2^{-k'(s_1-s_2)} \mu(Q_{\tau'}^{k',v'})$ , where we use the facts that  $p_2/p_1 < 1$  and  $s_1 - \omega/p_1 = s_2 - \omega/p_2$ . We obtain



$$\|f\|_{\dot{B}_{p_1}^{s_1,q}(X)} \lesssim \left\{ \sum_{k \in \mathbb{Z}} \left[ \sum_{k' \in \mathbb{Z}} 2^{(k-k')s_1 p_2} 2^{-|k-k'|p_2 \epsilon_1} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} 2^{k' s_2 p_2} \right. \right. \\ \left. \left. \times \mu(Q_{\tau'}^{k',v'}) |D_{k'}(f)(y_{\tau'}^{k',v'})|^{p_2} \right]^{q/p_2} \right\}^{1/q}.$$

Now we choose  $s_1 \in (-\epsilon_1, \epsilon_1)$ , applying the Hölder inequality for  $q/p_2 > 1$  and the  $q/p_2$ -inequality for  $q/p_2 \leq 1$  implies that the last term above is dominated by  $C\|f\|_{\dot{B}_{p_2}^{s_2,q}(X)}$ , which implies (1.8) for the case where  $p_1 > 1$ .

Case 2:  $p_1 \leq 1$ .

From (2.7) and the  $p_1$ -inequality and the  $p_2/p_1$ -inequality in (2.9), we deduce that

$$\|f\|_{\dot{B}_{p_1}^{s_1,q}(X)} \lesssim \left\{ \sum_{k \in \mathbb{Z}} 2^{ks_1 q} \left[ \sum_{k' \in \mathbb{Z}} 2^{-|k-k'| \epsilon p_1} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',v'})^{p_1} |D_{k'}(f)(y_{\tau'}^{k',v'})|^{p_1} \right. \right. \\ \left. \left. \times \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \mu(Q_{\tau}^{k,v}) \right. \right. \\ \left. \left. \times \left( \frac{1}{V_{2^{-(k \wedge k')}}(y_{\tau}^{k,v}) + V(y_{\tau}^{k,v}, y_{\tau'}^{k',v'})} \frac{2^{-(k \wedge k') \epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k,v}, y_{\tau'}^{k',v'}))^\epsilon} \right)^{p_1} \right]^{q/p_1} \right\}^{1/q} \\ \lesssim \left\{ \sum_{k \in \mathbb{Z}} 2^{ks_1 q} \left[ \sum_{k' \in \mathbb{Z}} 2^{-|k-k'| \epsilon p_1} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} V_{2^{-(k \wedge k')}}(y_{\tau'}^{k',v'})^{1-p_1} \right. \right. \\ \left. \left. \times \mu(Q_{\tau'}^{k',v'})^{p_1} |D_{k'}(f)(y_{\tau'}^{k',v'})|^{p_1} \right]^{q/p_1} \right\}^{1/q} \\ \lesssim \left\{ \sum_{k \in \mathbb{Z}} 2^{ks_1 q} \left[ \sum_{k' \in \mathbb{Z}} 2^{-|k-k'| \epsilon p_2} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} V_{2^{-(k \wedge k')}}(y_{\tau'}^{k',v'})^{p_2/p_1-1} \right. \right. \\ \left. \left. \times V_{2^{-(k \wedge k')}}(y_{\tau'}^{k',v'})^{1-p_2} \mu(Q_{\tau'}^{k',v'})^{p_2-1} \mu(Q_{\tau'}^{k',v'}) |D_{k'}(f)(y_{\tau'}^{k',v'})|^{p_2} \right]^{q/p_2} \right\}^{1/q},$$

where we used the fact that

$$\sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \mu(Q_{\tau}^{k,v}) \left( \frac{1}{V_{2^{-(k \wedge k')}}(y_{\tau}^{k,v}) + V(y_{\tau}^{k,v}, y_{\tau'}^{k',v'})} \frac{2^{-(k \wedge k') \epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k,v}, y_{\tau'}^{k',v'}))^\epsilon} \right)^{p_1} \\ \lesssim V_{2^{-(k \wedge k')}}(y_{\tau'}^{k',v'})^{1-p_1}.$$

Note that by the doubling property (1.2),

$$V_{2^{-(k \wedge k')}}(y_{\tau'}^{k',v'}) \lesssim \mu(B(y_{\tau'}^{k',v'}, 2^{-(k \wedge k')})) = \mu(B(y_{\tau'}^{k',v'}, 2^{k-(k \wedge k')} 2^{-k})) \\ \lesssim 2^{[k'-(k \wedge k')] \omega} \mu(B(y_{\tau'}^{k',v'}, 2^{-k})) \lesssim 2^{[k'-(k \wedge k')] \omega} \mu(Q_{\tau'}^{k',v'})$$

and thus

$$V_{2^{-(k \wedge k')}}(y_{\tau'}^{k',v'})^{1-p_2} \mu(Q_{\tau'}^{k',v'})^{p_2-1} \lesssim 2^{[k'-(k \wedge k')] \omega(1-p_2)}$$

for  $p_2 \leq 1$ . The density condition (1.7) implies that

$$V_{2^{-(k \wedge k')}}(y_{\tau'}^{k',v'})^{p_2/p_1-1} \lesssim 2^{-(k \wedge k')\omega p_2(\frac{1}{p_1}-\frac{1}{p_2})} \lesssim 2^{-(k \wedge k')p_2(s_1-s_2)}$$

for  $p_2/p_1 < 1$ . Therefore, we further obtain

$$\begin{aligned} \|f\|_{\dot{B}_{p_1}^{s_1,q}(X)} &\lesssim \left\{ \sum_{k \in \mathbb{Z}} \left[ \sum_{k' \in \mathbb{Z}} 2^{-|k-k'| \in p_2} 2^{ks_1 p_2} 2^{-k's_2 p_2} 2^{[k'-(k \wedge k')]\omega(1-p_2)} 2^{-(k \wedge k')p_2(s_1-s_2)} \right. \right. \\ &\quad \times \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} 2^{k's_2 p_2} \mu(Q_{\tau'}^{k',v'}) |D_{k'}(f)(y_{\tau'}^{k',v'})|^{p_2} \left. \right]^{q/p_2} \Big\}^{1/q}. \end{aligned}$$

Applying the Hölder inequality for  $q/p_2 > 1$  and the  $q/p_2$ -inequality for  $q/p_2 \leq 1$  implies that the last term above is dominated by  $C\|f\|_{\dot{B}_{p_2}^{s_2,q}(X)}$  when  $1 \geq p_1 > p_2$ , where the last inequality follows from the facts if  $s_1 < \epsilon$  and  $p_2 > \frac{\omega}{\omega+s_2+\epsilon}$ , then

$$\sum_{k \in \mathbb{Z}} [2^{-|k-k'| \in p_2} 2^{ks_1 p_2} 2^{-k's_2 p_2} 2^{[k'-(k \wedge k')]\omega(1-p_2)} 2^{-(k \wedge k')p_2(s_1-s_2)}]^{q/p_2} \wedge 1 \leq C$$

and

$$\sum_{k' \in \mathbb{Z}} [2^{-|k-k'| \in p_2} 2^{ks_1 p_2} 2^{-k's_2 p_2} 2^{[k'-(k \wedge k')]\omega(1-p_2)} 2^{-(k \wedge k')p_2(s_1-s_2)}]^{q/p_2} \wedge 1 \leq C.$$

This completes the proof of (1.8).

We now show (1.9). By the homogeneity of the norm  $\|\cdot\|_{\dot{F}_{p_2}^{s_2,q_2}(X)}$ , we may assume  $\|f\|_{\dot{F}_{p_2}^{s_2,q_2}(X)} = 1$  without loss of generality. By Lemma 2.3 and the estimate in (2.5), we have

$$\begin{aligned} \|f\|_{\dot{F}_{p_1}^{s_1,q_1}(X)} &\lesssim \left\| \left\{ \sum_{k \in \mathbb{Z}} 2^{ks_1 q_1} \left( \sum_{k' \in \mathbb{Z}} 2^{-|k-k'| \in \epsilon} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',v'}) |D_{k'}(f)(y_{\tau'}^{k',v'})| \right. \right. \right. \\ &\quad \times \frac{1}{V_{2^{-(k \wedge k')}}(y_{\tau}^{k,v}) + V(y_{\tau}^{k,v}, y_{\tau'}^{k',v'})} \\ &\quad \times \left. \left. \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k,v}, y_{\tau'}^{k',v'}))^\epsilon} \chi_{Q_{\tau}^{k,v}} \right) \right\}^{q_1/q_1} \right\|_{L^{p_1}(X)}. \end{aligned} \quad (2.10)$$

To estimate the last expression in (2.10), we claim that for  $\max\{\frac{\omega}{\omega+\epsilon}, \frac{\omega}{\omega+\epsilon+s_2}\} < r \leq 1$ ,

$$\begin{aligned} &\sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',v'}) |D_{k'}(f)(y_{\tau'}^{k',v'})| \\ &\quad \times \frac{1}{V_{2^{-(k \wedge k')}}(y_{\tau}^{k,v}) + V(y_{\tau}^{k,v}, y_{\tau'}^{k',v'})} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k,v}, y_{\tau'}^{k',v'}))^\epsilon} \chi_{Q_{\tau}^{k,v}}(x) \\ &\leq C 2^{-k'\omega(1-\frac{1}{r})} \mu(B)^{\frac{1}{r}-1} \inf_{y \in B} \left\{ M \left( \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} |D_{k'}(f)(y_{\tau'}^{k',v'})| \chi_{Q_{\tau'}^{k',v'}}(y) \right)^r \right\}^{\frac{1}{r}}, \end{aligned} \quad (2.11)$$

where  $B = B(x, 2^{-(k \wedge k')})$  and  $M$  is the Hardy-Littlewood maximal function. To prove (2.11), from (2.8), it follows that

$$\begin{aligned} & \sum_{\tau \in I_k} \sum_{v=1}^{N(k, \tau)} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k', \tau')} \mu(Q_{\tau'}^{k', v'}) |D_{k'}(f)(y_{\tau'}^{k', v'})| \\ & \quad \times \frac{1}{V_{2^{-(k \wedge k')}}(y_{\tau}^{k, v}) + V(y_{\tau}^{k, v}, y_{\tau'}^{k', v'})} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k, v}, y_{\tau'}^{k', v'}))^\epsilon} \chi_{Q_{\tau}^{k, v}}(x) \\ & \lesssim \sum_{\tau \in I_k} \sum_{v=1}^{N(k, \tau)} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k', \tau')} \mu(Q_{\tau'}^{k', v'}) |D_{k'}(f)(y_{\tau'}^{k', v'})| \\ & \quad \times \frac{1}{V_{2^{-(k \wedge k')}}(x) + V(x, y_{\tau'}^{k', v'})} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(x, y_{\tau'}^{k', v'}))^\epsilon} \chi_{Q_{\tau}^{k, v}}(x). \end{aligned}$$

By Christ's construction in [3], the dyadic cubes on spaces of homogeneous type are disjoint, and therefore

$$\sum_{\tau \in I_k} \sum_{v=1}^{N(k, \tau)} \chi_{Q_{\tau}^{k, v}}(x) = 1,$$

which implies that the last term above is dominated by

$$\begin{aligned} & \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k', \tau')} \mu(Q_{\tau'}^{k', v'}) |D_{k'}(f)(y_{\tau'}^{k', v'})| \frac{1}{V_{2^{-(k \wedge k')}}(x) + V(x, y_{\tau'}^{k', v'})} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(x, y_{\tau'}^{k', v'}))^\epsilon} \\ & \lesssim \left\{ \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k', \tau')} \mu(Q_{\tau'}^{k', v'})^r |D_{k'}(f)(y_{\tau'}^{k', v'})|^r \right. \\ & \quad \times \left[ \frac{1}{V_{2^{-(k \wedge k')}}(x) + V(x, y_{\tau'}^{k', v'})} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(x, y_{\tau'}^{k', v'}))^\epsilon} \right]^r \Bigg\}^{\frac{1}{r}} \\ & = \left\{ \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k', \tau')} \mu(Q_{\tau'}^{k', v'})^{r-1} |D_{k'}(f)(y_{\tau'}^{k', v'})|^r \right. \\ & \quad \times \int_X \left[ \frac{1}{V_{2^{-(k \wedge k')}}(x) + V(x, y_{\tau'}^{k', v'})} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(x, y_{\tau'}^{k', v'}))^\epsilon} \right]^r \chi_{Q_{\tau'}^{k', v'}}(y) d\mu(y) \Bigg\}^{\frac{1}{r}}. \end{aligned}$$

Note that the density condition (1.7) implies  $C2^{-k'\omega} \leq \mu(Q_{\tau'}^{k', v'})$ . The fact that  $r-1 \leq 0$  yields

$$\begin{aligned} & \left\{ \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k', \tau')} \mu(Q_{\tau'}^{k', v'})^{r-1} |D_{k'}(f)(y_{\tau'}^{k', v'})|^r \right. \\ & \quad \times \int_X \left[ \frac{1}{V_{2^{-(k \wedge k')}}(x) + V(x, y_{\tau'}^{k', v'})} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(x, y_{\tau'}^{k', v'}))^\epsilon} \right]^r \chi_{Q_{\tau'}^{k', v'}}(y) d\mu(y) \Bigg\}^{\frac{1}{r}} \end{aligned}$$

$$\begin{aligned}
&\lesssim 2^{-k'\omega(1-\frac{1}{r})} \left\{ \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} |D_{k'}(f)(y_{\tau'}^{k',v'})|^r \right. \\
&\quad \times \left. \int_X \left[ \frac{1}{V_{2^{-(k \wedge k')} } (y_{\tau'}^{k',v'}) + V(x,y)} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(x,y))^\epsilon} \right]^r \chi_{Q_{\tau'}^{k',v'}}(y) d\mu(y) \right\}^{\frac{1}{r}} \\
&\lesssim 2^{-k'\omega(1-\frac{1}{r})} \left\{ \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} \int_B |D_{k'}(f)(y_{\tau'}^{k',v'})|^r \right. \\
&\quad \times \left. \left[ \frac{1}{V_{2^{-(k \wedge k')} } (x) + V(x,y)} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(x,y))^\epsilon} \right]^r \chi_{Q_{\tau'}^{k',v'}}(y) d\mu(y) \right\}^{\frac{1}{r}} \\
&\quad + 2^{-k'\omega(1-\frac{1}{r})} \left\{ \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} |D_{k'}(f)(y_{\tau'}^{k',v'})|^r \right. \\
&\quad \times \left. \sum_{m=0}^{\infty} \int_{[2^{(m+1)B}] \setminus [2^m B]} \left[ \frac{1}{V_{2^{-(k \wedge k')} } (x) + V(x,y)} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(x,y))^\epsilon} \right]^r \chi_{Q_{\tau'}^{k',v'}}(y) d\mu(y) \right\}^{\frac{1}{r}} \\
&=: H_1 + H_2.
\end{aligned}$$

We first estimate the term of  $H_1$ , for any  $x' \in B$ , we then further have

$$\begin{aligned}
H_1 &\lesssim 2^{-k'\omega(1-\frac{1}{r})} \left\{ \mu(B)^{1-r} \frac{1}{\mu(B)} \int_B \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} |D_{k'}(f)(y_{\tau'}^{k',v'})|^r \chi_{Q_{\tau'}^{k',v'}}(y) d\mu(y) \right\}^{\frac{1}{r}} \\
&\lesssim 2^{-k'\omega(1-\frac{1}{r})} \mu(B)^{\frac{1}{r}-1} \left\{ M \left( \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} |D_{k'}(f)(y_{\tau'}^{k',v'})|^r \chi_{Q_{\tau'}^{k',v'}} \right)^r (x') \right\}^{\frac{1}{r}}.
\end{aligned}$$

Now we estimate  $H_2$ . For any  $x' \in B$ , it also follows that

$$\begin{aligned}
&\sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} \sum_{m=0}^{\infty} \int_{[2^{(m+1)B}] \setminus [2^m B]} |D_{k'}(f)(y_{\tau'}^{k',v'})|^r \\
&\quad \times \left[ \frac{1}{V_{2^{-(k \wedge k')} } (x) + V(x,y)} \frac{2^{-(k \wedge k')\epsilon}}{(2^{-(k \wedge k')} + d(x,y))^\epsilon} \right]^r \chi_{Q_{\tau'}^{k',v'}}(y) d\mu(y) \\
&\lesssim \mu(2^m B)^{-r} 2^{-m\epsilon r} \int_{2^{(m+1)B}} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} |D_{k'}(f)(y_{\tau'}^{k',v'})|^r \chi_{Q_{\tau'}^{k',v'}}(y) d\mu(y) \\
&\lesssim \mu(2^m B)^{-r} \mu(2^{m+1} B) 2^{-(m-1)\epsilon r} M \left( \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} |D_{k'}(f)(y_{\tau'}^{k',v'})|^r \chi_{Q_{\tau'}^{k',v'}} \right)^r (x') \\
&\lesssim \mu(B)^{1-r} 2^{-m[\epsilon r - \omega(1-r)]} M \left( \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k',\tau')} |D_{k'}(f)(y_{\tau'}^{k',v'})|^r \chi_{Q_{\tau'}^{k',v'}} \right)^r (x'),
\end{aligned}$$

where we use the doubling condition (1.2) which implies that  $\mu(2^{m+1}B)^{1-r} \leq C 2^{m\omega(1-r)} \times \mu(B)^{1-r}$ . Thus, for  $r > \frac{\omega}{\omega+\gamma}$ , the inequality above implies that

$$H_2 \leq C 2^{-k' \omega(1-\frac{1}{r})} \mu(B)^{\frac{1}{r}-1} \left\{ M \left( \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k', \tau')} |D_{k'}(f)(y_{\tau'}^{k', v'})| \chi_{Q_{\tau'}^{k', v'}} \right)^r (x') \right\}^{\frac{1}{r}}.$$

Combining the estimates of  $H_1$ ,  $H_2$  and the arbitrariness of  $x' \in B$  gives the proof of claim (2.11).

Applying inequality (2.11) yields

$$\begin{aligned} & \sum_{\tau \in I_k} \sum_{v=1}^{N(k, \tau)} |D_k(f)(y_{\tau}^{k, v})| \chi_{Q_{\tau}^{k, v}}(x) \\ & \lesssim \sum_{k' \in \mathbb{Z}} 2^{-|k-k'| \epsilon} \sum_{\tau \in I_k} \sum_{v=1}^{N(k, \tau)} \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k', \tau')} \mu(Q_{\tau'}^{k', v'}) |D_{k'}(f)(y_{\tau'}^{k', v'})| \\ & \quad \times \frac{1}{V_{2^{-(k \wedge k')} } (y_{\tau}^{k, v}) + V(y_{\tau}^{k, v}, y_{\tau'}^{k', v'})} \frac{2^{-(k \wedge k') \epsilon}}{(2^{-(k \wedge k')} + d(y_{\tau}^{k, v}, y_{\tau'}^{k', v'}))^\epsilon} \chi_{Q_{\tau}^{k, v}} \\ & \lesssim \sum_{k' \in \mathbb{Z}} 2^{-|k-k'| \epsilon} 2^{-k' \omega(1-\frac{1}{r})} \mu(B)^{\frac{1}{r}-1} \inf_{y \in B} \left\{ M \left( \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k', \tau')} |D_{k'}(f)(y_{\tau'}^{k', v'})| \chi_{Q_{\tau'}^{k', v'}} \right)^r (y) \right\}^{\frac{1}{r}} \\ & \lesssim \sum_{k' \in \mathbb{Z}} 2^{-k' s_2 - |k-k'| \epsilon - k' \omega(1-\frac{1}{r})} \\ & \quad \times \mu(B)^{\frac{1}{r}-1} \inf_{y \in B} \left\{ M \left( \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k', \tau')} 2^{k' s_2} |D_{k'}(f)(y_{\tau'}^{k', v'})| \chi_{Q_{\tau'}^{k', v'}} \right)^r (y) \right\}^{\frac{1}{r}} \end{aligned}$$

for  $\frac{\omega}{\omega+\gamma} < r \leq 1$ . Choose  $r$  satisfying  $\frac{\omega}{\omega+\gamma} < r < \max\{p_2, q_2\}$  and  $r \leq 1$ , and denote

$$F_{k'}(y) = \left\{ M \left( \sum_{\tau' \in I_{k'}} \sum_{v'=1}^{N(k', \tau')} 2^{k' s_2} |D_{k'}(f)(y_{\tau'}^{k', v'})| \chi_{Q_{\tau'}^{k', v'}} \right)^r (y) \right\}^{\frac{1}{r}}.$$

We now have

$$\begin{aligned} & \left\{ \sum_{k=-\infty}^N \sum_{\tau \in I_k} \sum_{v=1}^{N(k, \tau)} 2^{k s_1 q_1} (|D_k(f)(y_{\tau}^{k, v})| \chi_{Q_{\tau}^{k, v}}(x))^{q_1} \right\}^{1/q_1} \\ & \lesssim \left\{ \sum_{k=-\infty}^N \left( \sum_{k' \in \mathbb{Z}} 2^{k s_1 - k' s_2 - |k-k'| \epsilon - k' \omega(1-\frac{1}{r})} \mu(B)^{\frac{1}{r}-1} \inf_{y \in B} F_{k'}(y) \right)^{q_1} \right\}^{1/q_1}. \end{aligned}$$

The Fefferman-Stein vector-valued maximal function inequality [21] and the arbitrariness of  $y_{\tau'}^{k', v'}$  further yield that

$$\begin{aligned} \inf_{y \in B} F_{k'}(y) & \leq C \inf_{y \in B} \left\{ \sum_{k' \in \mathbb{Z}} (F_{k'}(y))^{q_2} \right\}^{1/q_2} \\ & \leq C \left\{ \mu(B)^{-1} \int_B \left( \sum_{k' \in \mathbb{Z}} (F_{k'}(x))^{q_2} \right)^{p_2/q_2} d\mu(x) \right\}^{1/p_2} \end{aligned}$$

$$\begin{aligned}
 &\leq C\mu(B)^{-1/p_2} \left\| \left( \sum_{k' \in \mathbb{Z}} (F_{k'}(y))^{q_2} \right)^{1/q_2} \right\|_{p_2} \\
 &\leq C\mu(B)^{-1/p_2} \left\| \left\{ \sum_{k' \in \mathbb{Z}} \left( \sum_{\tau' \in I_{k'}}^{N(k', \tau')} \sum_{v'=1}^{N(k', \tau')} 2^{k's_2} |D_{k'}(f)(y_{\tau'}^{k', v'})| \chi_{Q_{\tau'}^{k', v'}} \right)^{q_2} \right\}^{1/q_2} \right\|_{p_2} \\
 &\leq C\mu(B)^{-1/p_2},
 \end{aligned}$$

which implies that

$$\begin{aligned}
 &\left\{ \sum_{k=-\infty}^N \sum_{\tau \in I_k} \sum_{v=1}^{N(k, \tau)} 2^{ks_1 q_1} (|D_k(f)(y_{\tau}^{k, v})| \chi_{Q_{\tau}^{k, v}}(x))^{q_1} \right\}^{1/q_1} \\
 &\leq C \left\{ \sum_{k=-\infty}^N \left( \sum_{k' \in \mathbb{Z}} 2^{ks_1 - k's_2 - |k-k'| \epsilon - k' \omega(1 - \frac{1}{r})} \mu(B)^{\frac{1}{r} - 1} \mu(B)^{-1/p_2} \right)^{q_1} \right\}^{1/q_1}.
 \end{aligned}$$

Since  $s_2 - s_1 = \frac{\omega}{p_2} - \frac{\omega}{p_1} > 0$ , so  $p_1 > p_2 > \frac{p_2}{1+p_2}$ . The key point here is to choose  $r = \frac{p_2}{1+p_2}$  so that  $\mu(B)^{\frac{1}{r} - 1} \mu(B)^{-1/p_2} = 1$ . Note that  $r < p_2$  and  $r < 1$ . We assume  $q_2 > r = \frac{p_2}{1+p_2}$  for the moment and this assumption will be removed at the end of the proof. This yields

$$\begin{aligned}
 &\left\{ \sum_{k=-\infty}^N \sum_{\tau \in I_k} \sum_{v=1}^{N(k, \tau)} 2^{ks_1 q_1} (|D_k(f)(y_{\tau}^{k, v})| \chi_{Q_{\tau}^{k, v}}(x))^{q_1} \right\}^{1/q_1} \\
 &\lesssim \left\{ \sum_{k=-\infty}^N \left( \sum_{k' \in \mathbb{Z}} 2^{ks_1 - k's_2 - |k-k'| \epsilon - k' \omega(1 - \frac{1}{r})} \right)^{q_1} \right\}^{1/q_1} \\
 &\lesssim \left\{ \sum_{k=-\infty}^N \left( \sum_{k' \in \mathbb{Z}} 2^{-|k-k'| \epsilon} 2^{k' \frac{\omega}{p_1}} 2^{(k-k')s_1} \right)^{q_1} \right\}^{1/q_1} \\
 &\leq C \left\{ \sum_{k=-\infty}^N \left( \sum_{k'=-\infty}^k 2^{-(k-k') \epsilon} 2^{k' \frac{\omega}{p_1}} 2^{(k-k')s_1} \right)^{q_1} \right\}^{1/q_1} \\
 &\quad + C \left\{ \sum_{k=-\infty}^N \left( \sum_{k'=k}^{\infty} 2^{-(k'-k) \epsilon} 2^{k' \frac{\omega}{p_1}} 2^{(k-k')s_1} \right)^{q_1} \right\}^{1/q_1} \\
 &\lesssim \left\{ \sum_{k=-\infty}^N \left( 2^{k \frac{\omega}{p_1}} \right)^{q_1} \right\}^{1/q_1} \\
 &\leq C 2^{\frac{N\omega}{p_1}}, \tag{2.12}
 \end{aligned}$$

where  $\epsilon$  can be taken arbitrarily close to  $\theta$  and satisfies  $-\epsilon < s_1 - \frac{\omega}{p_1} < \epsilon$ .

On the other hand, if  $\frac{q_2}{q_1} \leq 1$ , since  $s_2 - s_1 = \frac{\omega}{p_2} - \frac{\omega}{p_1} > 0$ , applying the  $q_2/q_1$ -inequality, we have

$$\begin{aligned}
 &\left\{ \sum_{k=N+1}^{\infty} \sum_{\tau \in I_k} \sum_{v=1}^{N(k, \tau)} 2^{ks_1 q_1} (|D_k(f)(y_{\tau}^{k, v})| \chi_{Q_{\tau}^{k, v}}(x))^{q_1} \right\}^{1/q_1} \\
 &= \left\{ \sum_{k=N+1}^{\infty} 2^{k(s_1 - s_2) q_1} \sum_{\tau \in I_k} \sum_{v=1}^{N(k, \tau)} 2^{ks_2 q_1} (|D_k(f)(y_{\tau}^{k, v})| \chi_{Q_{\tau}^{k, v}}(x))^{q_1} \right\}^{1/q_1}
 \end{aligned}$$

$$\begin{aligned} &\lesssim \left\{ \sum_{k=N+1}^{\infty} 2^{k(s_1-s_2)q_2} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_2q_2} (|D_k(f)(y_{\tau}^{k,v})| \chi_{Q_{\tau}^{k,v}}(x))^{q_2} \right\}^{1/q_2} \\ &\lesssim 2^{N(\frac{\omega}{p_1} - \frac{\omega}{p_2})} \left\{ \sum_{k \in \mathbb{Z}} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_2q_2} (|D_k(f)(y_{\tau}^{k,v})| \chi_{Q_{\tau}^{k,v}}(x))^{q_2} \right\}^{1/q_2}. \end{aligned} \quad (2.13)$$

If  $\delta := \frac{q_2}{q_1} > 1$ , using the Hölder inequality also yields that

$$\begin{aligned} &\left\{ \sum_{k=N+1}^{\infty} 2^{k(s_1-s_2)q_1} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_2q_1} (|D_k(f)(y_{\tau}^{k,v})| \chi_{Q_{\tau}^{k,v}}(x))^{q_1} \right\}^{1/q_1} \\ &\lesssim \left\{ \left[ \sum_{k=N+1}^{\infty} 2^{k(s_1-s_2)q_1} \right]^{1/\delta'} \right. \\ &\quad \times \left. \left[ \sum_{k=N+1}^{\infty} 2^{k(s_1-s_2)q_1} \left\{ \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_2q_1} (|D_k(f)(y_{\tau}^{k,v})| \chi_{Q_{\tau}^{k,v}}(x))^{q_1} \right\}^{\delta} \right]^{1/\delta} \right\}^{1/q_1} \\ &\lesssim \left\{ \left[ \sum_{k=N+1}^{\infty} 2^{k(s_1-s_2)q_1} \right]^{1/\delta'} \right. \\ &\quad \times \left. \left[ \sum_{k=N+1}^{\infty} 2^{k(s_1-s_2)q_1} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_2q_2} (|D_k(f)(y_{\tau}^{k,v})| \chi_{Q_{\tau}^{k,v}}(x))^{q_2} \right]^{1/\delta} \right\}^{1/q_1} \\ &\lesssim 2^{N(\frac{\omega}{p_1} - \frac{\omega}{p_2})} \left\{ \sum_{k \in \mathbb{Z}} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_2q_2} (|D_k(f)(y_{\tau}^{k,v})| \chi_{Q_{\tau}^{k,v}}(x))^{q_2} \right\}^{1/q_2}. \end{aligned} \quad (2.14)$$

Combining the estimates in (2.13) and (2.14), we obtain

$$\begin{aligned} &\left\{ \sum_{k=N+1}^{\infty} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_1q_1} (|D_k(f)(y_{\tau}^{k,v})| \chi_{Q_{\tau}^{k,v}}(x))^{q_1} \right\}^{1/q_1} \\ &\leq C 2^{N(\frac{\omega}{p_1} - \frac{\omega}{p_2})} \left\{ \sum_{k \in \mathbb{Z}} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_2q_2} (|D_k(f)(y_{\tau}^{k,v})| \chi_{Q_{\tau}^{k,v}}(x))^{q_2} \right\}^{1/q_2}. \end{aligned} \quad (2.15)$$

Applying (2.12) and (2.15) yields

$$\begin{aligned} &\|f\|_{\dot{F}_{p_1}^{s_1, q_1}(X)}^{p_1} \\ &= p_1 \int_0^{\infty} t^{p_1-1} \mu \left\{ x : \left\{ \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_1q_1} (|D_k(f)(y_{\tau}^{k,v})| \chi_{Q_{\tau}^{k,v}}(x))^{q_1} \right\}^{1/q_1} > t \right\} dt \\ &= p_1 \sum_{N=-\infty}^{\infty} \int_{2^{C_2\omega N/p_1+1/q_1}}^{2^{C_2\omega(N+1)/p_1+1/q_1}} t^{p_1-1} \\ &\quad \times \mu \left\{ x : \left\{ \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_1q_1} |D_k(f)(y_{\tau}^{k,v})|^{q_1} \chi_{Q_{\tau}^{k,v}}(x) \right\}^{1/q_1} > t \right\} dt \end{aligned}$$

$$\begin{aligned}
&\leq p_1 \sum_{N=-\infty}^{\infty} \int_{2C2^{\omega N/p_1+1/q_1}}^{2C2^{\omega(N+1)/p_1+1/q_1}} t^{p_1-1} \\
&\quad \times \mu \left\{ x : \left\{ \sum_{k=-\infty}^N \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_1 q_1} |D_k(f)(y_{\tau}^{k,v})|^{q_1} \chi_{Q_{\tau}^{k,v}}(x) \right\}^{1/q_1} \right. \\
&\quad \left. + \left\{ \sum_{k=N+1}^{\infty} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_1 q_1} |D_k(f)(y_{\tau}^{k,v})|^{q_1} \chi_{Q_{\tau}^{k,v}}(x) \right\}^{1/q_1} > 2^{-1/q_1} t \right\} dt \\
&\leq p_1 \sum_{N=-\infty}^{\infty} \int_{2C2^{\omega N/p_1+1/q_1}}^{2C2^{\omega(N+1)/p_1+1/q_1}} t^{p_1-1} \\
&\quad \times \mu \left\{ x : \left\{ \sum_{k=N+1}^{\infty} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_1 q_1} |D_k(f)(y_{\tau}^{k,v})|^{q_1} \chi_{Q_{\tau}^{k,v}}(x) \right\}^{1/q_1} > 2^{-1/q_1} t/2 \right\} dt \\
&\leq p_1 \sum_{N=-\infty}^{\infty} \int_{2C2^{\omega N/p_1+1/q_1}}^{2C2^{\omega(N+1)/p_1+1/q_1}} t^{p_1-1} \\
&\quad \times \mu \left\{ x : \left\{ \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_2 q_2} |D_k(f)(y_{\tau}^{k,v})|^{q_2} \chi_{Q_{\tau}^{k,v}}(x) \right\}^{1/q_2} \right. \\
&\quad \left. > C2^{-N(\frac{\omega}{p_1} - \frac{\omega}{p_2})} 2^{-1/q_1} t/2 \right\} dt \\
&\leq p_1 \sum_{N=-\infty}^{\infty} \int_{2C2^{\omega N/p_1+1/q_1}}^{2C2^{\omega(N+1)/p_1+1/q_1}} t^{p_1-1} \\
&\quad \times \mu \left\{ x : \left\{ \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_2 q_2} |D_k(f)(y_{\tau}^{k,v})|^{q_2} \chi_{Q_{\tau}^{k,v}}(x) \right\}^{1/q_2} > Ct^{p_1/p_2} \right\} dt \\
&\leq p_1 \int_0^{\infty} t^{p_1-1} \\
&\quad \times \mu \left\{ x : \left\{ \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_2 q_2} |D_k(f)(y_{\tau}^{k,v})|^{q_2} \chi_{Q_{\tau}^{k,v}}(x) \right\}^{1/q_2} > Ct^{p_1/p_2} \right\} dt \\
&\leq p_2 \int_0^{\infty} u^{p_2-1} \mu \left\{ x : \left\{ \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{v=1}^{N(k,\tau)} 2^{ks_2 q_2} |D_k(f)(y_{\tau}^{k,v})|^{q_2} \chi_{Q_{\tau}^{k,v}}(x) \right\}^{1/q_2} > Cu \right\} du \\
&\leq C \|f\|_{\dot{F}_{p_2}^{s_2, q_2}(X)}^{p_2} \\
&\leq C,
\end{aligned}$$

where we used the fact that  $t \approx 2^{\omega N/p_1}$ .

This proves (1.9), i.e.,  $\dot{F}_{p_2}^{s_2, q_2} \hookrightarrow \dot{F}_{p_1}^{s_1, q_1}$  with  $q_2 > \frac{p_2}{1+p_2}$ . For any  $q$  with  $\max\{\frac{\omega}{\omega+\theta}, \frac{\omega}{\omega+\theta+s_2}\} < q \leq \frac{p_2}{1+p_2}$ , it is easy to see that  $\dot{F}_{p_2}^{s_2, q} \hookrightarrow \dot{F}_{p_2}^{s_2, q_2}$  for  $q_2 > \frac{p_2}{1+p_2}$ . The proof of Theorem 1.5 is completed.  $\square$

#### Competing interests

The author declares that they have no competing interests.



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