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Inequalities for the Casorati curvatures of slant submanifolds in quaternionic space forms

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Abstract
In this paper we prove two sharp inequalities that relate the normalized scalar curvature with the Casorati curvature for a slant submanifold in a quaternionic space form. Moreover, we show that in both cases, the equality at all points characterizes the invariantly quasi-umbilical submanifolds.

Keywords: scalar curvature; mean curvature; Casorati curvature; shape operator; quaternionic space form; slant submanifold; optimal inequality

1 Introduction

In a seminal paper published in the early 1990s, Chen [1] established a sharp inequality for a submanifold in a real space form using the scalar curvature and the sectional curvature, both being intrinsic invariants, and squared mean curvature, the main extrinsic invariant, initiating the theory of δ-invariants or the so-called Chen invariants; this turned out to be one of the most interesting modern research topic in differential geometry of submanifolds. These inequalities were further extended to many classes of submanifolds in different ambient spaces (for an extensive and comprehensive survey on this topic see [2]). For example, in a quaternionic Kähler setting, Chen-like inequalities were proved in [3–11] and a set of open problems in the field was proposed recently in [12]. Moreover, new optimal inequalities involving δ-invariants were recently proved in [13–20]. We also note that some interesting inequalities for the length of the second fundamental form of the warped product submanifolds were obtained recently in [21–25].

On the other hand, the Casorati curvature of a submanifold in a Riemannian manifold is an extrinsic invariant defined as the normalized square of the length of the second fundamental form. It is well known that this notion extends the concept of the principal direction of a hypersurface of a Riemannian manifold to submanifolds of a Riemannian manifold and it was preferred by Casorati over the traditional Gauss curvature because corresponds better with the common intuition of curvature [26] (see also [27–30] for the geometrical meaning and the importance of the Casorati curvatures). Therefore it is of great interest to obtain optimal inequalities for the Casorati curvatures of submanifolds in different ambient spaces. We note that in [31], Decu, Haesen and Verstraelen obtained some optimal inequalities involving the scalar curvature and the Casorati curvature of a Riemannian submanifold in a real space form and the holomorphic sectional curvature and the Casorati curvature of a Kähler hypersurface in a complex space form. Moreover, the same authors proved in [32] an inequality in which the scalar curvature is estimated from above by the normalized Casorati curvatures, while Ghişoiu obtained in [33] some inequalities for the Casorati curvatures of slant submanifolds in complex space forms.

In this paper we generalize these inequalities in a quaternionic setting, proving the following result which also solves the Problem 6.7 from [12].

Theorem 1.1 *Let M^n be a θ-slant proper submanifold of a quaternionic space form $\overline{M}^{4m}(c)$. Then*:

(i) *The normalized δ-Casorati curvature $\delta_c(n-1)$ satisfies*

$$\rho \leq \delta_c(n-1) + \frac{c}{4}\left(1 + \frac{9}{n-1}\cos^2\theta\right). \tag{1}$$

Moreover, the equality sign holds if and only if M^n is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}^{4m}(c)$, such that with respect to suitable orthonormal tangent frame $\{\xi_1, \dots, \xi_n\}$ and normal orthonormal frame $\{\xi_{n+1}, \dots, \xi_{4m}\}$, the shape operators $A_r \equiv A_{\xi_r}$, $r \in \{n+1, \dots, 4m\}$, take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 & 0 \\ 0 & a & 0 & \cdots & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 \\ 0 & 0 & 0 & \cdots & 0 & 2a \end{pmatrix}, \qquad A_{n+2} = \cdots = A_{4m} = 0. \tag{2}$$

(ii) *The normalized δ-Casorati curvature $\hat{\delta}_c(n-1)$ satisfies*

$$\rho \leq \hat{\delta}_c(n-1) + \frac{c}{4}\left(1 + \frac{9}{n-1}\cos^2\theta\right). \tag{3}$$

Moreover, the equality sign holds if and only if M^n is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}^{4m}(c)$, such that with respect to suitable orthonormal tangent frame $\{\xi_1, \dots, \xi_n\}$ and normal orthonormal frame $\{\xi_{n+1}, \dots, \xi_{4m}\}$, the shape operators $A_r \equiv A_{\xi_r}$, $r \in \{n+1, \dots, 4m\}$, take the following forms:

$$A_{n+1} = \begin{pmatrix} 2a & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2a & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2a & 0 \\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix}, \qquad A_{n+2} = \cdots = A_{4m} = 0. \tag{4}$$

2 Preliminaries

2.1 Riemannian invariants

In this subsection we recall some basic concepts in Riemannian geometry, using mainly [34].

Let $(\overline{M},\overline{g})$ be an m-dimensional Riemannian manifold. For an n-dimensional Riemannian submanifold M of $(\overline{M},\overline{g})$, we denote by g the metric tensor induced on M. If $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$ and ∇ is the covariant differentiation induced on M, then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall X,Y \in \Gamma(TM)$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \quad \forall X \in \Gamma(TM), \forall N \in \Gamma\bigl(TM^{\perp}\bigr),$$

where h is the second fundamental form of M, $\nabla^{\perp}$ is the connection on the normal bundle and A_N is the shape operator of M with respect to N. The shape operator A_N is related to h by

$$g(A_N X, Y) = \overline{g}\bigl(h(X,Y),N\bigr)$$

for all $X,Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$.

If we denote by $\overline{R}$ and R the curvature tensor fields of $\overline{\nabla}$ and ∇, then we have the Gauss equation:

$$\begin{aligned}\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) &+ \overline{g}\bigl(h(X,W),h(Y,Z)\bigr)\\ &- \overline{g}\bigl(h(X,Z),h(Y,W)\bigr)\end{aligned} \tag{5}$$

for all $X,Y,Z,W \in \Gamma(TM)$.

We denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$. If $\{e_1,\ldots,e_n\}$ is an orthonormal basis of the tangent space T_pM and $\{e_{n+1},\ldots,e_m\}$ is an orthonormal basis of the normal space $T_p^{\perp}M$, then the scalar curvature τ at p is given by

$$\tau(p) = \sum_{1\le i<j\le n} K(e_i \wedge e_j)$$

and the normalized scalar curvature ρ of M is defined as

$$\rho = \frac{2\tau}{n(n-1)}.$$

We denote by H the mean curvature vector, that is,

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i)$$

and we also set

$$h_{ij}^{\alpha} = g\big(h(e_i, e_j), e_\alpha\big), \quad i, j \in \{1, \dots, n\}, \alpha \in \{n+1, \dots, m\}.$$

Then the squared mean curvature of the submanifold M in $\overline{M}$ is defined by

$$\|H\|^2 = \frac{1}{n^2} \sum_{\alpha=n+1}^{m} \left(\sum_{i=1}^{n} h_{ii}^{\alpha} \right)^2$$

and the squared norm of h over dimension n is denoted by $\mathcal{C}$ and is called the Casorati curvature of the submanifold M. Therefore we have

$$\mathcal{C} = \frac{1}{n} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{n} \big(h_{ij}^{\alpha}\big)^2.$$

The submanifold M is called totally geodesic if the second fundamental form vanishes identically and totally umbilical if there is a real number λ such that $h(X, Y) = \lambda g(X, Y)H$ for any tangent vectors X, Y on M. If $H = 0$, then the submanifold M is said to be minimal.

The submanifold M is called invariantly quasi-umbilical if there exist $m - n$ mutually orthogonal unit normal vectors $\xi_{n+1}, \dots, \xi_m$ such that the shape operators with respect to all directions ξ_α have an eigenvalue of multiplicity $n - 1$ and that for each ξ_α the distinguished eigendirection is the same [35].

Suppose now that L is an r-dimensional subspace of T_pM, $r \geq 2$ and let $\{e_1, \dots, e_r\}$ be an orthonormal basis of L. Then the scalar curvature $\tau(L)$ of the r-plane section L is given by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq r} K(e_\alpha \wedge e_\beta)$$

and the Casorati curvature $\mathcal{C}(L)$ of the subspace L is defined as

$$\mathcal{C}(L) = \frac{1}{r} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{r} \big(h_{ij}^{\alpha}\big)^2.$$

The normalized δ-Casorati curvature $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ are given by

$$\big[\delta_c(n-1)\big]_p = \frac{1}{2}\mathcal{C}_p + \frac{n+1}{2n(n-1)} \inf\big\{\mathcal{C}(L) | L \text{ a hyperplane of } T_pM\big\}$$

and

$$\big[\hat{\delta}_c(n-1)\big]_p = 2\mathcal{C}_p - \frac{2n-1}{2n} \sup\big\{\mathcal{C}(L) | L \text{ a hyperplane of } T_pM\big\}.$$

2.2 Quaternionic Kähler manifolds

We give in this subsection a quick review of basic definitions and properties of manifolds endowed with quaternionic Kähler structures, using mainly [36].

Let $\overline{M}$ be a differentiable manifold and assume that there is a rank 3-subbundle σ of $\mathrm{End}(T\overline{M})$ such that a local basis $\{J_1, J_2, J_3\}$ exists on sections of σ satisfying for all $\alpha \in$

$\{1,2,3\}$:

$$J_\alpha^2 = -\operatorname{Id}, \qquad J_\alpha J_{\alpha+1} = -J_{\alpha+1}J_\alpha = J_{\alpha+2},$$

where Id denotes the identity tensor field of type $(1,1)$ on M and the indices are taken from $\{1,2,3\}$ modulo 3. Then the bundle σ is called an almost quaternionic structure on M and $\{J_1,J_2,J_3\}$ is called a canonical local basis of σ. Moreover, $(\overline{M},\sigma)$ is said to be an almost quaternionic manifold. It is easy to see that any almost quaternionic manifold is of dimension $4m$, $m \geq 1$.

A Riemannian metric $\overline{g}$ on $\overline{M}$ is said to be adapted to the almost quaternionic structure σ if it satisfies

$$\overline{g}(J_\alpha X, J_\alpha Y) = \overline{g}(X,Y), \quad \forall \alpha \in \{1,2,3\}$$

for all vector fields X, Y on $\overline{M}$ and any canonical local basis $\{J_1,J_2,J_3\}$ of σ. Moreover, $(\overline{M},\sigma,\overline{g})$ is said to be an almost quaternionic Hermitian manifold.

If the bundle σ is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ of $\overline{g}$, then $(\overline{M},\sigma,\overline{g})$ is said to be a quaternionic Kähler manifold. Equivalently, locally defined 1-forms ω_1, ω_2, ω_3 exist such that we have for all $\alpha \in \{1,2,3\}$:

$$\overline{\nabla}_X J_\alpha = \omega_{\alpha+2}(X)J_{\alpha+1} - \omega_{\alpha+1}(X)J_{\alpha+2}$$

for any vector field X on $\overline{M}$, where the indices are taken from $\{1,2,3\}$ modulo 3.

We remark that any quaternionic Kähler manifold is an Einstein manifold, provided that $\dim M > 4$.

Let $(\overline{M},\sigma,\overline{g})$ be a quaternionic Kähler manifold and let X be a non-null vector field on $\overline{M}$. Then the 4-plane spanned by $\{X, J_1X, J_2X, J_3X\}$, denoted by $Q(X)$, is called a quaternionic 4-plane. Any 2-plane in $Q(X)$ is called a quaternionic plane. The sectional curvature of a quaternionic plane is called a quaternionic sectional curvature. A quaternionic Kähler manifold is a quaternionic space form if its quaternionic sectional curvatures are equal to a constant, say c. It is well known that a quaternionic Kähler manifold $(\overline{M},\sigma,\overline{g})$ is a quaternionic space form, denoted $\overline{M}(c)$, if and only if its curvature tensor is given by

$$\begin{aligned}\overline{R}(X,Y)Z = \frac{c}{4}\Bigg\{&\overline{g}(Z,Y)X - \overline{g}(X,Z)Y + \sum_{\alpha=1}^{3}\big[\overline{g}(Z,J_\alpha Y)J_\alpha X - \\ &- \overline{g}(Z,J_\alpha X)J_\alpha Y + 2\overline{g}(X,J_\alpha Y)J_\alpha Z\big]\Bigg\}\end{aligned} \tag{6}$$

for all vector fields X, Y, Z on $\overline{M}$ and any local basis $\{J_1,J_2,J_3\}$ of σ.

A submanifold M in a quaternionic Kähler manifold $(\overline{M},\sigma,\overline{g})$ is called a quaternionic submanifold [37] (resp. a totally real submanifold [38]) if each tangent space of M is carried into itself (resp. into the normal space) by each section in σ. In [39], the author introduced the concept of slant submanifolds as a natural generalization of both quaternionic and totally real submanifolds. A submanifold M of a quaternionic Kähler manifold $(\overline{M},\sigma,\overline{g})$ is said to be a slant submanifold if for each non-zero vector X tangent to M at

p, the angle $\theta(X)$ between $J_\alpha(X)$ and T_pM, $\alpha \in \{1,2,3\}$ is constant, *i.e.* it does not depend on the choice of $p \in M$ and $X \in T_pM$. We can easily see that quaternionic submanifolds are slant submanifolds with $\theta = 0$ and totally real submanifolds are slant submanifolds with $\theta = \frac{\pi}{2}$. A slant submanifold of a quaternionic Käler manifold is said to be proper (or θ-slant proper) if it is neither quaternionic nor totally real. We note that another natural generalization of both quaternionic and totally real submanifolds in a quaternionic Kähler manifold is given by quaternionic CR-submanifolds. A submanifold M of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$ is said to be a quaternionic CR-submanifold if there exist two orthogonal complementary distributions D and $D^\perp$ on M such that D is invariant under quaternionic structure and $D^\perp$ is totally real (see [40]). It is clear that, although quaternionic CR-submanifolds are also the generalization of both quaternionic and totally real submanifolds, there exists no inclusion between the two classes of quaternionic CR-submanifolds and slant submanifolds.

We also note that we have the next characterization of slant submanifolds in quaternionic Kähler manifolds.

Theorem 2.1 [39] *Let M be a submanifold of a quaternionic Kähler manifold $\overline{M}$. Then M is slant if and only if there exists a constant $\lambda \in [-1, 0]$ such that*

$$P_\beta P_\alpha X = \lambda X, \quad \forall X \in \Gamma(TM), \alpha, \beta \in \{1,2,3\}, \tag{7}$$

where $P_\alpha X$ denote the tangential component of $J_\alpha X$. Furthermore, in such a case, if θ is the slant angle of M, then it satisfies $\lambda = -\cos^2\theta$.

From the above theorem it follows easily that

$$g(P_\alpha X, P_\beta Y) = \cos^2\theta g(X, Y) \tag{8}$$

for $X, Y \in \Gamma(TM)$ and $\alpha, \beta \in \{1,2,3\}$.

Moreover, every proper slant submanifold of a quaternionic Kähler manifold is of even dimension $n = 2s \geq 2$ and we can choose a canonical orthonormal local frame, called an adapted slant frame, as follows: $\{e_1, e_2 = \sec\theta P_\alpha e_1, \ldots, e_{2s-1}, e_{2s} = \sec\theta P_\alpha e_{2s-1}\}$, where α is 1, 2 or 3 (see [41]).

3 Proof of Theorem 1.1

(i) Since $\overline{M}^{4m}(c)$ is a quaternionic space form, from (6) and Gauss equation (5) we can easily obtain:

$$n^2\|H\|^2 = 2\tau(p) + \|h\|^2 - \frac{n(n-1)c}{4} - \frac{3c}{4}\sum_{\beta=1}^{3}\sum_{i,j=1}^{n} g^2(P_\beta e_i, e_j). \tag{9}$$

Choosing now an adapted slant basis

$$\{e_1, e_2 = \sec\theta P_\alpha e_1, \ldots, e_{2s-1}, e_{2s} = \sec\theta P_\alpha e_{2s-1}\}$$

of T_pM, $p \in M$, where $2s = n$ and making use of (7) and (8), we derive

$$g^2(P_\beta e_i, e_{i+1}) = g^2(P_\beta e_{i+1}, e_i) = \cos^2\theta \quad \text{for } i = 1, 3, \ldots, 2s-1 \tag{10}$$

and

$$g(P_\beta e_i, e_j) = 0 \quad \text{for } (i,j) \notin \bigl\{(2l-1, 2l), (2l, 2l-1) | l \in \{1, 2, \ldots, s\}\bigr\}. \tag{11}$$

From (9), (10), and (11) we deduce that

$$2\tau(p) = n^2\|H\|^2 - n\mathcal{C} + \frac{c}{4}\bigl[n(n-1) + 9n\cos^2\theta\bigr]. \tag{12}$$

We define now the following function, denoted by $\mathcal{P}$, which is a quadratic polynomial in the components of the second fundamental form:

$$\mathcal{P} = \frac{1}{2}n(n-1)\mathcal{C} + \frac{1}{2}(n+1)\mathcal{C}(L) - 2\tau(p) + \frac{c}{4}\bigl[n(n-1) + 9n\cos^2\theta\bigr],$$

where L is a hyperplane of T_pM. We can assume without loss of generality that L is spanned by $e_1, \ldots, e_{n-1}$. Then we have

$$\begin{aligned}\mathcal{P} &= \frac{n-1}{2}\sum_{\alpha=n+1}^{4m}\sum_{i,j=1}^{n}\bigl(h_{ij}^\alpha\bigr)^2 + \frac{n+1}{2(n-1)}\sum_{\alpha=n+1}^{4m}\sum_{i,j=1}^{n-1}\bigl(h_{ij}^\alpha\bigr)^2 \\ &\quad - 2\tau(p) + \frac{c}{4}\bigl[n(n-1) + 9n\cos^2\theta\bigr].\end{aligned} \tag{13}$$

From (12) and (13), we derive

$$\begin{aligned}\mathcal{P} &= \frac{n+1}{2}\sum_{\alpha=n+1}^{4m}\sum_{i,j=1}^{n}\bigl(h_{ij}^\alpha\bigr)^2 \\ &\quad + \frac{n+1}{2(n-1)}\sum_{\alpha=n+1}^{4m}\sum_{i,j=1}^{n-1}\bigl(h_{ij}^\alpha\bigr)^2 - \sum_{\alpha=n+1}^{4m}\left(\sum_{i=1}^{n}h_{ii}^\alpha\right)^2\end{aligned}$$

and now we obtain easily that

$$\begin{aligned}\mathcal{P} &= \sum_{\alpha=n+1}^{4m}\sum_{i=1}^{n-1}\left[\frac{n^2-n+2}{2(n-1)}\bigl(h_{ii}^\alpha\bigr)^2 + (n+1)\bigl(h_{in}^\alpha\bigr)^2\right] \\ &\quad + \sum_{\alpha=n+1}^{4m}\left[\frac{n(n+1)}{n-1}\sum_{i<j=1}^{n-1}\bigl(h_{ij}^\alpha\bigr)^2 - 2\sum_{i<j=1}^{n}h_{ii}^\alpha h_{jj}^\alpha + \frac{n-1}{2}\bigl(h_{nn}^\alpha\bigr)^2\right].\end{aligned} \tag{14}$$

From (14) it follows that the critical points

$$h^c = \bigl(h_{11}^{n+1}, h_{12}^{n+1}, \ldots, h_{nn}^{n+1}, \ldots, h_{11}^{4m}, h_{12}^{4m}, \ldots, h_{nn}^{4m}\bigr)$$

of $\mathcal{P}$ are the solutions of the following system of linear homogeneous equations:

$$\begin{cases}\frac{\partial\mathcal{P}}{\partial h_{ii}^\alpha} = \frac{n(n+1)}{n-1}h_{ii}^\alpha - 2\sum_{k=1}^{n}h_{kk}^\alpha = 0, \\ \frac{\partial\mathcal{P}}{\partial h_{nn}^\alpha} = (n-1)h_{nn}^\alpha - 2\sum_{k=1}^{n-1}h_{kk}^\alpha = 0, \\ \frac{\partial\mathcal{P}}{\partial h_{ij}^\alpha} = \frac{2n(n+1)}{n-1}h_{ij}^\alpha = 0, \\ \frac{\partial\mathcal{P}}{\partial h_{in}^\alpha} = 2(n+1)h_{in}^\alpha = 0,\end{cases} \tag{15}$$

with $i, j \in \{1, \ldots, n-1\}$, $i \neq j$, and $\alpha \in \{n+1, \ldots, 4m\}$.

From (15) it follows that every solutions h^c has $h_{ij}^\alpha = 0$ for $i \neq j$ and the determinant which corresponds to the first two sets of equations of the above system is zero (there exist solutions for non-totally geodesic submanifolds). Moreover, the Hessian matrix of $\mathcal{P}$ has the eigenvalues

$$\lambda_{11} = 0, \qquad \lambda_{22} = \frac{2n^2 - 5n + 5}{n-1}, \qquad \lambda_{33} = \cdots = \lambda_{nn} = \frac{n(n+1)}{n-1},$$
$$\lambda_{ij} = \frac{2n(n+1)}{n-1}, \qquad \lambda_{in} = 2(n+1), \quad \forall i,j \in \{1,\ldots,n-1\}, i \neq j.$$

Thus, it follows that $\mathcal{P}$ is parabolic and reaches a minimum $\mathcal{P}(h^c)$ for each solution h^c of the system (15). But inserting (15) in (14) we obtain $\mathcal{P}(h^c) = 0$. So $\mathcal{P} \geq 0$, and this implies

$$2\tau(p) \leq \frac{1}{2}n(n-1)\mathcal{C} + \frac{1}{2}(n+1)\mathcal{C}(L) + \frac{c}{4}\big[n(n-1) + 9n\cos^2\theta\big].$$

Hence we deduce that

$$\rho \leq \frac{1}{2}\mathcal{C} + \frac{n+1}{2n(n-1)}\mathcal{C}(L) + \frac{c}{4}\left[1 + \frac{9}{n-1}\cos^2\theta\right]$$

for every tangent hyperplane L of M. Taking now the infimum over all tangent hyperplane L we obtain (1).

Moreover, we can easily see now that the equality sign holds in the inequality (1) if and only if

$$h_{ij}^\alpha = 0, \quad \forall i,j \in \{1,\ldots,n\}, i \neq j \text{ and } \alpha \in \{n+1,\ldots,4m\} \tag{16}$$

and

$$h_{nn}^\alpha = 2h_{11}^\alpha = 2h_{22}^\alpha = \cdots = 2h_{n-1,n-1}^\alpha, \quad \forall \alpha \in \{n+1,\ldots,4m\}. \tag{17}$$

From (16) and (17) we conclude that the equality sign holds in the inequality (1) if and only if the submanifold M is invariantly quasi-umbilical with trivial normal connection in $\overline{M}$, such that with respect to suitable orthonormal tangent and normal orthonormal frames, the shape operators take the forms (2).

(ii) can be proved in a similar way, considering the following quadratic polynomial in the components of the second fundamental form:

$$\mathcal{Q} = 2n(n-1)\mathcal{C} - \frac{1}{2}\big(2n - 1(n-1)\big)\mathcal{C}(L) - 2\tau(p) + \frac{c}{4}\big[n(n-1) + 9n\cos^2\theta\big],$$

where L is a hyperplane of T_pM.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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