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Tail properties and approximate distribution and expansion for extreme of LGMD

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Abstract
We introduce logarithmic generalized Maxwell distribution motivated by Vodă (Math. Rep. 11:171-179, 2009), which is an extension of the generalized Maxwell distribution. Some interesting properties of this distribution are studied and the asymptotic distribution of the partial maximum of an i.i.d. sequence from the logarithmic generalized Maxwell distribution is gained. The expansion of the limit distribution from the normalized maxima is established under the optimal norming constants, which shows the rate of convergence of the distribution for normalized maximum to the extreme limit.

MSC: Primary 60G70; secondary 60F05

Keywords: limit distribution; logarithmic generalized Maxwell distribution; Mills' ratio; maximum; tail properties

1 Introduction

The generalized Maxwell distribution (GMD for short), a generalization of ordinary Maxwell (or classical Maxwell) distribution, was proposed by Vodă [1]. With the rapid development of economy and science and technology, some of the existing distribution functions cannot meet the needs of research. For example, for some skewed data, it is appropriate to describe and fit them only by using some logarithmic models. Therefore, the recent development of some new distribution functions and the study of logarithmic case of the distribution functions have become hot issues in the statistical field. For more details, please refer to [2–8]. In this paper, we define the logarithmic generalized Maxwell distribution (for brevity LGMD), which is a natural prolongation of the generalized Maxwell distribution. In addition to the previously mentioned, one motivation of thinking of LGMD is to obtain more efficient results as parameter estimators when random models were supposed with the LGMD error terms instead of normal ones. Other aspects, like compressive sensing, we hope the LGMD could be used to model impulsive noise [9].

The GMD has a variety of applications in statistics, physics, and chemistry. The probability density function (pdf) and the cumulative distribution function (cdf) of the GMD with the parameter $k > 0$ are respectively,

$$g_k(x) = \frac{k}{2^{k/2}\sigma^{2+1/k}\Gamma(1+k/2)} x^{2k} \exp\left(-\frac{x^{2k}}{2\sigma^2}\right)$$

and

$$G_k(x) = \int_{-\infty}^{x} g_k(t)\,\mathrm{d}t$$

for $x \in R$, where σ is a positive constant and $\Gamma(\cdot)$ is the gamma function.

Mills [10] gave a well-known inequality and Mills' ratio conclusion for the standard Gauss cdf $\Phi(x)$ with respect to its pdf $\phi(x)$ as follows:

$$x^{-1}\left(1+x^{-2}\right)^{-1}\phi(x) < \Phi(-x) < x^{-1}\phi(x) \tag{1.1}$$

for $x > 0$, and

$$\frac{\Phi(-x)}{\phi(x)} \sim \frac{1}{x}, \tag{1.2}$$

as $x \to \infty$.

Peng *et al.* [11] extended the Mills results to the case of the general error distribution:

$$\frac{2\lambda^\nu}{\nu} x^{1-\nu}\left(1 + \frac{2(\nu-1)\lambda^\nu}{\nu} x^{-\nu}\right)^{-1} < \frac{T_\nu(-x)}{t_\nu(x)} < \frac{2\lambda^\nu}{\nu} x^{1-\nu} \tag{1.3}$$

for $\nu > 1$ and $x > 0$, and for $\nu > 0$

$$\frac{T_\nu(-x)}{t_\nu(x)} \sim \frac{2\lambda^\nu}{\nu} x^{1-\nu}, \tag{1.4}$$

as $x \to \infty$, where $\lambda = [\frac{2^{-2/\nu}\Gamma(1/\nu)}{\Gamma(3/\nu)}]^{1/2}$, and $T_\nu(x)$ is the general error cdf with pdf $t_\nu(x)$. Huang and Chen [12] investigated similar results of GMD, *viz.*,

$$\frac{\sigma^2}{k} x^{1-2k} < \frac{1-G_k(x)}{g_k(x)} < \frac{\sigma^2}{k} x^{1-2k}\left(1 + \left(\frac{\sigma^2}{k} x^{2k} - 1\right)^{-1}\right) \tag{1.5}$$

for $k > 1/2$, $\sigma > 0$ and $x > 0$, and for $k > 0$,

$$\frac{1-G_k(x)}{g_k(x)} \sim \frac{\sigma^2}{k} x^{1-2k}, \tag{1.6}$$

as $x \to \infty$. The above-mentioned Mill type inequalities such as (1.1), (1.3), and (1.5) and Mills' type ratios such as (1.2), (1.4), and (1.6) play an important role in considering some tail behavior and extremes of economic and financial data.

The present paper is to derive the Mills' inequality, Mills' ratio, and the distributional tail expression for the LGMD. As an important application, the asymptotic distribution of the partial maximum of i.i.d. variables with common LGMD is investigated. As another significant application, with appropriate normalized constants, the distributional expansion of the normalized maxima from LGMD is obtained. Moreover, we indicate that rate of convergence of the distribution of normalized maxima to corresponding extreme value limit is of the order of $O(1/(\log n)^{1-1/(2k)})$.

First of all, we provide the definition of LGMD.

Definition 1.1 Set X stand for a random variable which obeys the GMD. Set $Y = \exp(X)$. Then Y is termed obeying the LGMD, denoted by $Y \sim \mathrm{LGMD}(k)$ with parameter $k > 0$.

Easily check that the pdf is

$$f_k(x) = \frac{kx^{-1}}{2^{k/2}\sigma^{2+1/k}\Gamma(1+k/2)}(\log x)^{2k}\exp\left(-\frac{(\log x)^{2k}}{2\sigma^2}\right)$$

for $x > 0$, where parameter $k > 0$, and σ is a positive constant. Suppose that

$$F_k(x) = \int_0^x f_k(t)\,\mathrm{d}t$$

for $x > 0$. Observe that the LGMD decreases to the logarithmic Maxwell distribution when $k = 1$.

The rest of the article is organized as follows. In Section 2, we derive some interesting results including Mills-type ratios and tail behaviors of LGMD. In Section 3, we discuss the asymptotic distribution of normalized maxima of i.i.d. random variables following the LGMD and the suitable norming constants. We generalize the result to the case of a finite blending of LGMDs. In Section 4, we establish the asymptotic expansion of the distribution of the normalized maximum from LGMD under optimal choice of norming constants. As a byproduct, we obtain the convergence speed of the distribution of the normalized partial maxima to its limit.

2 Mills' ratio and tail properties of LGMD

In this part, we obtain some significant results including Mills' inequality, Mills' ratio of LGMD.

As to LGMD and GMD, observe that $1 - G_k(\log x) = 1 - F_k(x)$ and

$$\frac{1 - G_k(\log x)}{x^{-1}g_k(\log x)} = \frac{1 - F_k(x)}{f_k(x)}.$$

Hence, by Lemma 2.2 and Theorem 2.1 in Huang and Chen [12], the two results below follow.

Theorem 2.1 *Suppose that F_k and f_k respectively represent the cdf and pdf of LGMD with parameter $k > 1/2$. We have the inequality below, for all $x > 1$,*

$$\frac{\sigma^2}{k}x(\log x)^{1-2k} < \frac{1 - F_k(x)}{f_k(x)} < \frac{\sigma^2}{k}x(\log x)^{1-2k}\left(1 + \left(\frac{\sigma^2}{k}(\log x)^{2k} - 1\right)^{-1}\right), \tag{2.1}$$

where σ is a positive constant.

Corollary 2.1 *For fixed $k > 0$, as $x \to \infty$, we have*

$$\frac{1 - F_k(x)}{f_k(x)} \sim \frac{\sigma^2}{k}x(\log x)^{1-2k}. \tag{2.2}$$

Remark 2.1 Since the LGMD(k) are reduced to the logarithmic Maxwell distribution as $k = 1$, so by Theorem 2.1 and Corollary 2.1, we derive the Mill's inequality and Mills' ratio of the logarithmic Maxwell distribution, *viz.*,

$$\sigma^2 x(\log x)^{-1} f_1(x) < 1 - F_1(x) < \sigma^2 x(\log x)^{-1}\left(1 + \left(\sigma^2(\log x)^2 - 1\right)^{-1}\right) f_1(x)$$

for $x > 1$, and

$$\frac{1 - F_1(x)}{f_1(x)} \sim \frac{\sigma^2 x}{\log x},$$

as $x \to \infty$.

Remark 2.2 For $k > 1/2$, Corollary 2.1 gives $F_k \in D(\Lambda)$, *i.e.*, there are norming constants $\alpha_n > 0$ and $\beta_n \in \mathbb{R}$ which ensure $F_k^n(\alpha_n x + \beta_n)$ converges to $\exp(-\exp(-x))$, as $n \to \infty$. Since

$$\frac{(\mathrm{d}/\mathrm{d}x) f_k(x)}{f_k(x)} = -\frac{1}{x}\left(1 - \frac{2k}{\log x} + \frac{k}{\sigma^2}(\log x)^{2k-1}\right),$$

by Corollary 2.1, we obtain

$$\frac{1 - F_k(x)}{f_k(x)} \frac{(\mathrm{d}/\mathrm{d}x) f_k(x)}{f_k(x)} \to -1,$$

as $x \to \infty$. Hence, applying Proposition 1.18 in Resnick [13], we obtain $F_k \in D(\Lambda)$. As to how to choose the norming constants α_n and β_n will be explored by Theorem 3.2.

Finner *et al.* [14] investigated the asymptotic property of the ratio of the Student t and Gauss distributions as the degrees of freedom $u = u(x)$ satisfies

$$\lim_{x\to\infty} \frac{x^4}{u} = \beta \in [0, \infty). \tag{2.3}$$

The main motivation of the work is to consider the false discovery rate in multiple testing problems with large numbers of hypotheses and extremely small critical values for the smallest ordered p value; for details, see Finner *et al.* [15]. In the following, we investigate the asymptotic property of the ratio of pdfs and the ratio of the tails of the LGMD and the logarithmic Maxwell distribution. Firstly, we think over the situation of $k \to 1$. Secondly, we think over the situation of $x \to \infty$ for fixed k.

Theorem 2.2 *For $k > 0$, let $x = x(k)$ be such that*

$$k - 1 = \frac{\gamma}{2(\log x)^2 \log\log x} \tag{2.4}$$

for some $\gamma \in \mathbb{R}$. We obtain

$$\lim_{k\to 1} \frac{f_1(x)}{f_k(x)} = \exp\left(\frac{\gamma}{2\sigma^2}\right) \tag{2.5}$$

and

$$\lim_{k\to 1}\frac{1-F_1(x)}{1-F_k(x)} = \exp\left(\frac{\gamma}{2\sigma^2}\right). \tag{2.6}$$

Proof Observe that $\frac{2^{(k+1)/2}\sigma^{2+1/k}\Gamma(1+k/2)}{k\sigma^3\pi^{1/2}} \to 1$ as $k \to 1$, therefore

$$\begin{aligned}\lim_{k\to 1}\frac{f_1(x)}{f_k(x)} &= \lim_{k\to 1}(\log x)^{2-2k}\exp\left(\frac{(\log x)^{2k}}{2\sigma^2}-\frac{(\log x)^2}{2\sigma^2}\right)\\ &= \lim_{k\to 1}\exp\left(\frac{(\log x)^2}{2\sigma^2}\bigl((\log x)^{2k-2}-1\bigr)\right)\\ &= \lim_{k\to 1}\exp\left(\frac{(\log x)^2}{2\sigma^2}\bigl(\exp\bigl((2k-2)\log\log x\bigr)-1\bigr)\right)\\ &= \lim_{k\to 1}\exp\left(\frac{(\log x)^2}{2\sigma^2}\left(\exp\left(\frac{\gamma}{(\log x)^2}\right)-1\right)\right)\\ &= \exp\left(\frac{\gamma}{2\sigma^2}\right).\end{aligned}$$

By (2.4), it is easy to check that $x \to \infty$ as $k \to 1$. Again applying (2.4), we have

$$\begin{aligned}(\log x)^{2-2k} &= \exp\bigl(2(1-k)\log\log x\bigr)\\ &= \exp\left(\frac{\gamma}{2(\log x)^2}\right) \to 1, \quad \text{as } k \to 1.\end{aligned} \tag{2.7}$$

Combining (2.7), Corollary 2.1, Remark 2.1, and (2.5), representation (2.6) can be derived. □

Theorem 2.3 *For fixed k, we have*

$$\frac{f_1(x)}{f_k(\exp((\log x)^{1/k}))} = \frac{2^{(k+1)/2}\Gamma(1+k/2)\exp((\log x)^{1/k})}{\pi^{1/2}k\sigma^{1-1/k}x} \tag{2.8}$$

and

$$\lim_{x\to\infty}\frac{(\log x)^{1/k-1}(1-F_1(x))}{1-F_k(\exp((\log x)^{1/k}))} = \frac{2^{(k+1)/2}\Gamma(1+k/2)}{\pi^{1/2}\sigma^{1-1/k}}. \tag{2.9}$$

Proof It is easy to verify (2.8) by fundamental calculation. By Corollary 2.1, Remark 2.1, and (2.8), we have

$$\begin{aligned}&\lim_{x\to\infty}\frac{(\log x)^{1/k-1}(1-F_1(x))}{1-F_k(\exp((\log x)^{1/k}))}\\ &\quad= \lim_{x\to\infty}\frac{kx}{\exp((\log x)^{1/k})}\frac{f_1(x)}{f_k(\exp((\log x)^{1/k}))}\\ &\quad= \frac{2^{(k+1)/2}\Gamma(1+k/2)}{\pi^{1/2}\sigma^{1-1/k}}.\end{aligned}$$

Hence (2.9) follows. □

3 Limiting distribution of the maxima

By applying Corollary 2.1, we could establish the distributional tail representation for the LGMD.

Theorem 3.1 *Under the conditions of Theorem* 2.1, *we have*

$$1-F_k(x)=c(x)\exp\biggl(-\int_e^x \frac{g(t)}{f(t)}\,\mathrm{d}t\biggr)$$

for large enough x, *where*

$$c(x)=\frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\exp\bigl(-1/\bigl(2\sigma^2\bigr)\bigr)\bigl(1+\theta_1(x)\bigr)$$

and

$$f(t)=\frac{\sigma^2}{k}t(\log t)^{1-2k},\qquad g(t)=1-\frac{\sigma^2}{k}(\log t)^{-2k},$$

where $\theta_1(x)\to 0$ *as* $x\to\infty$.

Proof For large enough x, by Corollary 2.1, we have

$$\begin{aligned}
1-F_k(x) &= \frac{\sigma^2}{k}(\log x)^{1-2k}xf_k(x)\bigl(1+\theta_1(x)\bigr)\\
&= \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\exp\biggl(\log\log x-\frac{(\log x)^{2k}}{2\sigma^2}\biggr)\bigl(1+\theta_1(x)\bigr)\\
&= \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\exp\biggl(-\frac{1}{2\sigma^2}\biggr)\exp\biggl(-\int_e^x\biggl(\frac{k(\log t)^{2k-1}}{\sigma^2 t}-\frac{1}{t\log t}\biggr)\mathrm{d}t\biggr)\\
&\quad\times\bigl(1+\theta_1(x)\bigr)\\
&= \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\exp\biggl(-\frac{1}{2\sigma^2}\biggr)\exp\biggl(-\int_e^x\frac{1-k^{-1}\sigma^2(\log t)^{-2k}}{k^{-1}\sigma^2 t(\log t)^{1-2k}}\,\mathrm{d}t\biggr)\\
&\quad\times\bigl(1+\theta_1(x)\bigr)\\
&= c(x)\exp\biggl(-\int_e^x\frac{g(t)}{f(t)}\,\mathrm{d}t\biggr),
\end{aligned}$$

where $\theta_1(x)\to 0$ as $x\to\infty$. The desired result follows. □

Remark 3.1 As $\lim_{t\to\infty}g(t)=1$, $f(t)>0$ on $[1,\infty)$ is absolutely continuous function and $\lim_{t\to\infty}f'(t)=0$ in Theorem 3.1, an application of Theorem 3.1 and Corollary 1.7 in Resnick [13] shows $F_k\in D(\Lambda)$, and the norming constants a_n and b_n can be chosen by

$$\frac{1}{1-F_k(b_n)}=n,\qquad a_n=f(b_n) \tag{3.1}$$

such that

$$\lim_{n\to\infty}F_k^n(a_nx+b_n)=\Lambda(x), \tag{3.2}$$

where $D(\Lambda)$ denotes the domain of attraction $\Lambda(x)=\exp(-\exp(-x))$.

Here we establish the asymptotic distribution of the normalized maximum of a sequence of i.i.d. random variables following LGMD. Remark 2.2 and Theorem 3.1 showed that the distribution of partial maximum converges to $\Lambda(x)$. So, the following task is to look for the associated suitable norming constants.

Theorem 3.2 *Suppose that* $\{X_n, n \geq 1\}$ *be an i.i.d. sequence from the LGMD with* $k > 1/2$. *Let* $M_n = \max\{X_1, X_2, \dots, X_n\}$. *We have*

$$\lim_{n\to\infty} P(M_n \leq \alpha_n x + \beta_n) = \exp\bigl(-\exp(-x)\bigr),$$

where

$$\alpha_n = \frac{\sigma^2 \exp(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)})(1 + \frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}(\log\log n - (k^2-1)\log 2 - 2k\log\Gamma(1+k/2)))}{k(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)} + \log(1 + \frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}(\log\log n - (k^2-1)\log 2 - 2k\log\Gamma(1+k/2))))^{2k-1}}$$

and

$$\begin{aligned}\beta_n = {}& \exp\bigl(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)}\bigr)\Biggl(1 + \frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}\bigl(\log\log n - \bigl(k^2-1\bigr)\log 2 \\ & - 2k\log\Gamma(1+k/2)\bigr)\Biggr).\end{aligned}$$

Proof Since $F_k \in D(\Lambda)$, there must be norming constants $a_n > 0$ and $b_n \in \mathbb{R}$ which make sure that $\lim_{n\to\infty} P((M_n - b_n)/a_n \leq x) = \exp(-\exp(-x))$. By Proposition 1.1 in Resnick [13] and Theorem 3.1, we can make choice of the norming constants a_n and b_n satisfying the equations: $b_n = (1/(1-F_k))^{\leftarrow}(n)$ and $a_n = f(b_n)$. Note that $F_k(x)$ is continuous, then $1 - F_k(b_n) = n^{-1}$. By Corollary 2.1, we have

$$nk^{-1}\sigma^2(\log b_n)^{1-2k} b_n f_k(b_n) \to 1,$$

as $n \to \infty$, *viz.*,

$$n2^{-\frac{k}{2}}\sigma^{-\frac{1}{k}}\Gamma^{-1}\left(1 + \frac{k}{2}\right)\log b_n \exp\left(-\frac{(\log b_n)^{2k}}{2\sigma^2}\right) \to 1,$$

as $n \to \infty$, and so

$$\log n - \frac{k}{2}\log 2 - \frac{1}{k}\log\sigma - \log\Gamma\left(1 + \frac{k}{2}\right) + \log\log b_n - \frac{(\log b_n)^{2k}}{2\sigma^2} \to 0, \tag{3.3}$$

as $n \to \infty$, from which one deduces

$$\frac{(\log b_n)^{2k}}{2\sigma^2 \log n} \to 1,$$

as $n \to \infty$, thus

$$2k\log\log b_n - \log 2 - 2\log\sigma - \log\log n \to 0,$$

as $n \to \infty$, hence

$$\log\log b_n = \frac{1}{2k}(\log 2 + 2\log\sigma + \log\log n) + o(1).$$

Putting the equality above into (3.3), we have

$$(\log b_n)^{2k} = 2\sigma^2\left(\log n + \frac{1}{2k}\log\log n - \frac{k^2-1}{2k}\log 2 - \log\Gamma\left(1+\frac{k}{2}\right)\right) + o(1),$$

from which one induces that

$$\log b_n = 2^{\frac{1}{2k}}\sigma^{\frac{1}{k}}(\log n)^{\frac{1}{2k}}\left(1 + \frac{\log\log n - (k^2-1)\log 2 - 2k\log\Gamma(1+\frac{k}{2})}{2^2k^2\log n} + o\big((\log n)^{-1}\big)\right),$$

therefore

$$\begin{aligned} b_n &= \exp\big(2^{\frac{1}{2k}}\sigma^{\frac{1}{k}}(\log n)^{\frac{1}{2k}}\big)\left(1 + \frac{\sigma^{\frac{1}{k}}(\log n)^{\frac{1}{2k}-1}}{2^{2-\frac{1}{2k}}k^2}\left(\log\log n - \big(k^2-1\big)\log 2\right.\right. \\ &\quad \left.\left. - 2k\log\Gamma\left(1+\frac{k}{2}\right)\right) + o\big((\log n)^{\frac{1}{2k}-1}\big)\right) \\ &= \beta_n + o\big((\log n)^{\frac{1}{2k}-1}\exp\big(2^{\frac{1}{2k}}\sigma^{\frac{1}{k}}(\log n)^{\frac{1}{2k}}\big)\big), \end{aligned}$$

where

$$\begin{aligned} \beta_n &= \exp\big(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)}\big)\left(1 + \frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}\big(\log\log n - \big(k^2-1\big)\log 2\right. \\ &\quad \left. - 2k\log\Gamma(1+k/2)\big)\right). \end{aligned}$$

Hence, we have

$$\begin{aligned} \alpha_n &= f(\beta_n) \\ &= \frac{\sigma^2\exp(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)})(1 + \frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}(\log\log n - (k^2-1)\log 2 - 2k\log\Gamma(1+k/2)))}{k(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)} + \log(1 + \frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}(\log\log n - (k^2-1)\log 2 - 2k\log\Gamma(1+k/2))))^{2k-1}}. \end{aligned}$$

It is easy to check that $\lim_{n\to\infty}\alpha_n/a_n = 1$ and $\lim_{n\to\infty}(b_n-\beta_n)/\alpha_n = 0$. Hence, by Theorem 1.2.3 in Leadbetter *et al.* [16], the proof is complete. □

Remark 3.2 Theorem 3.2 shows that the limit distribution of the normalized maximum from the logarithmic Maxwell distribution is the extreme value distribution $\exp(-\exp(-x))$ with norming constants

$$\alpha_n = \frac{\sigma^2\exp(2^{1/2}\sigma(\log n)^{1/2})(1 + \frac{\sigma}{2^{3/2}(\log n)^{1/2}}(\log\log n - 2\log(\pi^{1/2}/2)))}{2^{1/2}\sigma(\log n)^{1/2} + \log(1 + \frac{\sigma}{2^{3/2}(\log n)^{1/2}}(\log\log n - 2\log(\pi^{1/2}/2)))}$$

and

$$\beta_n = \exp\big(2^{1/2}\sigma(\log n)^{1/2}\big)\left(1 + \frac{\sigma}{2^{3/2}(\log n)^{1/2}}\big(\log\log n - 2\log\big(\pi^{1/2}/2\big)\big)\right).$$

At the end of this section, we generalize the result of Theorem 3.2 to the situation of a finite blending of LGMDs.

Finite mixture distributions (or models) have been widely applied in various areas such as Chemistry [17] and image and video databases [18]. Specifically, related extreme statistical scholars have studied them. Mladenović [19] have considered extreme values of the sequences of independent random variables with common mixed distributions containing normal, Cauchy and uniform distributions. Peng *et al.* [20] have investigated the limit distribution and its corresponding uniform rate of convergence for a finite mixed of exponential distribution.

If the distribution function (df) F of a random variable ξ have

$$F(x) = p_1 F_1(x) + p_2 F_2(x) + \cdots + p_r F_r(x),$$

we say that ξ obeys a finite mixed distribution F, where F_i, $1 \le i \le r$ stand for different dfs of the mixture components. The weight coefficients satisfy the condition that $p_i > 0$, $i = 1, 2, \ldots, r$ and $\sum_{j=1}^{r} p_j = 1$.

Next, we think of the extreme value distribution from a finite blending with constituent dfs F_{k_i} obeying LGMD(k_i), where the parameter $k_i > 1$ for $1 \le i \le r$ and $k_i \neq k_j$ for $i \neq j$. Denote the cumulative df of the finite blending by

$$F(x) = p_1 F_{k_1}(x) + p_2 F_{k_2}(x) + \cdots + p_r F_{k_r}(x) \tag{3.4}$$

for $x > 0$.

Theorem 3.3 *Suppose that* $\{Z_n, n \ge 1\}$ *be a sequence of i.i.d. random variables following the common df* F *given by* (3.4). *Set* $M_n = \max\{Z_k, 1 \le k \le n\}$. *Now*

$$\lim_{n\to\infty} P\left(\frac{M_n - \beta_n}{\alpha_n} \le x\right) = \exp\bigl(-\exp(-x)\bigr)$$

holds with the norming constants

$$\alpha_n = \frac{\sigma^{1/k}\exp(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)})}{2^{1-1/(2k)}k(\log n)^{1-1/(2k)}}$$

and

$$\beta_n = \exp\bigl(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)}\bigr)\Biggl(1 + \frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}\bigl(\log\log n + 2k\log p$$
$$- \bigl(k^2-1\bigr)\log 2 - 2k\log\Gamma(1+k/2)\bigr)\Biggr),$$

where $\sigma = \max\{\sigma_1, \ldots, \sigma_r\}$, *and* $p = p_{i_1} + \cdots + p_{i_j}$, $i_s \in \{i, \sigma_i = \sigma$ *and* $k = k_i\}$, $1 \le s \le j \le r$, *and* $k = \min\{k_1, \ldots, k_r\}$.

Proof By (3.4), we have

$$1 - F(x) = \sum_{i=1}^{r} p_i\bigl(1 - F_{k_i}(x)\bigr).$$

By Theorem 2.1, we have

$$\begin{aligned}
&\sum_{i=1}^{r} \frac{p_i\sigma_i^2}{k_i}(\log x)^{1-2k_i} x f_{k_i}(x) \\
&\quad < 1 - F(x) < \sum_{i=1}^{r} \frac{p_i\sigma_i^2}{k_i}(\log x)^{1-2k_i} x \left(1 + \left(\frac{\sigma_i^2}{k_i}(\log x)^{2k_i} - 1\right)^{-1}\right) f_{k_i}(x)
\end{aligned}$$

for all $x > 1$, according to the definition of f_k, which implies

$$\begin{aligned}
&\frac{p \log x}{2^{\frac{k}{2}} \sigma^{\frac{1}{k}} \Gamma(1 + \frac{k}{2})} \exp\left(-\frac{(\log x)^{2k}}{2\sigma^2}\right)\bigl(1 + A_k(x)\bigr) \\
&\quad < 1 - F(x) \\
&\quad < \frac{p \log x}{2^{\frac{k}{2}} \sigma^{\frac{1}{k}} \Gamma(1 + \frac{k}{2})} \left(1 + \left(\frac{\sigma^2}{k}(\log x)^{2k} - 1\right)^{-1}\right) \exp\left(-\frac{(\log x)^{2k}}{2\sigma^2}\right)\bigl(1 + B_k(x)\bigr),
\end{aligned} \tag{3.5}$$

where

$$A_k(x) = \sum_{k_i \neq k} \frac{2^{\frac{k}{2}} p_i \sigma^{\frac{1}{k}} \Gamma(1 + \frac{k}{2})}{2^{\frac{k_i}{2}} p \sigma^{\frac{1}{k_i}} \Gamma(1 + \frac{k_i}{2})} \exp\left(\frac{(\log x)^{2k}}{2\sigma^2} - \frac{(\log x)^{2k_i}}{2\sigma_i^2}\right) \to 0 \tag{3.6}$$

and

$$\begin{aligned}
B_k(x) &= \sum_{k_i \neq k} \frac{2^{\frac{k}{2}} p_i \sigma^{\frac{1}{k}} \Gamma(1 + \frac{k}{2})}{2^{\frac{k_i}{2}} p \sigma^{\frac{1}{k_i}} \Gamma(1 + \frac{k_i}{2})} \frac{1 + (\frac{\sigma_i^2}{k_i}(\log x)^{2k_i} - 1)^{-1}}{1 + (\frac{\sigma^2}{k}(\log x)^{2k} - 1)^{-1}} \exp\left(\frac{(\log x)^{2k}}{2\sigma^2} - \frac{(\log x)^{2k_i}}{2\sigma_i^2}\right) \\
&\to 0
\end{aligned} \tag{3.7}$$

as $x \to \infty$ since $k = \min\{k_1, k_2, \ldots, k_r\}$. Combining (3.5)-(3.7) with (2.2), for large enough x, we obtain

$$1 - F(x) \sim p\bigl(1 - F_k(x)\bigr) \tag{3.8}$$

as $x \to \infty$, where F_k represents the cdf of the LGMD(k), and σ and p are defined by Theorem 3.3. By Proposition 1.19 in Resnick [13], we can derive $F \in D(\Lambda)$. The norming constants can be obtained by Theorem 3.2 and (3.8). The desired result follows. □

4 Asymptotic expansion of maximum

In this section, we establish an high-order expansion of the distribution of the extreme from the LGMD sample.

Theorem 4.1 *For the norming constants a_n and b_n given by* (3.1), *we have*

$$\lim_{n\to\infty} (\log b_n)^{\lambda}\Bigl((\log b_n)^{2k-1}\bigl(F_k^n(a_n x + b_n) - \Lambda(x)\bigr) - I(x)\Lambda(x)\Bigr) = l(x)\Lambda(x),$$

where

$$l(x)=\begin{cases} J_k(x)+\frac{1}{2}I^2(x), & \textit{if } \frac{1}{2}<k<1,\\ J_k(x)+w(x), & \textit{if } k=1,\\ J_k(x), & \textit{if } k>1,\end{cases}$$

$$I(x)=\frac{1}{2}k^{-1}\sigma^2x^2e^{-x},\qquad w(x)=\frac{1}{4}\sigma^4x^4e^{-2x},$$

and

$$J_k(x)=\begin{cases} k^{-2}\sigma^4x^3(\frac{1}{3}-\frac{1}{4}x)e^{-x}, & \textit{if } \frac{1}{2}<k<1,\\ \sigma^2x(1+\frac{1}{2}x+\frac{1}{3}\sigma^2x^2-\frac{1}{4}\sigma^2x^3)e^{-x}, & \textit{if } k=1,\\ k^{-1}\sigma^2x(\frac{1}{2}(2k-1)x+1)e^{-x}, & \textit{if } k>1.\end{cases}$$

Corollary 4.1 *Under the condition of Theorem* 4.1, *we have*

$$F_k^n(a_nx+b_n)-\Lambda(x)\sim\frac{I(x)\Lambda(x)}{(2\sigma^2\log n)^{1-1/(2k)}}$$

for large n, *where* $I(x)$ *is given by Theorem* 4.1.

Proof The result directly follows from Theorem 4.1. The detailed proof is omitted. □

In order to prove Theorem 4.1, we need several lemmas. The following lemma shows a decomposition of the distributional tail representation of the LGMD.

Lemma 4.1 *Let* $F_k(x)$ *denote the cdf of the LMGD. For large* x, *we have*

$$\begin{aligned} 1-F_k(x)={}&\frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\exp\bigl(-1/\bigl(2\sigma^2\bigr)\bigr)\bigl[1+k^{-1}\sigma^2(\log x)^{-2k}\\ &+k^{-2}(1-2k)\sigma^4(\log x)^{-4k}+O\bigl((\log x)^{-6k}\bigr)\bigr]\exp\biggl(-\int_e^x\frac{g(t)}{f(t)}\,\mathrm{d}t\biggr)\end{aligned}$$

with $f(t)$ *and* $g(t)$ *given by Theorem* 3.1.

Proof By integration by parts, we have

$$\begin{aligned} 1-F_k(x)={}&\frac{k}{2^{\frac{1}{2}}\Gamma(1+\frac{k}{2})}\int_{\log x/\sigma^{\frac{1}{k}}}^{\infty}s^{2k}\exp\biggl(-\frac{1}{2}s^{2k}\biggr)\,\mathrm{d}s\\ ={}&\frac{\log x}{2^{\frac{k}{2}}\sigma^{\frac{1}{k}}\Gamma(1+\frac{k}{2})}\exp\biggl(-\frac{(\log x)^{2k}}{2\sigma^2}\biggr)\bigl[1+k^{-1}\sigma^2(\log x)^{-2k}\\ &+k^{-2}(1-2k)\sigma^4(\log x)^{-4k}+k^{-3}(1-2k)(1-4k)\sigma^6(\log x)^{-6k}\bigr]\\ &+\frac{(1-2k)(1-4k)(1-6k)}{2^{\frac{k}{2}}k^3\Gamma(1+\frac{k}{2})}\int_{\log x/\sigma^{\frac{1}{k}}}^{\infty}s^{-6k}\exp\biggl(-\frac{1}{2}s^{2k}\biggr)\,\mathrm{d}s. \end{aligned}\tag{4.1}$$

Using L'Hospital's rules yields

$$\lim_{n\to\infty} \frac{\int_{\log x/\sigma^{\frac{1}{k}}}^{\infty} s^{-6k} \exp(-\frac{1}{2}s^{2k})\,\mathrm{d}s}{(\log x)^{1-6k}\exp(-\frac{(\log x)^{2k}}{2\sigma^2})} = 0. \tag{4.2}$$

One easily checks that

$$\begin{aligned} &\frac{\log x}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)} \exp\biggl(-\frac{(\log x)^{2k}}{2\sigma^2}\biggr) \\ &\quad = \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)} \exp\biggl(-\frac{1}{2\sigma^2}\biggr) \exp\biggl(-\int_e^x \frac{g(t)}{f(t)}\,\mathrm{d}t\biggr) \end{aligned} \tag{4.3}$$

with $f(t)$ and $g(t)$ determined by Theorem 3.1. Combining with (4.1)-(4.3), we complete the proof. □

Lemma 4.2 *Set*

$$B_n(x) = \frac{1 + k^{-1}\sigma^2(\log b_n)^{-2k} + k^{-2}(1-2k)\sigma^4(\log b_n)^{-4k} + O((\log b_n)^{-6k})}{1 + k^{-1}\sigma^2(\log(a_nx+b_n))^{-2k} + k^{-2}(1-2k)\sigma^4(\log(a_nx+b_n))^{-4k} + O((\log(a_nx+b_n))^{-6k})}$$

with the norming constants a_n and b_n given by (3.1), *then*

$$B_n(x) - 1 = 2k^{-1}\sigma^4(\log b_n)^{-4k}x + O\bigl((\log b_n)^{1-6k}\bigr).$$

Proof By (3.2), we have $n(1-F_k(a_nx+b_n)) \to e^{-x}$ as $n\to\infty$, with the norming constants a_n and b_n given by (3.1). It is not difficult to verify that $\lim_{n\to\infty} B_n(x) = 1$ and

$$\begin{aligned} B_n(x) - 1 = {}& \bigl[k^{-1}\sigma^2\bigl((\log b_n)^{-2k} - \bigl(\log(a_nx+b_n)\bigr)^{-2k}\bigr) \\ & + k^{-2}(1-2k)\sigma^4\bigl((\log b_n)^{-4k} - \bigl(\log(a_nx+b_n)\bigr)^{-4k}\bigr) \\ & + O\bigl((\log b_n)^{-6k}\bigr)\bigr]\bigl(1+o(1)\bigr). \end{aligned} \tag{4.4}$$

For large n we have

$$\begin{aligned} (\log b_n)^{-2k} - \bigl(\log(a_nx+b_n)\bigr)^{-2k} = {}& 2\sigma^2(\log b_n)^{-4k}x - k^{-1}\sigma^4(\log b_n)^{1-6k}x^2 \\ & + O\bigl((\log b_n)^{2-8k}\bigr) + O\bigl((\log b_n)^{-6k}\bigr) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} (\log b_n)^{-4k} - \bigl(\log(a_nx+b_n)\bigr)^{-4k} = {}& 4\sigma^2(\log b_n)^{-6k}x - 2k^{-1}\sigma^4(\log b_n)^{1-8k}x^2 \\ & + O\bigl((\log b_n)^{2-10k}\bigr) + O\bigl((\log b_n)^{-8k}\bigr). \end{aligned} \tag{4.6}$$

By (4.4)-(4.6), the desired result follows. □

Lemma 4.3 *Set $\lambda = 1 \wedge (2k-1)$ to denote the minimum of $\{1, 2k-1\}$ and $\nu_n(x) = n\log F_k(a_nx+b_n) + e^{-x}$ with norming constants a_n and b_n given by* (3.1). *Then*

$$\lim_{n\to\infty} (\log b_n)^{\lambda}\bigl((\log b_n)^{2k-1}\nu_n(x) - I(x)\bigr) = J_k(x), \tag{4.7}$$

with $I(x)$, $J_k(x)$ given by Theorem 4.1.

Proof For any positive integers m and $i > 1$, by Corollary 2.1 and the fact that $1/(1-F_k(a_nx+b_n)) = n$, we have

$$\lim_{n\to\infty} \frac{(1-F_k(a_nx+b_n))^i}{n^{-1}(\log b_n)^{-mk}} = 0. \tag{4.8}$$

For any $x \in \mathbb{R}$ and $a_n = k^{-1}\sigma^2 b_n(\log b_n)^{1-2k}$, we have

$$(\log b_n)^{2k-1}\left(\frac{ka_n}{\sigma^2(a_nx+b_n)(\log(a_nx+b_n))^{1-2k}} - 1\right) \to -k^{-1}\sigma^2 x \tag{4.9}$$

and

$$\frac{a_n(\log b_n)^{2k-1}}{(a_nx+b_n)\log(a_nx+b_n)} \to 0, \tag{4.10}$$

as $n \to \infty$. Here set

$$C_n(x) = \frac{ka_n}{\sigma^2(a_nx+b_n)(\log(a_nx+b_n))^{1-2k}} - \frac{a_n}{(a_nx+b_n)\log(a_nx+b_n)} - 1.$$

By Lemmas 4.1, 4.2, (4.9), and (4.10), we have

$$\begin{aligned}
&\frac{1-F_k(b_n)}{1-F_k(a_nx+b_n)}e^{-x} \\
&\quad = B_n(x)\exp\left(\int_0^x \left(\frac{ka_n}{\sigma^2(a_ns+b_n)(\log(a_ns+b_n))^{1-2k}} - \frac{a_n}{(a_ns+b_n)\log(a_ns+b_n)} - 1\right) \mathrm{d}s\right) \\
&\quad = B_n(x)\exp\left(\int_0^x C_n(s)\,\mathrm{d}s\right) \\
&\quad = B_n(x)\left(1 + \int_0^x C_n(s)\,\mathrm{d}s + \frac{1}{2}\left(\int_0^x C_n(s)\,\mathrm{d}s\right)^2\big(1+o(1)\big)\right).
\end{aligned} \tag{4.11}$$

By (4.8)-(4.11), Lemma 4.2, and the dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{n\to\infty}(\log b_n)^{2k-1}v_n(x) \\
&\quad = \lim_{n\to\infty}\frac{\log F_k(a_nx+b_n) + n^{-1}e^{-x}}{n^{-1}(\log b_n)^{1-2k}} \\
&\quad = \lim_{n\to\infty}\frac{-(1-F_k(a_nx+b_n)) - \frac{1}{2}(1-F_k(a_nx+b_n))^2(1+o(1)) + (1-F_k(b_n))e^{-x}}{n^{-1}(\log b_n)^{1-2k}} \\
&\quad = \lim_{n\to\infty}\frac{1-F_k(a_nx+b_n)}{n^{-1}}\frac{\frac{1-F_k(b_n)}{1-F_k(a_nx+b_n)}e^{-x} - 1}{(\log b_n)^{1-2k}} \\
&\quad = e^{-x}\lim_{n\to\infty}(\log b_n)^{2k-1}\left(B_n(x) + B_n(x)\int_0^x C_n(s)\,\mathrm{d}s\big(1+o(1)\big) - 1\right) \\
&\quad = e^{-x}\lim_{n\to\infty}(\log b_n)^{2k-1}\int_0^x C_n(s)\,\mathrm{d}s \\
&\quad = -\frac{1}{2}k^{-1}\sigma^2x^2e^{-x} \\
&\quad =: I(x).
\end{aligned} \tag{4.12}$$

For all $x \in \mathbb{R}$,

$$\begin{aligned}&\frac{ka_n}{\sigma^2(a_ns+b_n)(\log(a_ns+b_n))^{1-2k}} - 1 + k^{-1}\sigma^2 s(\log b_n)^{1-2k}\\&\quad= \left(1+k^{-1}\sigma^2(\log b_n)^{1-2k}s\right)^{-1}\left((2k-1)\left(k^{-1}\sigma^2(\log b_n)^{-2k}s - \frac{1}{2}k^{-2}\sigma^4(\log b_n)^{1-4k}s^2\right)\right.\\&\qquad\left.+ k^{-2}\sigma^4(\log b_n)^{2-4k}s^2 + O\left((\log b_n)^{2-6k}\right)\right)\end{aligned}$$

for large n, which implies

$$\begin{aligned}&(\log b_n)^{2k-1+\lambda}\left(\frac{ka_n}{\sigma^2(a_ns+b_n)(\log(a_ns+b_n))^{1-2k}} - 1 + k^{-1}\sigma^2 s(\log b_n)^{1-2k}\right)\\&\quad\to \begin{cases} k^{-2}\sigma^4 s^2, & \text{if } \frac{1}{2} < k < 1,\\ \sigma^2 s(1+\sigma^2 s), & \text{if } k = 1,\\ (2k-1)k^{-1}\sigma^2 s, & \text{if } k > 1,\end{cases}\end{aligned}\tag{4.13}$$

and

$$\frac{a_n(\log b_n)^{2k-1+\lambda}}{(a_ns+b_n)\log(a_ns+b_n)} \to \begin{cases} 0, & \text{if } \frac{1}{2} < k < 1,\\ \sigma^2, & \text{if } k = 1,\\ k^{-1}\sigma^2, & \text{if } k > 1,\end{cases}\tag{4.14}$$

as $n \to \infty$.

By (4.13), (4.14), and Lemma 4.2, we have

$$\begin{aligned}&\lim_{n\to\infty}(\log b_n)^{\lambda}\left((\log b_n)^{2k-1}v_n(x) - I(x)\right)\\&\quad= \lim_{n\to\infty}\frac{-(1-F_k(a_nx+b_n)) + (1-F_k(b_n))e^{-x}(1-I(x)e^{x}(\log b_n)^{1-2k})}{n^{-1}(\log b_n)^{1-2k-\lambda}}\\&\quad= \lim_{n\to\infty}\frac{1-F_k(a_nx+b_n)}{n^{-1}}\frac{\frac{1-F_k(b_n)}{1-F_k(a_nx+b_n)}e^{-x}(1-I(x)e^{x}(\log b_n)^{1-2k}) - 1}{(\log b_n)^{1-2k-\lambda}}\\&\quad= e^{-x}\lim_{n\to\infty}\left[(\log b_n)^{2k+\lambda-1}\left(B_n(x)-1\right) + B_n(x)(\log b_n)^{2k+\lambda-1}\right.\\&\qquad\times\int_0^x\left(C_n(s) + k^{-1}\sigma^2 s(\log b_n)^{1-2k}\right)\mathrm{d}s - B_n(x)I(x)e^{x}(\log b_n)^{\lambda}\int_0^x C_n(s)\,\mathrm{d}s\\&\qquad\left.+\frac{1}{2}B_n(x)(\log b_n)^{2k+\lambda-1}\left(1-I(x)e^{x}(\log b_n)^{1-2k}\right)\left(\int_0^x C_n(s)\,\mathrm{d}s\right)^2\left(1+o(1)\right)\right]\\&\quad= \begin{cases} k^{-2}\sigma^4x^3(\frac{1}{3}-\frac{1}{4}x)e^{-x}, & \text{if } \frac{1}{2} < k < 1,\\ \sigma^2x(1+\frac{1}{2}x+\frac{1}{3}\sigma^2x^2-\frac{1}{4}\sigma^2x^3)e^{-x}, & \text{if } k = 1,\\ k^{-1}\sigma^2x(\frac{1}{2}(2k-1)x+1)e^{-x}, & \text{if } k > 1\end{cases}\\&\quad=: J_k(x),\end{aligned}$$

with $\lambda = 1 \wedge (2k-1)$. The proof is completed. □

Proof of Theorem 4.1 By Lemma 4.3, we have

$$(\log b_n)^{2k-1+\lambda} v_n^2(x) \to \begin{cases} I^2(x), & \text{if } \frac{1}{2} < k \le 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{4.15}$$

as $n \to \infty$. Once again by Lemma 4.3, we have

$$\begin{aligned} &(\log b_n)^{\lambda}\bigl((\log b_n)^{2k-1}\bigl(F_k^n(a_n x + b_n) - \Lambda(x)\bigr) - I(x)\Lambda(x)\bigr) \\ &\quad = (\log b_n)^{\lambda}\bigl((\log b_n)^{2k-1}\bigl(\exp\bigl(u_n(x)\bigr) - 1\bigr) - I(x)\bigr)\Lambda(x) \\ &\quad = \left((\log b_n)^{\lambda}\bigl((\log b_n)^{2k-1}v_n(x) - I(x)\bigr) + (\log b_n)^{2k+\lambda-1}v_n^2(x)\left(\frac{1}{2} + O\bigl(v_n(x)\bigr)\right)\right)\Lambda(x) \\ &\quad \to l(x)\Lambda(x), \end{aligned}$$

where $l(x)$ is provided by Theorem 4.1. The proof is completed. □

5 Conclusion

Motivated by Vodă [1], we put forward the logarithmic generalized Maxwell distribution. We discuss tail properties and the limit distribution of the distribution. We extend the results to the case of a finite mixture distribution. With the optimal norming constants, we establish the high-order expansion of the distribution of maxima from logarithmic generalized Maxwell distribution, by which we derive the convergence rate of the distribution of maximum to the associate extreme limit.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JH obtained the theorem and completed the proof. JW, GL, and JH corrected and improved the final version. All authors read and approved the final manuscript.

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