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Some means inequalities for positive operators in Hilbert spaces

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Abstract

In this paper, we obtain two refinements of the ordering relations among Heinz means with different parameters via the Taylor series of some hyperbolic functions and by the way, we derive new generalizations of Heinz operator inequalities. Moreover, we establish a matrix version of Heinz inequality for the Hilbert-Schmidt norm. Finally, we introduce a weighted multivariate geometric mean and show that the weighted multivariate operator geometric mean possess several attractive properties and means inequalities.

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Keywords: means inequalities; positive linear operators; Hilbert space

1 Introduction

Since Heinz proved a series of useful norm inequalities, which are closely related to the Cordes inequality and the Furuta inequality, in 1951, many researchers have devoted themselves to sharpening the Heinz inequalities and extending the Heinz norm inequalities to more general cases with the help of a Bernstein type inequality for nonselfadjoint operators, the convexity of norm functions, the Jensen functional and its properties, the Hermite-Hadamard inequality, and so on. With this kind of research, the study of various means inequalities, such as the geometric mean, the arithmetic mean, the Heinz mean, arithmetic-geometric means, and Arithmetic-Geometric-Harmonic (A-G-H) weighted means, has received much attention and development too. For recent interesting work in this area, we refer the reader to [1–10] and references therein.

Based on [1–8], in this paper, we are concerned with the further refinements of the geometric mean and the Heinz mean for operators in Hilbert spaces. Our purpose is to derive some new generalizations of Heinz operator inequalities by refining the ordering relations among Heinz means with different parameters, and of the geometric mean by investigating geometric means of several operator variables in a weighted setting. Moreover, we will obtain a matrix version of the Heinz inequality for the Hilbert-Schmidt norm.

Throughout this paper, $\mathfrak{B}^{++}(\mathcal{H})$ stands for the set of all bounded positive invertible operators on a Hilbert space \mathcal{H} , $\mathfrak{B}(\mathcal{H})$ is the set of all bounded linear operators on \mathcal{H} , and $\mathfrak{B}(\mathcal{H})_{sa}$ is a convex domain of selfadjoint operators in $\mathfrak{B}(\mathcal{H})$. For any $T, S \in \mathfrak{B}^{++}(\mathcal{H})$ and $\nu \in [0, 1]$, we write

$$T \nabla_{\nu} S := (1 - \nu)T + \nu S$$

and

$$T\sharp_{\nu}S := T^{\frac{1}{2}}(T^{-\frac{1}{2}}ST^{-\frac{1}{2}})^{\nu}T^{\frac{1}{2}}$$

as the geometric mean of T and S . When $\nu = 1/2$ we write $T\nabla S$ and $T\sharp S$ in short, respectively. We refer the reader to Kubo and Ando [6] for more information on the means of positive linear operators.

Recall that, for any $a, b \geq 0$, the number

$$H_{\nu}(a, b) = \frac{a^{1-\nu}b^{\nu} + a^{\nu}b^{1-\nu}}{2}, \quad 0 \leq \nu \leq 1,$$

is called the Heinz mean $H_{\nu}(a, b)$ of a and b . It is clear that

$$\begin{aligned} H_0(a, b) &= H_1(a, b) = \frac{a+b}{2}, & H_{1/2}(a, b) &= \sqrt{ab}, \\ H_{\nu}(a, b) &= H_{1-\nu}(a, b), & \nu &\in [0, 1], \\ \sqrt{ab} &\leq H_{\nu}(a, b) \leq \frac{a+b}{2}, & \nu &\in [0, 1]. \end{aligned}$$

For any $T, S \in \mathfrak{B}^{++}(\mathcal{H})$ and $\nu \in [0, 1]$, the operator

$$H_{\nu}(T, S) = \frac{T\sharp_{\nu}S + T\sharp_{1-\nu}S}{2}$$

is called the Heinz operator mean of T and S . Clearly,

$$T\sharp S \leq H_{\nu}(T, S) \leq T\nabla S,$$

that is, the Heinz operator mean interpolates between the geometric mean and the arithmetic mean.

2 Improved Heinz means inequalities

In a very recent work [8], we establish the following inequalities:

$$H_t(a, b) \leq \left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right)a^{\frac{1}{2}}b^{\frac{1}{2}} + \frac{(1-2t)^2}{(1-2s)^2}H_s(a, b) \tag{2.1}$$

and

$$H_t(T, S) \leq \left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right)T\sharp S + \frac{(1-2t)^2}{(1-2s)^2}H_s(T, S)$$

hold for $s, t \in [0, 1]$ satisfying $|s - \frac{1}{2}| \geq |t - \frac{1}{2}|, s \neq \frac{1}{2}$.

In this section, we improve the result and give two theorems as follows.

Theorem 2.1 *Suppose $T, S \in \mathfrak{B}^{++}(\mathcal{H})$, and let $s, t \in [0, 1]$ satisfy*

$$s \neq \frac{1}{2}, \quad \left|s - \frac{1}{2}\right| \geq \left|t - \frac{1}{2}\right|.$$

Then

$$\begin{aligned} & \left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right) T_{\sharp}^{\#} S + \frac{(1-2t)^2}{(1-2s)^2} H_s(T, S) \\ & \leq (1 - (1-2t)^2) T_{\sharp}^{\#} S + (1-2t)^2 T \nabla S. \end{aligned} \tag{2.2}$$

Proof Writing

$$\alpha = 1 - 2s, \quad \beta = 1 - 2t,$$

we see

$$\alpha \neq 0, \quad 0 \leq |\beta| \leq |\alpha| \leq 1.$$

In view of the Taylor series of $\cosh x$, we deduce that

$$\begin{aligned} & (1 - \beta^2) + \beta^2 \cosh x - \left[\left(1 - \frac{\beta^2}{\alpha^2}\right) + \frac{\beta^2}{\alpha^2} \cosh \alpha x \right] \\ & = \frac{\beta^2}{\alpha^2} (1 - \alpha^2 + \alpha^2 \cosh x - \cosh \alpha x) \\ & = \frac{\beta^2}{\alpha^2} \left[1 - \alpha^2 + \left(\alpha^2 + \frac{\alpha^2 x^2}{2!} + \frac{\alpha^2 x^4}{4!} + \frac{\alpha^2 x^6}{6!} + \dots \right) \right. \\ & \quad \left. - \left(1 + \frac{\alpha^2 x^2}{2!} + \frac{\alpha^4 x^4}{4!} + \frac{\alpha^6 x^6}{6!} + \dots \right) \right] \\ & = \frac{\beta^2}{\alpha^2} \left[\left(\frac{\alpha^2 x^4}{4!} + \frac{\alpha^2 x^6}{6!} + \dots \right) - \left(\frac{\alpha^4 x^4}{4!} + \frac{\alpha^6 x^6}{6!} + \dots \right) \right] \\ & \geq 0, \end{aligned}$$

i.e.,

$$\left(1 - \frac{\beta^2}{\alpha^2}\right) + \frac{\beta^2}{\alpha^2} \cosh \alpha x \leq (1 - \beta^2) + \beta^2 \cosh x.$$

With $\frac{x-y}{2}$ instead of x , we have

$$\begin{aligned} & \left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right) + \frac{(1-2t)^2}{(1-2s)^2} \cosh \left((1-2s) \frac{x-y}{2} \right) \\ & \leq (1 - (1-2t)^2) + (1-2t)^2 \cosh \left(\frac{x-y}{2} \right). \end{aligned}$$

Let $a = e^x, b = e^y$. Then

$$\begin{aligned} & \left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right) \sqrt{ab} + \frac{(1-2t)^2}{(1-2s)^2} H_s(a, b) \\ & \leq (1 - (1-2t)^2) \sqrt{ab} + (1-2t)^2 \frac{a+b}{2}. \end{aligned} \tag{2.3}$$

Taking $a = x$ and $b = 1$ in the inequality (2.2), we get

$$\left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right)x^{1/2} + \frac{(1-2t)^2}{(1-2s)^2} \left(\frac{x^{1-s} + x^s}{2}\right) \leq (1 - (1-2t)^2)x^{1/2} + (1-2t)^2 \frac{x+1}{2}.$$

With the positive operator $T^{-\frac{1}{2}}ST^{-\frac{1}{2}}$ instead of x , we have

$$\left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right)T\sharp S + \frac{(1-2t)^2}{(1-2s)^2}H_s(T, S) \leq (1 - (1-2t)^2)T\sharp S + (1-2t)^2 T\nabla S.$$

The proof is completed. □

For the functions $F_\nu : \mathbb{R}_+ \rightarrow \mathbb{R}$ ($\nu \in [0, 1]$) defined by

$$F_\nu(x) = \begin{cases} \frac{x^\nu - x^{1-\nu}}{\log x}, & x > 0, x \neq 1, \\ 2\nu - 1, & x = 1, \end{cases}$$

we have the following result.

Theorem 2.2 *Suppose $T, S \in \mathfrak{B}^{++}(\mathcal{H})$ and let $s, t \in [0, 1]$ satisfy*

$$s, t \neq \frac{1}{2}, \quad \left|s - \frac{1}{2}\right| \geq \left|t - \frac{1}{2}\right|.$$

Then

$$\begin{aligned} & \frac{1}{2t-1} T^{\frac{1}{2}} F_t(T^{-\frac{1}{2}}ST^{-\frac{1}{2}}) T^{\frac{1}{2}} \\ & \leq \left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right) T\sharp S + \frac{(1-2t)^2}{(1-2s)^2} \frac{1}{2s-1} T^{\frac{1}{2}} F_s(T^{-\frac{1}{2}}ST^{-\frac{1}{2}}) T^{\frac{1}{2}}. \end{aligned}$$

Proof Writing $\alpha = 1 - 2s$, $\beta = 1 - 2t$, we have

$$\alpha \neq 0, \quad \beta \neq 0, \quad 0 \leq |\beta| \leq |\alpha| \leq 1.$$

It follows from the Taylor series of $\sinh x$ that

$$\begin{aligned} \frac{\sinh \beta x}{\beta x} &= 1 + \frac{\beta^2 x^2}{3!} + \frac{\beta^4 x^4}{5!} + \frac{\beta^6 x^6}{7!} + \dots \\ &\leq 1 - \frac{\beta^2}{\alpha^2} + \frac{\beta^2}{\alpha^2} + \frac{\beta^2}{\alpha^2} \left(\frac{\alpha^2 x^2}{3!} + \frac{\alpha^4 x^4}{5!} + \frac{\alpha^6 x^6}{7!} + \dots\right) \\ &\leq 1 - \frac{\beta^2}{\alpha^2} + \frac{\beta^2}{\alpha^2} \left(1 + \frac{\alpha^2 x^2}{3!} + \frac{\alpha^4 x^4}{5!} + \frac{\alpha^6 x^6}{7!} + \dots\right) \\ &= \left(1 - \frac{\beta^2}{\alpha^2}\right) + \frac{\beta^2}{\alpha^2} \left(\frac{\sinh \alpha x}{\alpha x}\right). \end{aligned}$$

With $\frac{x-y}{2}$ instead of x , we know that, for any $s, t \in [0, 1]$ satisfying $|s - \frac{1}{2}| \geq |t - \frac{1}{2}|$, $s, t \neq \frac{1}{2}$,

$$\frac{\sinh((1-2t)(\frac{x-y}{2}))}{(1-2t)(\frac{x-y}{2})} \leq \left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right) + \frac{(1-2t)^2}{(1-2s)^2} \left(\frac{\sinh((1-2s)(\frac{x-y}{2}))}{(1-2s)(\frac{x-y}{2})}\right).$$

Put $a = e^x, b = e^y$. Then we get

$$\begin{aligned} & \frac{a^{1-t}b^t - a^t b^{1-t}}{(1-2t)(\log a - \log b)} \\ & \leq \left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right) a^{\frac{1}{2}} b^{\frac{1}{2}} + \frac{(1-2t)^2}{(1-2s)^2} \left(\frac{a^{1-t}b^t - a^t b^{1-t}}{(1-2t)(\log a - \log b)}\right). \end{aligned} \tag{2.4}$$

Letting $a = x$ and $b = 1$ in inequality (2.3), we see that

$$\frac{x^t - x^{1-t}}{(2t-1)\log x} \leq \left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right) x^{\frac{1}{2}} + \frac{(1-2t)^2}{(1-2s)^2} \left(\frac{x^t - x^{1-t}}{(2t-1)\log x}\right),$$

that is,

$$\frac{1}{2t-1} F_t(x) \leq \left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right) x^{\frac{1}{2}} + \frac{(1-2t)^2}{(1-2s)^2} \frac{1}{2s-1} F_s(x).$$

With the positive operator $T^{-\frac{1}{2}}ST^{-\frac{1}{2}}$ instead of x , we have

$$\begin{aligned} & \frac{1}{2t-1} F_t(T^{-\frac{1}{2}}ST^{-\frac{1}{2}}) \\ & \leq \left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right) (T^{-\frac{1}{2}}ST^{-\frac{1}{2}})^{\frac{1}{2}} + \frac{(1-2t)^2}{(1-2s)^2} \frac{1}{2s-1} F_s(T^{-\frac{1}{2}}ST^{-\frac{1}{2}}). \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{1}{2t-1} T^{\frac{1}{2}} F_t(T^{-\frac{1}{2}}ST^{-\frac{1}{2}}) T^{\frac{1}{2}} \\ & \leq \left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right) T^{\sharp} S + \frac{(1-2t)^2}{(1-2s)^2} \frac{1}{2s-1} T^{\frac{1}{2}} F_s(T^{-\frac{1}{2}}ST^{-\frac{1}{2}}) T^{\frac{1}{2}}. \end{aligned}$$

□

3 Heinz inequality for the Hilbert-Schmidt norm

In this section, we let M_n be the Hilbert space of $n \times n$ complex matrices and let $\|\cdot\|$ stand for any unitarily invariant norm on M_n , i.e. $\|UTV\| = \|T\|$ for all $T \in M_n$ and for all unitary matrices $U, V \in M_n$. We suppose that $T, S, X \in M_n$ with T and S being positive semidefinite. For $T = [a_{ij}] \in M_n$, the Hilbert-Schmidt norm of T is defined by

$$\|T\|_2 = \left(\sum_{ij=1}^n |a_{ij}|^2\right)^{1/2}.$$

It is well known that the Hilbert-Schmidt norm is unitarily invariant.

Next, we prove the following matrix version of Heinz inequality for the Hilbert-Schmidt norm.

Theorem 3.1 *Let $s, t \in [0, 1]$ satisfy*

$$s \neq \frac{1}{2}, \quad \left|s - \frac{1}{2}\right| \geq \left|t - \frac{1}{2}\right|.$$

Then

$$\begin{aligned} & \left\| \frac{T^t X S^{1-t} + T^{1-t} X S^t}{2} \right\|_2 \\ & \leq \left\| \left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right) T^{1/2} X S^{1/2} + \frac{(1-2t)^2}{(1-2s)^2} \frac{T^s X S^{1-s} + T^{1-s} X S^s}{2} \right\|_2 \\ & \leq \left\| (1 - (1-2t)^2) T^{1/2} X S^{1/2} + (1-2t)^2 \frac{TX + XS}{2} \right\|_2. \end{aligned}$$

Proof Noting that T and S are positive semidefinite, we know by the spectral theorem that there exist unitary matrices $U, V \in M_n$ such that

$$T = U \Lambda_1 U^* \quad \text{and} \quad S = V \Lambda_2 V^*,$$

where

$$\Lambda_1 = \text{diag}(\lambda_1, \dots, \lambda_n), \quad \Lambda_2 = \text{diag}(\mu_1, \dots, \mu_n), \quad \lambda_i, \mu_i \geq 0, i = 1, \dots, n.$$

Put

$$Y = U^* X V = [y_{ij}].$$

Then we have

$$\begin{aligned} \frac{T^t X S^{1-t} + T^{1-t} X S^t}{2} &= \frac{(U \Lambda_1 U^*)^t X (V \Lambda_2 V^*)^{1-t} + (U \Lambda_1 U^*)^{1-t} X (V \Lambda_2 V^*)^t}{2} \\ &= \frac{(U \Lambda_1^t U^*) X (V \Lambda_2^{1-t} V^*) + (U \Lambda_1^{1-t} U^*) X (V \Lambda_2^t V^*)}{2} \\ &= \frac{U \Lambda_1^t (U^* X V) \Lambda_2^{1-t} V^* + U \Lambda_1^{1-t} (U^* X V) \Lambda_2^t V^*}{2} \\ &= U \left(\frac{\Lambda_1^t Y \Lambda_2^{1-t} + \Lambda_1^{1-t} Y \Lambda_2^t}{2} \right) V^*. \end{aligned}$$

Hence,

$$\begin{aligned} \left\| \frac{T^t X S^{1-t} + T^{1-t} X S^t}{2} \right\|_2^2 &= \left\| \frac{\Lambda_1^t Y \Lambda_2^{1-t} + \Lambda_1^{1-t} Y \Lambda_2^t}{2} \right\|_2^2 \\ &= \sum_{i,j=1}^n \left(\frac{\lambda_i^t \mu_j^{1-t} + \lambda_i^{1-t} \mu_j^t}{2} \right)^2 |y_{ij}|^2. \end{aligned}$$

By a similar argument to the above, we deduce that

$$\begin{aligned} & \left\| \left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right) T^{1/2} X S^{1/2} + \frac{(1-2t)^2}{(1-2s)^2} \frac{T^s X S^{1-s} + T^{1-s} X S^s}{2} \right\|_2^2 \\ &= \sum_{i,j=1}^n \left(\left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right) \sqrt{\lambda_i \mu_j} + \frac{(1-2t)^2}{(1-2s)^2} \frac{\lambda_i^s \mu_j^{1-s} + \lambda_i^{1-s} \mu_j^s}{2} \right)^2 |y_{ij}|^2 \end{aligned}$$

and

$$\begin{aligned} & \left\| (1 - (1 - 2t)^2) T^{1/2} X S^{1/2} + (1 - 2t)^2 \frac{TX + XS}{2} \right\|_2^2 \\ &= \sum_{i,j=1}^n \left((1 - (1 - 2t)^2) \sqrt{\lambda_i \mu_j} + (1 - 2t)^2 \frac{\lambda_i + \mu_j}{2} \right)^2 |y_{ij}|^2. \end{aligned}$$

By virtue of the inequalities (2.1) and (2.2), we obtain

$$\begin{aligned} & \sum_{i,j=1}^n \left(\frac{\lambda_i^t \mu_j^{1-t} + \lambda_i^{1-t} \mu_j^t}{2} \right)^2 |y_{ij}|^2 \\ & \leq \sum_{i,j=1}^n \left(\left(1 - \frac{(1 - 2t)^2}{(1 - 2s)^2} \right) \sqrt{\lambda_i \mu_j} + \left(\frac{(1 - 2t)^2}{(1 - 2s)^2} \right) \frac{\lambda_i^s \mu_j^{1-s} + \lambda_i^{1-s} \mu_j^s}{2} \right)^2 |y_{ij}|^2 \\ & \leq \sum_{i,j=1}^n \left((1 - (1 - 2t)^2) \sqrt{\lambda_i \mu_j} + (1 - 2t)^2 \frac{\lambda_i + \mu_j}{2} \right)^2 |y_{ij}|^2. \end{aligned}$$

Thus, the proof is completed. □

4 The inductive weighted geometric means and means inequalities

Let $F : \mathcal{D} \rightarrow \mathfrak{B}(\mathcal{H})$ be a mapping of k variables defined in a convex domain $\mathcal{D} \subseteq \mathfrak{B}(\mathcal{H})^k$.

Recall from Hansen [7] that F is regular if:

- (i) The domain \mathcal{D} is invariant under unitary transformations of \mathcal{H} and

$$F(U^* T_1 U, \dots, U^* T_k U) = U^* F(T_1, \dots, T_k) U$$

for every $(T_1, \dots, T_k) \in \mathcal{D}$ and every unitary U on \mathcal{H} .

- (ii) Let P and Q be mutually orthogonal projections acting on \mathcal{H} and take arbitrary k -tuples (T_1, \dots, T_k) and (S_1, \dots, S_k) of operators in $\mathfrak{B}(\mathcal{H})$ such that the compressed tuples (PT_1P, \dots, PT_kP) and (QS_1Q, \dots, QS_kQ) are in the domain \mathcal{D} . Then the k -tuple of diagonal block matrices

$$(PT_1P + QS_1Q, \dots, PT_kP + QS_kQ)$$

is also in the domain \mathcal{D} and

$$\begin{aligned} & F(PT_1P + QS_1Q, \dots, PT_kP + QS_kQ) \\ &= PF(PT_1P, \dots, PT_kP)P + QF(QS_1Q, \dots, QS_kQ)Q. \end{aligned}$$

Recall also from Hansen [7] that the perspective of a regular operator mapping of several variables is defined as

$$\mathcal{P}_F(T_1, \dots, T_k, S) = S^{1/2} F(S^{-1/2} T_1 S^{-1/2}, \dots, S^{-1/2} T_k S^{-1/2}) S^{1/2}.$$

Hansen [7] proves that the perspective \mathcal{P}_F of a convex regular map $F : \mathcal{D}_+^k \rightarrow \mathfrak{B}(\mathcal{H})$ is regular and convex.

Write

$$\mathcal{D}_+^k = \{(T_1, \dots, T_k) \mid T_1, \dots, T_k > 0\}.$$

Now we prove another two properties of \mathcal{P}_F .

Theorem 4.1 *Suppose that $F : \mathcal{D}_+^k \rightarrow \mathfrak{B}(\mathcal{H})_{\text{sa}}$ is regular, concave, and continuous. Then the perspective function \mathcal{P}_F is monotone.*

Proof Let T_i and S_i be positive invertible operators such that $T_i \leq S_i$ for $i = 1, \dots, k + 1$. If $S_i - T_i$ is invertible for each $i = 1, \dots, k + 1$ and $\lambda \in (0, 1)$, then we have

$$\lambda S_i = \lambda T_i + (1 - \lambda)W_i, \quad i = 1, \dots, k + 1,$$

where

$$W_i = \lambda(1 - \lambda)^{-1}(S_i - T_i), \quad i = 1, \dots, k + 1,$$

are positive and invertible. Thus, the concavity of \mathcal{P}_F implies that

$$\begin{aligned} \mathcal{P}_F(\lambda S_1, \dots, \lambda S_{k+1}) &\geq \lambda \mathcal{P}_F(T_1, \dots, T_{k+1}) + (1 - \lambda) \mathcal{P}_F(W_1, \dots, W_{k+1}) \\ &\geq \lambda \mathcal{P}_F(T_1, \dots, T_{k+1}). \end{aligned}$$

For $\lambda \rightarrow 1$, by continuity, we get

$$\mathcal{P}_F(S_1, \dots, S_{k+1}) \geq \mathcal{P}_F(T_1, \dots, T_{k+1}).$$

Generally, choose $0 < \nu < 1$ such that

$$\nu T_i < T_i \leq S_i, \quad i = 1, \dots, k + 1.$$

Then we have

$$\mathcal{P}_F(\mu T_1, \dots, \mu T_{k+1}) \leq \mathcal{P}_F(S_1, \dots, S_{k+1}).$$

Letting $\nu \rightarrow 1$, we get the conclusion. □

Theorem 4.2 *Suppose that $F : \mathcal{D}_+^k \rightarrow \mathfrak{B}(\mathcal{H})_{\text{sa}}$ is a regular, concave, and positively homogeneous. Then the perspective function \mathcal{P}_F satisfies the property of congruence invariance:*

$$\mathcal{P}_F(W^* T_1 W, \dots, W^* T_{k+1} W) = W^* \mathcal{P}_F(T_1, \dots, T_{k+1}) W \tag{4.1}$$

for any invertible operator W on \mathcal{H} .

Proof It follows from Theorem 3.2 of [7] that the perspective function \mathcal{P}_F is concave. Moreover, since F is positively homogeneous, it is easy to prove that \mathcal{P}_F is also positively homogeneous. Hence, by Proposition 2.3 in [7], we get the conclusion. □

Let

$$\beta = (\beta_1, \dots, \beta_k) \in [0, 1]^k$$

with $\sum_{i=1}^k \beta_i = 1$, and let $\alpha_{k+1} \in [0, 1]$ and

$$\alpha = (\alpha_1, \dots, \alpha_k, \alpha_{k+1}) = (\beta_1(1 - \alpha_{k+1}), \dots, \beta_k(1 - \alpha_{k+1}), \alpha_{k+1}).$$

Then, simulated by the significant work of Hansen [7], we construct a sequence of weighted multivariate geometric means $G_1^\alpha, G_2^\alpha, \dots$ as follows.

- (i) Let $G_1^\alpha(T) = T$ for each positive definite invertible operator T .
- (ii) To each weighted geometric mean G_k^β of k variables we associate an auxiliary mapping $A_k : \mathcal{D}_+^k \rightarrow \mathfrak{B}(\mathcal{H})$ such that A_k is regular and concave, and

$$A_k(T_1, \dots, T_k) = G_k^\beta(T_1, \dots, T_k)^{(1-\alpha_{k+1})} = (T_1^{\beta_1} \dots T_k^{\beta_k})^{(1-\alpha_{k+1})}$$

for positive T_1, \dots, T_k , where β is the weight associated to T_1, \dots, T_k .

- (iii) Define the geometric mean $G_{k+1}^\alpha : \mathcal{D}_+^{k+1} \rightarrow \mathfrak{B}(\mathcal{H})$ of $k + 1$ variables as

$$G_{k+1}^\alpha(T_1, \dots, T_{k+1}) = \mathcal{P}_{A_k}(T_1, \dots, T_{k+1}),$$

where

$$\mathcal{P}_{A_k}(T_1, \dots, T_k, T_{k+1}) = T_{k+1}^{1/2} A_k(T_{k+1}^{-1/2} T_1 T_{k+1}^{-1/2}, \dots, T_{k+1}^{-1/2} T_k T_{k+1}^{-1/2}) T_{k+1}^{1/2}.$$

Particularly, the geometric means of two variables

$$G_2^\alpha(T_1, T_2) = T_2^{\frac{1}{2}} (T_2^{-\frac{1}{2}} T_1 T_2^{-\frac{1}{2}})^{\alpha_1} T_2^{\frac{1}{2}}$$

coincide with the weighted geometric means of two variables $T_1 \#_{\alpha_1} T_2$ in the sense of Kubo and Ando [6], where $\alpha = (\alpha_1, \alpha_2)$ satisfy $\alpha_1 + \alpha_2 = 1$.

In the above procedure, α_i is determined by β_i and α_{k+1} in the following sense:

$$\alpha_i = \beta_i(1 - \alpha_{k+1}) \quad (i = 1, \dots, k).$$

Conversely for fixed $\alpha = (\alpha_1, \dots, \alpha_{k+1})$, we can set

$$\beta_i = \frac{\alpha_i}{1 - \alpha_{k+1}}, \quad i = 1, 2, \dots, k, \alpha_{k+1} \neq 1,$$

and hence trace back to the case of $k = 1$. Therefore for fixed weight we can define the corresponding weighted geometric mean.

Theorem 4.3 *The means $G_k^\alpha : \mathcal{D}_+^k \rightarrow \mathfrak{B}(\mathcal{H})^+$ constructed as above are regular, positively homogeneous, concave, and they satisfy*

$$G_{k+1}^\alpha(T_1, \dots, T_k, 1) = G_k^\beta(\mathbb{T})^{(1-\alpha_{k+1})} \tag{4.2}$$

for $\mathbb{T} = (T_1, \dots, T_k) \in \mathcal{D}_+^k$.

Proof By the definition of G_k^α , we know that G_k^α for each $k = 2, 3, \dots$ is the perspective of a regular positively homogeneous map. Therefore, G_k^α are regular and positively homogeneous. Moreover, since G_{k+1}^α is the perspective of $(G_k^\beta)^{1-\alpha_{k+1}}$, we see that (4.2) holds.

Next, we prove that G_k^α is concave. Clearly, G_1^α is concave. Assume that G_k^β is concave for some k and the corresponding weight β . For $\alpha_{k+1} \in [0, 1]$, the map $x \rightarrow x^{1-\alpha_{k+1}}$ is operator monotone (increasing) and operator concave. Then we have

$$\begin{aligned} (G_k^\beta(\lambda \mathbb{T} + (1-\lambda)\mathbb{S}))^{(1-\alpha_{k+1})} &\geq (\lambda(G_k^\beta \mathbb{T}) + (1-\lambda)(G_k^\beta \mathbb{S}))^{(1-\alpha_{k+1})} \\ &\geq \lambda(G_k^\beta \mathbb{T})^{(1-\alpha_{k+1})} + (1-\lambda)(G_k^\beta \mathbb{S})^{(1-\alpha_{k+1})}, \end{aligned}$$

where $\mathbb{T} = (T_1, \dots, T_k)$, $\mathbb{S} = (S_1, \dots, S_k)$. So the auxiliary mapping

$$A_k(T_1, \dots, T_k) = G_k^\beta(T_1, \dots, T_k)^{1-\alpha_{k+1}}$$

is concave. Then by Theorem 3.2 in [7] we see that its perspective G_{k+1}^α is also concave. By induction, we know that G_k^α is concave for all $k = 1, 2, \dots$. \square

Remark 4.4 A similar analysis to Theorem 4.3 in [7] shows that the above conditions uniquely determine the Geometric means G_k^α for $k = 1, 2, \dots$ by setting $G_1^\alpha(T) = T$.

Theorem 4.5 Set $\mathbb{T} = (T_1, \dots, T_k) \in \mathcal{D}_+^k$. The means G_k^α constructed as above have the following properties:

- (P1) (consistency with scalars) $G_k^\alpha(\mathbb{T}) = T_1^{\alpha_1} \dots T_k^{\alpha_k}$ if the T_i 's commute;
- (P2) (joint homogeneity) $G_k^\alpha(t_1 T_1, \dots, t_k T_k) = t_1^{\alpha_1} \dots t_k^{\alpha_k} G_k^\alpha(\mathbb{T})$ for $t_i > 0$;
- (P3) (monotonicity) if $B_i \leq T_i$ for all $1 \leq i \leq k$, then $G_k^\alpha(\mathbb{B}) \leq G_k^\alpha(\mathbb{T})$;
- (P4) (congruence invariance) $G_k^\alpha(W^* T_1 W, \dots, W^* T_k W) = W^* G_k^\alpha(\mathbb{T}) W$ for any invertible operator W on \mathcal{H} ;
- (P5) (self-duality) $G_k^\alpha(\mathbb{T}^{-1}) = G_k^\alpha(\mathbb{T})^{-1}$;
- (P6) (A-G-H weighted mean inequalities) $(\sum_{i=1}^k \alpha_i T_i^{-1})^{-1} \leq G_k^\alpha(\mathbb{T}) \leq \sum_{i=1}^k \alpha_i T_i$;
- (P7) (determinant identity) $\det G_k^\alpha(\mathbb{T}) = \prod_{i=1}^k (\det T_i)^{\alpha_i}$.

Proof If T_1 and T_2 commute, then

$$G_2^\alpha(T_1, T_2) = T_2^{\frac{1}{2}} (T_2^{-\frac{1}{2}} T_1 T_2^{-\frac{1}{2}})^{\alpha_1} T_2^{\frac{1}{2}} = T_1^{\alpha_1} T_2^{\alpha_2}.$$

Hence, (P1) holds for $k = 1, 2$. Now assume that (P1) holds for some $k > 2$. Since

$$\begin{aligned} G_{k+1}^\alpha(T_1, \dots, T_{k+1}) &= \mathcal{P}_{F_k}(T_1, \dots, T_{k+1}) \\ &= T_{k+1}^{\frac{1}{2}} F_k(T_{k+1}^{-\frac{1}{2}} T_1 T_{k+1}^{-\frac{1}{2}}, \dots, T_{k+1}^{-\frac{1}{2}} T_k T_{k+1}^{-\frac{1}{2}}) T_{k+1}^{\frac{1}{2}} \\ &= T_{k+1}^{\frac{1}{2}} G_k^\beta(T_{k+1}^{-\frac{1}{2}} T_1 T_{k+1}^{-\frac{1}{2}}, \dots, T_{k+1}^{-\frac{1}{2}} T_k T_{k+1}^{-\frac{1}{2}})^{(1-\alpha_{k+1})} T_{k+1}^{\frac{1}{2}} \\ &= T_{k+1}^{\frac{1}{2}} ((T_{k+1}^{-\frac{1}{2}} T_1 T_{k+1}^{-\frac{1}{2}})^{\beta_1} \dots (T_{k+1}^{-\frac{1}{2}} T_k T_{k+1}^{-\frac{1}{2}})^{\beta_k})^{(1-\alpha_{k+1})} T_{k+1}^{\frac{1}{2}} \\ &= T_1^{\beta_1(1-\alpha_{k+1})} \dots T_k^{\beta_k(1-\alpha_{k+1})} T_{k+1}^{1-(1-\alpha_{k+1})} \\ &= T_1^{\alpha_1} \dots T_k^{\alpha_k} T_{k+1}^{\alpha_{k+1}}, \end{aligned}$$

we see that (P1) also holds for $k + 1$. By induction, we know that (P1) holds for $k = 1, 2, \dots$

It is easy to verify (P2) holds for $k = 1$ and $k = 2$. Assume that (P2) holds for some $k > 2$. Then we have

$$\begin{aligned} &G_{k+1}^\alpha(t_1 T_1, \dots, t_k T_k, t_{k+1} T_{k+1}) \\ &= t_{k+1} T_{k+1}^{\frac{1}{2}} F_k(t_1 t_{k+1}^{-1} T_{k+1}^{-\frac{1}{2}} T_1 T_{k+1}^{-\frac{1}{2}}, \dots, t_k t_{k+1}^{-1} T_{k+1}^{-\frac{1}{2}} T_k T_{k+1}^{-\frac{1}{2}}) T_{k+1}^{\frac{1}{2}} \\ &= t_{k+1} T_{k+1}^{\frac{1}{2}} G_k^\beta(t_1 t_{k+1}^{-1} T_{k+1}^{-\frac{1}{2}} T_1 T_{k+1}^{-\frac{1}{2}}, \dots, t_k t_{k+1}^{-1} T_{k+1}^{-\frac{1}{2}} T_k T_{k+1}^{-\frac{1}{2}})^{(1-\alpha_{k+1})} T_{k+1}^{\frac{1}{2}}. \end{aligned}$$

By induction, we get

$$\begin{aligned} &G_{k+1}^\alpha(t_1 T_1, \dots, t_k T_k, t_{k+1} T_{k+1}) \\ &= t_{k+1} (t_{k+1}^{-1} t_1^{\beta_1} \dots t_k^{\beta_k})^{(1-\alpha_{k+1})} G_{k+1}^\alpha(T_1, \dots, T_k, T_{k+1}) \\ &= t_1^{\alpha_1} \dots t_k^{\alpha_k} t_{k+1}^{\alpha_{k+1}} G_{k+1}^\alpha(T_1, \dots, T_k, T_{k+1}). \end{aligned}$$

Hence (P2) is true.

(P3) and (P4) follow from Theorems 4.1 and 4.2.

Clearly, (P5) is true for $k = 1$ and $k = 2$. Assume that (P5) is true for some $k > 2$. Then we have

$$\begin{aligned} &G_{k+1}^\alpha(T_1^{-1}, \dots, T_k^{-1}, T_{k+1}^{-1}) \\ &= T_{k+1}^{-\frac{1}{2}} F_k(T_{k+1}^{\frac{1}{2}} T_1^{-1} T_{k+1}^{\frac{1}{2}}, \dots, T_{k+1}^{\frac{1}{2}} T_k^{-1} T_{k+1}^{\frac{1}{2}}) T_{k+1}^{-\frac{1}{2}} \\ &= T_{k+1}^{-\frac{1}{2}} G_k^\beta(T_{k+1}^{\frac{1}{2}} T_1^{-1} T_{k+1}^{\frac{1}{2}}, \dots, T_{k+1}^{\frac{1}{2}} T_k^{-1} T_{k+1}^{\frac{1}{2}})^{(1-\alpha_{k+1})} T_{k+1}^{-\frac{1}{2}}. \end{aligned}$$

By induction, we get

$$\begin{aligned} &G_{k+1}^\alpha(T_1^{-1}, \dots, T_k^{-1}, T_{k+1}^{-1}) \\ &= T_{k+1}^{-\frac{1}{2}} G_k^\beta(T_{k+1}^{-\frac{1}{2}} T_1 T_{k+1}^{-\frac{1}{2}}, \dots, T_{k+1}^{-\frac{1}{2}} T_k T_{k+1}^{-\frac{1}{2}})^{-(1-\alpha_{k+1})} T_{k+1}^{-\frac{1}{2}} \\ &= (T_{k+1}^{\frac{1}{2}} G_k^\beta(T_{k+1}^{-\frac{1}{2}} T_1 T_{k+1}^{-\frac{1}{2}}, \dots, T_{k+1}^{-\frac{1}{2}} T_k T_{k+1}^{-\frac{1}{2}})^{(1-\alpha_{k+1})} T_{k+1}^{\frac{1}{2}})^{-1} \\ &= G_{k+1}^\alpha(T_1, \dots, T_k, T_{k+1})^{-1}, \end{aligned}$$

which verifies (P5).

The A-G-H weighted mean inequality, i.e. the arithmetic-geometric-harmonic weighted mean inequality reads

$$\left(\sum_{i=1}^k \alpha_i T_i^{-1} \right)^{-1} \leq G_k^\alpha(T_1, \dots, T_k) \leq \sum_{i=1}^k \alpha_i T_i$$

for arbitrary $(T_1, \dots, T_k) \in \mathcal{D}_+^k$. Firstly, we show the second inequality. It is easy to see the second inequality holds for $k = 1$. Assume the inequality holds for some k . Then, by virtue of

$$X^p \leq 1 + p(X - 1)$$

for $X \in \mathcal{D}^1$ (the set of positive operators) and $p \in [0, 1]$, we obtain

$$\begin{aligned} F_k(T_1, \dots, T_k) &= G_k^\beta(T_1, \dots, T_k)^{(1-\alpha_{k+1})} \\ &\leq 1 + (1 - \alpha_{k+1})(G_k^\beta(T_1, \dots, T_k) - 1) \\ &\leq 1 + (1 - \alpha_{k+1})\left(\sum_{i=1}^k \beta_i T_i - 1\right) \\ &= \sum_{i=1}^k (1 - \alpha_{k+1})\beta_i T_i + \alpha_{k+1} = \sum_{i=1}^k \alpha_i T_i + \alpha_{k+1}. \end{aligned}$$

Now taking perspective, we have

$$\begin{aligned} G_{k+1}^\alpha(T_1, \dots, T_k, T_{k+1}) &= \mathcal{P}_{F_k}(T_1, \dots, T_k, T_{k+1}) \\ &= T_{k+1}^{\frac{1}{2}} F_k(T_{k+1}^{-\frac{1}{2}} T_1 T_{k+1}^{-\frac{1}{2}}, \dots, T_{k+1}^{-\frac{1}{2}} T_k T_{k+1}^{-\frac{1}{2}}) T_{k+1}^{\frac{1}{2}} \\ &\leq T_{k+1}^{\frac{1}{2}} \left(\sum_{i=1}^k \alpha_i T_{k+1}^{-\frac{1}{2}} T_i T_{k+1}^{-\frac{1}{2}} + \alpha_{k+1}\right) T_{k+1}^{\frac{1}{2}} \\ &= \sum_{i=1}^{k+1} \alpha_i T_i. \end{aligned}$$

By induction, the second inequality is proved. Next, it follows from the second inequality that

$$G_k^\alpha(T_1^{-1}, \dots, T_k^{-1}) \leq \sum_{i=1}^k \alpha_i T_i^{-1}.$$

By inversion and using the self-duality (P7) of the weighted geometric mean, we get

$$\left(\sum_{i=1}^k \alpha_i T_i^{-1}\right)^{-1} \leq G_k^\alpha(T_1^{-1}, \dots, T_k^{-1})^{-1} = G_k^\alpha(T_1, \dots, T_k).$$

Hence the property (P6) holds.

For $T \in \mathcal{D}^1$ and $p \in \mathbb{R}$, we have $\det T^p = (\det T)^p$ due to $\det T = \exp(\text{Tr} \log T)$. For $k = 1$ and $k = 2$, (P7) is obviously correct. Assume that (P7) holds for some $k > 2$. Then, using

$$G_{k+1}^\alpha(T_1, \dots, T_k, T_{k+1}) = T_{k+1}^{\frac{1}{2}} G_k^\beta(T_{k+1}^{-\frac{1}{2}} T_1 T_{k+1}^{-\frac{1}{2}}, \dots, T_{k+1}^{-\frac{1}{2}} T_k T_{k+1}^{-\frac{1}{2}})^{(1-\alpha_{k+1})} T_{k+1}^{\frac{1}{2}},$$

we infer that

$$\begin{aligned} \det G_{k+1}^\alpha(T_1, \dots, T_k, T_{k+1}) &= \det T_{k+1} (\det T_{k+1}^{-\beta_1} \det T_1^{\beta_1} \dots \det T_{k+1}^{-\beta_k} \det T_k^{\beta_k})^{(1-\alpha_{k+1})} \\ &= \prod_{i=1}^{k+1} (\det T_i)^{\alpha_i}, \end{aligned}$$

which means that (P7) is true. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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