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Full Length Research Paper

Some new classes of analytic functions defined using convolution

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In this paper, we introduce and study some new subclasses of analytic functions defined in the open unit disc using the convolution technique. Inclusion results, radius problems and several other properties of these classes are discussed.

Key words: Convolution integral operator, functions with positive real part, alpha-starlike, bounded Mocanu variation, univalent.

INTRODUCTION

Let A denote the class of functions $f(z)$ given by:

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \qquad (1)$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. Let $P_k(\gamma)$ be the class of functions $p(z)$ defined in E, satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \gamma}{1-\gamma} \right| d\theta \le k\pi, \qquad (2)$$

where $z = re^{i\theta}$, $k \ge 2$ and $0 \le \gamma < 1$. When $\gamma = 0$, we obtain the class P_k defined in Pinchuk (1971) and for $k = 2$, $\gamma = 0$, we have the class P of functions with positive real part. We can write Equation 2 as:

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1-2\gamma) z e^{-it}}{1 - z e^{-it}} d\mu(t),$$

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that;

$$\int_0^{2\pi} d\mu(t) = 2, \qquad \int_0^{2\pi} |d\mu(t)| \le k.$$

From Equation 2, we can write, for $p \in P_k(\gamma)$,

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \; p_1, p_2 \in P_2(\gamma) = P(\gamma), \qquad (3)$$

where $P(\gamma)$ is the class of functions with positive real part greater than γ.

By S, K, S^* and C, we denote the subclasses of A, which consist of univalent, close-to-convex, starlike and convex functions in E, respectively. The class A is closed under the convolution $*$ (or Hadamard product).

$$(f * g)(z) = \sum_{m=0}^{\infty} a_m b_m z^{m+1}, \qquad f, g \in A$$

where $f(z)=\sum_{m=0}^{\infty} a_m z^{m+1}, \qquad g(z)=\sum_{m=0}^{\infty} b_m z^{m+1}.$

For $f_\lambda(z)=\dfrac{z}{(1-z)^{\lambda+1}},\ \lambda\geq 0,$ we have chosen a suitable branch so that $f_\lambda \in A.$

Let $f\in A$ be given by Equation 1, with the properties that $a_m \neq 0$ for all m and $\lim_{m\to\infty}|a_m|^{\frac{1}{m}}=1.$ Then we can define $f_\lambda^{(-1)}$ as the unique and well-defined function in A such that

$$\left(f_\lambda * f_\lambda^{(-1)}\right)(z)=\frac{z}{1-z}, \quad z\in E. \qquad (4)$$

We define

$$I_\lambda f(z)=\left(f_\lambda^{(-1)} * f\right)(z)=\left[\frac{z}{(1-z)^{\lambda+1}}\right]^{(-1)} * f(z), \quad \lambda\geq 0.$$

Remark

We can use hypergeometric functions to define $I_\lambda f$ as follows. Since $(1-z)^{-a}={}_2F_1(1,1;a;z),$ where ${}_2F_1$ is the hyper geometric function, we have, for $a>1,$

$$\left[\frac{1}{(1-z)^a}\right]^{(-1)}=\left[{}_2F_1(1,1;a;z)\right]^{(-1)}..$$

Therefore

$$I_\lambda f(z)=\left[z\,{}_2F_1(1,1;\lambda+1,z)\right]^{-1} * f(z) \;=z\,{}_2F_1(1,\lambda+1;1,z)$$

Using Equation 4 and 5, we can easily derive,

$$(\lambda+1)I_\lambda f(z)-\lambda I_{\lambda+1}f(z)=z\left(I_{\lambda+1}f(z)\right)' \qquad (6)$$

Definition

Let $f\in A$ and let for $\left(\dfrac{I_{\lambda+1}f(z)}{z}\right)\neq 0, \quad \left(\dfrac{I_{\lambda+2}f(z)}{z}\right)\neq 0, \quad z\in E,$

$$f)=\alpha(\lambda+2)\left[\frac{I_{\lambda+1}f(z)}{I_{\lambda+2}f(z)}-\frac{\lambda+1}{\lambda+2}\right]+(1-\alpha)(\lambda+1)\left[\frac{I_\lambda f(z)}{I_{\lambda+1}f(z)}-\frac{\lambda}{\lambda+1}\right]. \qquad (7)$$

Then $f\in M_k^*(\alpha,\lambda,\gamma)$ if and only if $J^*(\alpha,\lambda,f)\in P_k(\gamma),$ for $k\geq 2,\ 0\leq\gamma<1$ and $z\in E.$ As special cases, we note the following:

1. $M_2^*(0,0,0)=C,\ M_2(0,1,0)=S^*,$ and $M_2^*(\alpha,1,\gamma)\subset M_\alpha\subset S,$ where M_α is the class of alpha-starlike functions (Goodman, 1983).
2. $M_k^*(0,1,\gamma)=V_k(\gamma)\subset V_k,$ where V_k is the well-known class of analytic functions with bounded boundary rotation and $M_k^*(0,0,0)=R_k,$ consists of the functions with bounded radius rotation (Goodman, 1983).
3. $M_k^*(\alpha,0,0)=M_k(\alpha),$ represents the class of functions with bounded Mocanu variation (Goodman, 1983).

For different values of parameters k,α and $\lambda,$ we obtain several other subclasses of analytic functions (Noor, 1995, 1999, 2007; Noor and Noor., 2003).

PRELIMINARY RESULTS

Lemma 1

Let $u=u_1+iu_2$, $v=v_1+iv_2$ and $\psi(u,v)$ be a complex-valued function satisfying the conditions (Miller, 1975):

1. $\psi(u,v)$ is continuous in a domain $D\subset C^2,$
2. $(1,0)\in D$ and $\psi(1,0)>0,$
3. $\operatorname{Re}\psi(iu_2,v_1)\leq 0$ whenever $(iu_2,v_1)\in D,$ and $v_1\leq -\dfrac{1}{2}\left(1+u_2{}^2\right)$

If $h(z)=1+\sum_{m=1}^{\infty}c_m z^m$ is a function, analytic in E such that $(h(z),zh'(z))\in D$ and $\operatorname{Re}\{\psi(h(z),zh'(z))\}>0$ for $z\in E,$ then $\operatorname{Re}h(z)>0$ in $E.$

Lemma 2

Let $\beta > 0,\ \beta + \delta > 0$ and $\gamma \in [\gamma_0, 1),$ (Miller et al., 2000) where $\gamma_0 = Max\left\{\dfrac{\beta - \delta - n}{2\beta},\ -\dfrac{\delta}{\beta}\right\}.$ If

$$\left\{p(z) + \frac{nzp'(z)}{\beta p(z) + \delta}\right\} \in P(\gamma), \quad (z \in E),$$

then $p \in P(\gamma_1),$ where

$$\gamma_1 = \frac{(\beta+\delta)}{\left\{{}_2F_1\left(\frac{2\beta}{n}(1-\gamma); 1; \frac{\beta+\delta+1}{n}; \frac{1}{2}\right)\right\}\beta} - \frac{\delta}{\beta}.$$

The value of γ_1 is best possible.

Lemma 3

Let $h(z)$ be analytic function in E with $h(0) = 1$ and $\operatorname{Re} h(z) > 0 \text{ in } E.$ Then, for $|z| = r, z \in E$:

1. $\dfrac{1-r}{1+r} \le \operatorname{Re} h(z) \le |h(z)| \le \dfrac{1+r}{1-r},$
2. $|h'(z)| \le \dfrac{2 \operatorname{Re} h(z)}{1 - r^2}.$

This result is well-known (Goodman, 1983).

Lemma 4

Let $h(z)$ be an analytic function in E with $h(0) = 1$ and $\operatorname{Re} h(z) > 0 \text{ in } E.$ Then, for $s > 0$ and $\nu \ne -1$ (complex), $\operatorname{Re}\left\{h(z) + \dfrac{szh'(z)}{h(z) + \nu}\right\} > 0$ for $|z| < r_0,$ where r_0 is given by:

$$r_0 = \frac{|\nu + 1|}{\sqrt{A + \left(A^2 + \left|\nu^2 - 1\right|^2\right)^{\frac{1}{2}}}}, \quad A = 2(s+1)^2 + |\nu|^2 - 1, \quad (8)$$

and this result is best possible (Ruscheweyh and Singh, 1976).

Lemma 5

Let $f \in A$ with $\dfrac{f(z) f'(z)}{z} \ne 0 \text{ in } E.$ Then $f(z)$ is univalent (Bazilevic) in E if and only if, for $0 \le \theta_1 < \theta_2 \le 2\pi$ and $0 < r < 1,$ we have

$$\int_{\theta_1}^{\theta_2} \left[\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} + (\beta - 1)\frac{zf'(z)}{f(z)}\right\} - \alpha \operatorname{Im} \frac{zf'(z)}{f(z)}\right] d\theta > -\pi, \quad z = re^{i\theta}, \beta > 0$$

and α is real (Shiel-Small, 1972).

Lemma 6

If ϕ is prestarlike of order $\beta \le 1, \quad g \in S^*,$ then for each analytic function $h,$ $\dfrac{\phi * hg}{\phi * g}(E) \subset \bar{C}_0\, h(E),$ where $\bar{C}_0\, h(E)$ denotes the closed convex hull of $h(E)$ (Ruscheweyh, 1982).

MAIN RESULTS

Theorem

For

$\alpha > 0, 0 \le \gamma < 1,\ \lambda \ge 0, k \ge 2,\ \gamma = \max\left[\dfrac{1 - \lambda - \alpha}{2}, -\lambda\right],$

$$M_k^*(\alpha, \lambda, \gamma) \subset M_k^*(0, \lambda, \gamma_1) = R_k(\lambda, \gamma_1),$$

where

$$\gamma_1 = \left[\frac{(1+\lambda)}{{}_2F_1\left(\frac{2}{\alpha}(1-\gamma); 1; \frac{1+\alpha+\lambda}{\alpha}; \frac{1}{2}\right)} - \lambda\right]. \quad (9)$$

The value of γ_1 is best possible.

Proof

Let $J^*(\alpha, \lambda, f)$ be defined by Equation 7. Then, using the identity of Equation 6, we have:

$$J^*(\alpha, \lambda, f) = \alpha \frac{z\left(I_{\lambda+1} f(z)\right)'}{I_{\lambda+1} f(z)} + (1-\alpha)\frac{z\left(I_\lambda f(z)\right)'}{I_\lambda f(z)}. \quad (10)$$

Set

$$z\frac{\left(I_\lambda f(z)\right)'}{I_\lambda f(z)}=H(z)=\left(\frac{k}{4}+\frac{1}{2}\right)H_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)H_2(z). \quad (11)$$

We note that $H(z)$ is analytic in E and $H(0)=1$. We want to show that $H\in P_k(\gamma_1)$ in E. Now, from Equations 6 and 10, we have:

$$J^*(\alpha,\lambda,f(z))=H(z)+\alpha\frac{zH'(z)}{H(z)+\lambda}. \quad (12)$$

Define

$$\phi_{\alpha,\lambda}(z)=\alpha\frac{z}{(1-z)^{\lambda+2}}+(1-\alpha)\frac{z}{(1-z)^{\lambda+1}} \quad (13)$$

Using convolution techniques, it follows that $\left(H*\frac{\phi_{\alpha,\lambda}}{z}\right)(z)=H(z)+\alpha\frac{zH'(z)}{H(z)+\lambda}$,
and so, from Equations 11 and 13 we have:

$$\begin{aligned}J^*(\alpha,\lambda,f(z))&=H(z)+\alpha\frac{zH'(z)}{H(z)+\lambda} \quad (14)\\&=\left(\frac{k}{4}+\frac{1}{2}\right)\left(H_1(z)+\alpha\frac{zH_1'(z)}{H_1(z)+\lambda}\right)-\left(\frac{k}{4}-\frac{1}{2}\right)\left(H_2(z)+\alpha\frac{zH_2'(z)}{H_2(z)+\lambda}\right)\end{aligned}$$

Since $f\in M_k^*(\alpha,\lambda,\gamma)$, it follows that $\left[H_i(z)+\alpha\frac{zH_i'(z)}{H_i(z)+\lambda}\right]\in P(\gamma),\ i=1,2,\ z\in E.$
Thus, from Lemma 2, it follows that $H_i\in P(\gamma_1)$ and consequently $H\in P_k(\gamma_1)$ in E where γ_1 is given by Equation 9. This completes the proof.

Theorem 2

Let $f\in M_k^*(\alpha,\lambda,\gamma)$. Then $I_\lambda f$ is univalent, if $k\le\frac{2(\alpha-\gamma+1)}{1-\gamma},\ \alpha>0,\ 0\le\gamma<1.$

Proof

Since $f\in M_k^*(\alpha,\lambda,\gamma)$, it follows that $J^*(\alpha,\lambda,f)\in P_k(\gamma), z\in E.$ Therefore, with $z=re^{i\theta},\ k\ge2$

$$\int_0^{2\pi}\left|\operatorname{Re}\frac{J^*(\alpha,\lambda,f)-\gamma}{1-\gamma}\right|d\theta\le k\pi,\qquad \int_0^{2\pi}\left|\operatorname{Re}\frac{J^*(\alpha,\lambda,f)-\gamma}{1-\gamma}\right|d\theta=2\pi,$$

and these together imply

$$\int_{\theta_1}^{\theta_2}\operatorname{Re}J^*(\alpha,\lambda,f)\,d\theta>-\left\{\frac{(1-\gamma)k+2(\gamma-1)}{2}\right\}\pi,\quad 0\le\theta_1<\theta_2\le2\pi,\quad 0\le\gamma<1.$$

This is equivalent to

$$\int_{\theta_1}^{\theta_2}\operatorname{Re}\left\{\frac{z\left(I_{\lambda+1}f(z)\right)'}{I_{\lambda+1}f(z)}+\left(\frac{1}{\alpha}-1\right)\frac{z\left(I_\lambda f(z)\right)'}{I_\lambda f(z)}\right\}d\theta>-\left\{\frac{(1-\gamma)k+2(\gamma-1)}{2}\right\}\pi,$$

now, the required result follows by using Lemma 5.

Theorem 3

Let $-1<\lambda_1<\lambda_2.$ Then $M_k^*(\alpha,\lambda_2,0)\subset M_k^*(\alpha,\lambda_1,0)$.

Proof

Define

$$\phi(z)=z+\sum_{m=2}^{\infty}\frac{(\lambda_1+1)(\lambda_1+2)\cdots(\lambda_1+m-1)}{(\lambda_2+1)(\lambda_2+2)\cdots(\lambda_2+m-1)}z^m,\quad z\in E.$$

Then $\phi\in A$ and, for $z\in E$,

$$\frac{z}{(1-z)^{\lambda_2+1}}*\phi(z)=\frac{z}{(1-z)^{\lambda_1+1}}\ (-1<\lambda_1<\lambda_2) \quad (15)$$

This implies $\frac{z}{(1-z)^{\lambda_2+1}}*\phi(z)\in S^*\left(\frac{1-\lambda_1}{2}\right)\subset C^*\left(\frac{1-\lambda_2}{2}\right)$,

and therefore, $\phi(z)$ is prestarlike of order $\left(\frac{1-\lambda_2}{2}\right)$. Now let $f\in M_k^*(\alpha,\lambda_2,0), z\in E.$ Writing $H(z)=\left(\frac{k}{4}+\frac{1}{2}\right)H_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)H_2(z)$, we have

$$\begin{aligned}J^*(\alpha,\lambda_1,f(z))&=\left\{\frac{\phi(z)}{z}\right\}*J^*(\alpha,\lambda_2,f(z))\\&=\left(\frac{k}{4}+\frac{1}{2}\right)\left(\frac{\phi(z)}{z}*H_1(z)\right)\\&\quad-\left(\frac{k}{4}-\frac{1}{2}\right)\left(\frac{\phi(z)}{z}*H_2(z)\right),\qquad H_i\in P,\quad i=1,2,\quad z\in E.\end{aligned}$$

Since $H_i\in P$, $H_i(0)=1$ and H_i is analytic in E for $i=1,2,$ there exist $s_i\in S*$ such that $H_i(z)=\frac{zs_i'(z)}{s_i(z)}.$

Therefore,

$$z\left(\frac{\phi}{z}*H_i\right) = z\left(\frac{\phi}{z}*\frac{zs_i'}{s_i}\right)$$

$$= \frac{\phi * \frac{zs_i'}{s_i} s_i}{\phi * s_i}$$

$$= \frac{\phi * H_i s_i}{\phi * s_i}, \quad s_i \in S*$$

Using Lemma 6, we note that $(\phi * H_i) \in P$, and this implies that $\phi * J^*(\alpha, \lambda_2, f) = J^*(\alpha, \lambda_1, f) \in P_k$ for $z \in E$, and therefore $f \in M_k^*(\alpha, \lambda_1, 0) \text{ in } E.$ This completes the proof.

Theorem 4

Let $f \in M_k^*(0, \lambda, \gamma) \text{ for } z \in E.$ Then, $f \in M_k^*(\alpha, \lambda, \gamma) \text{ for } |z| < r_0$, where r_0 is given by Equation 8 with $\nu = \frac{\lambda+\gamma}{1-\gamma}$, $s = \frac{\alpha}{1-\gamma}$ and $A = 2(s+1)^2 + |\nu|^2 - 1,$ and this radius is exact.

Proof

Let $H(z)$ be an analytic function as defined by Equation 11. Since $f \in M_k^*(0, \lambda, \gamma)$, it follows that $H_i \in P(\gamma), i = 1, 2.$ Therefore, with $H_i(z) = (1-\gamma)h_i(z) + \gamma,\ h_i \in P,\ i = 1, 2,$ we have for $z \in E$,

$$\frac{1}{1-\gamma}\left[J^*(\alpha, \lambda, f) - \gamma\right] = \left(\frac{k}{4}+\frac{1}{2}\right)\left(h_1(z) + \frac{\alpha z h_1'(z)}{(1-\gamma)h_1(z)+\lambda+\gamma}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(h_2(z) + \frac{\alpha z h_2'(z)}{(1-\gamma)h_2(z)+\lambda+\gamma}\right)$$

Now using Lemma2.4 with $\nu = \frac{\lambda+\gamma}{1-\gamma}(\gamma \neq -1),\ s = \frac{\alpha}{1-\gamma} > 0,$ we can see that

$\left(h_i + \frac{\alpha z h_i'}{(1-\gamma)h_i + \lambda + \gamma}\right) \in P, \quad i = 1, 2$, for $|z| < r_0$ and consequently $f \in M_k^*(\alpha, \lambda, \gamma)$, for $|z| < r_0$ where r_0 is given by Equation 8.

As a special case, we note that with $\alpha = 1,\ \gamma = 0 \text{ and } \lambda = 0,$ we have $f \in R_k$. Then, from Theorem 4, it follows that $f \in V_k$ for $|z| < r_0 = \frac{1}{\sqrt{7+\sqrt{48}}} \approx 0.268 \approx 2 - \sqrt{3}.$ When we choose $k = 2,$ it gives us the radius of convexity for starlike functions.

Theorem 5

For $0 \le \alpha_2 < \alpha_1,\quad M_k^*(\alpha_1, \lambda, \gamma) \subset M_k^*(\alpha_2, \lambda, \gamma)$.

Proof

For $\alpha_2 = 0,$ the proof is immediate from Theorem 1. Therefore we let $\alpha_2 > 0$ and $f \in M_k^*(\alpha_1, \lambda, \gamma).$ Then we can write,

$$J^*(\alpha_2, \lambda, f_1(z)) = \frac{\alpha_2}{\alpha_1} J^*(\alpha_1, \lambda, f(z)) + \left(1 - \frac{\alpha_2}{\alpha_1}\right)\frac{z(I_\lambda f(z))'}{I_\lambda f(z)}$$

$$= \frac{\alpha_2}{\alpha_1} H(z) + \left(1 - \frac{\alpha_2}{\alpha_1}\right) p(z),$$

where $H \in P_k(\gamma),$ since $f \in M_k^*(\alpha_1, \lambda, \gamma)$ and $p \in P_k(\gamma),$ by Theorem 1.

It is known (Noor, 1992), that $P_k(\gamma)$ is a convex set and this implies that $J^*(\alpha_2, \lambda, f(z)) \in P_k(\gamma), z \in E.$ This completes the proof.

Theorem 6

$M_k^*(0, n, 0) \subset M_k^*(0, n+1, \sigma),\ n \in N_0 = \{0, 1, 2, \cdots\}$ and σ is given as:

$$\sigma = \sigma_n = \frac{2}{(2n+1) + \sqrt{4n^2+4n+9}}. \qquad (16)$$

Proof

Set
$$z\frac{(I_{n+1}f(z))'}{I_{n+1}f(z)} = h(z) = \left(\frac{k}{4}+\frac{1}{2}\right)\{(1-\sigma)h_1(z)+\sigma\} - \left(\frac{k}{4}-\frac{1}{2}\right)\{(1-\sigma)h_2(z)+\sigma\}, \qquad (17)$$

where $h(z)$ is analytic in E and $h(0)=1.$ From Equation 17 and identity of Equation 6 with $\lambda=n,$ we have:

$$z\frac{\left(I_n f(z)\right)'}{I_n f(z)}=\left\{h(z)+\frac{zh'(z)}{h(z)+n}\right\}\in P_k(n), \qquad z\in E$$

with $\phi_n(z)=\frac{1}{2}\left[\frac{z}{(1-z)^{n+1}}+\frac{z}{(1-z)^{n+2}}\right],$ and using convolution technique, we note that

$$\left(h*\frac{\phi_n}{z}\right)(z)=h(z)+\frac{zh'(z)}{h(z)+n}. \qquad (18)$$

Therefore, from Equations 17 and 18, it follows that,

$$\left[(1-\sigma)h_i(z)+\sigma+\frac{(1-\sigma)zh_i'(z)}{(1-\sigma)h_i(z)+\sigma+n}\right]\in P, \quad i=1,2, \ z\in E.$$

We form the functional $\psi(u,v)$ by choosing $u=h_i(z), v=zh_i'(z).$ Thus,

$$\psi(u,v)=(1-\sigma)u+\sigma+\frac{(1-\sigma)v}{(1-\sigma)u+(\sigma+n)}.$$

The first two conditions of Lemma 1 are clearly satisfied. We verify the condition 3 as follows:

$$\begin{aligned}\operatorname{Re}\psi(iu_2,v_1)&=\sigma+\frac{(1-\sigma)(\sigma+n)v_1}{(\sigma+n)^2+(1-\sigma)^2u_2^2}\\ &\le\sigma-\frac{(1-\sigma)(\sigma+n)(1+u_2^2)}{2\left[(\sigma+n)^2+(1-\sigma)^2u_2^2\right]}, \qquad v_1\le-\frac{1}{2}(1+u_2^2)\\ &=\frac{A_1+B_1u_2^2}{2C},\end{aligned}$$

where

$$\begin{aligned}A_1&=2\sigma(\sigma+n)^2-(1-\sigma)(\sigma+n),\\ B_1&=2\sigma(1-\sigma)^2-(1-\sigma)(\sigma+n),\\ C&=(\sigma+n)^2+(1-\sigma)^2u_2^2>0.\end{aligned}$$

We note that $\operatorname{Re}\psi(iu_2,v_1)\le 0$ if and only if $A_1\le 0$ and $B_1\le 0.$ From $A_1\le 0,$ we obtain $\sigma=\sigma_n$ as given by Equation 16 and $B_1\le 0$ gives us $0<\sigma_n<1.$ Thus, applying Lemma 1, we have $h_i\in P$ for $z\in E,$ and consequently $h\in P_k(\sigma)$ in E where σ is given by Equation 16. This completes the proof.

For $n=0,$ we have a result proved in Noor et al. (2009) that $f\in V_k \Rightarrow f\in R_k\left(\frac{1}{2}\right)$in $E.$ The case, $n=0,$ and $k=2,$ gives us a well-known result that every convex function is starlike function of order $\frac{1}{2}.$

Theorem 7

Let $\phi\in C$ and $f\in M_2^*(0,\lambda,\gamma).$Then $(\phi*f)\in M_2^*(0,\lambda,\gamma)$ for $z\in E.$

Proof

Since $I_\lambda(\phi*f)=\phi*(I_\lambda f),$ we have,

$$\begin{aligned}\frac{z\left[I_\lambda(\phi*f)\right]'}{\left[I_\lambda(\phi*f)\right]}&=\frac{\phi*z(I_\lambda f)'}{\phi*(I_\lambda f)}=\frac{\phi*\frac{z(I_\lambda f)'}{I_\lambda f}I_\lambda f}{\phi*(I_\lambda f)}\\ &=\frac{\phi*F(I_\lambda f)}{\phi*(I_\lambda f)}, \quad F\in P(\gamma), \ z\in E\end{aligned}$$

We use Lemma 6 to obtain $z\frac{\left[I_\lambda(\phi*f)\right]'}{\left[I_\lambda(\phi*f)\right]}\in P(\gamma)$ in E and this implies that $(\phi*f)\in M_2^*(0,\lambda,\gamma),\ z\in E.$

We give some applications of Theorem 7 as follows:

Corollary 1: The classes $M_\alpha^*(0,\lambda,\gamma)$ are invariant under the following integral operators:

(*i*) $f_1(z)=\int_0^z\frac{f(t)}{t}dt,$ (*ii*) $f_2(z)=\frac{2}{z}\int_0^z f(t)dt,$ (Libera's operator)

(iii) $f_3(z)=\int_0^z\frac{f(t)-f(xt)}{t-xt}dt, |x|\le 1, x\ne 1,$ (*iv*) $f_4(z)=\frac{1+c}{z^c}\int_0^z t^{c-1}f(t)dt, \operatorname{Re}c>0$

Proof

Let,

$$\phi_1(z) = -\log(1-z), \qquad \phi_2(z) = -2\frac{[z+\log(1-z)]}{z}$$

$$\phi_3(z) = \frac{1}{1-x}\log\left(\frac{1-xz}{1-z}\right), \qquad \phi_4(z) = \sum_{m=1}^{\infty}\frac{1+c}{m+c}z^m, \ \operatorname{Re} c > 0.$$

It can easily be verified that ϕ_i is convex for each $i = 1,2,3,4.$ Now the proof follows immediately since we can write $f_i = f * \phi_i,\ i = 1,2,3,4.$

Definition for $n \in N_0,$

$$\begin{aligned} L_n(f) &= \frac{n+1}{z^n}\int_0^z t^{n-1} f(t)\,dt, \\ &= \left(z + \sum_{m=0}^{\infty}\frac{n+1}{n+m+1}z^m\right) * f(z) \\ &= \left(z\,{}_2F_1(1, n+1; n+2, z)\right) * f(z) \qquad (19) \\ &= \frac{z}{(1-z)^{n+1}} * \left[\frac{z}{(1-z)^{n+2}}\right]^{(-1)} * f(z) \\ &= f_n(z) * \left(f_{n+1}(z)\right)^{(-1)} * f(z). \end{aligned}$$

This shows that,

$$I_n L_n(f) = I_{n+1}(f) \qquad (20)$$

From Equations 19 and 20, we have the Theorem 8.

Theorem 8

Let
$f \in M_k^*(\alpha, n+1, \gamma),\ n \in N_0.$ Then $L_n(f) \in M_k^*(\alpha, n, \gamma)$ for $z \in E.$
We now prove the following radius problems as Theorem 9.

Theorem 9

Let $f \in M_k^*(0, n+1, 0),\ n \in N_0.$ Then $f \in M_k^*(0, n, 0),$ for $|z| < r_n$, where,

$$r_n = \frac{(n+1)}{2+\sqrt{n^2+3}} \qquad (21)$$

This result is sharp.

Proof

Let $z\dfrac{\left(I_{n+1}f(z)\right)'}{I_{n+1}f(z)} = H(z).$ Then $H \in P_k$ in $E.$ Using identity of Equation 6 with $\lambda = n,$ we have:

$$(n+1)\frac{I_n f(z)}{I_{n+1}f(z)} = H(z) + n \qquad (22)$$

Differentiating Equation 22 logarithmically and writing $H(z) = \left(\frac{k}{4}+\frac{1}{2}\right)H_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)H_2(z),$ we have:

$$\begin{aligned} z\frac{\left(I_n f(z)\right)'}{I_n f(z)} &= H(z) + \frac{zH'(z)}{H(z)+n} \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left(H_1(z) + \frac{zH_1'(z)}{H_1(z)+n}\right) \qquad (23) \\ &\quad - \left(\frac{k}{4}-\frac{1}{2}\right)\left(H_2(z) + \frac{zH_2'(z)}{H_2(z)+n}\right). \end{aligned}$$

For $i = 1,2$ and $H_i \in P$ in $E,$ we have:

$$\begin{aligned} \operatorname{Re}\left\{H_i(z) + \frac{zH_i'(z)}{H_i(z)+n}\right\} &\geq \operatorname{Re} H_i(z)\left\{1 - \frac{2r}{1-r^2}\cdot\frac{1}{\frac{1-r}{1+r}+n}\right\} \qquad (24) \\ &= \operatorname{Re} H_i(z)\left\{\frac{(1+n) - 4r + (1-n)r^2}{(1-r)^2 + n(1-r^2)}\right\}, \end{aligned}$$

where we have used Lemma 3. The right hand side of Equation 24 is positive for $|z| < r_n,$ and r_n is given by Equation 21. By taking $H_i(z) = \dfrac{1+z}{1-z},$ we see that value of r_n is exact. Hence, from Equation 23 and 24, it follows that $z\dfrac{\left(I_n f(z)\right)'}{I_n f(z)} \in P_k$ for $|z| < r_n$ and this completes the proof.

Conclusion

In this paper, we have used the convolution technique to introduce some new subclasses of analytic functions in the unit disc. We have obtained several results such as inclusions results and radius problems for these classes of analytic functions. We have also discussed some special cases of our results. These results may stimulate further research in this field.

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