Li and Guo *Journal of Inequalities and Applications* (2015) 2015:132
DOI 10.1186/s13660-015-0658-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

On a product-type operator from Zygmund-type spaces to Bloch-Orlicz spaces

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Abstract
The boundedness and compactness of a product-type operator from Zygmund-type spaces to Bloch-Orlicz spaces are investigated in this paper.

MSC: Primary 47B38; secondary 47B33; 47B37

Keywords: product-type operator; boundedness; compactness; Zygmund-type spaces; Bloch-Orlicz spaces

1 Introduction

Let $\mathcal{D}$ denote the unit disk in the complex plane $\mathcal{C}$, and let $\mathcal{H}(\mathcal{D})$ be the space of all holomorphic functions on $\mathcal{D}$ with the topology of uniform convergence on compacts of $\mathcal{D}$.

For $0 < \alpha < \infty$, the α-Bloch space, denoted by $\mathcal{B}^\alpha$, consists of all functions $f \in \mathcal{H}(\mathcal{D})$ such that

$$\sup_{z\in\mathcal{D}}\left(1-|z|^2\right)^\alpha \left|f'(z)\right| < \infty.$$

By $\mathcal{Z}^\alpha$ we denote the Zygmund-type space consisting of those functions $f \in \mathcal{H}(\mathcal{D})$ satisfying

$$\sup_{z\in\mathcal{D}}\left(1-|z|^2\right)^\alpha \left|f''(z)\right| < \infty.$$

$\mathcal{B}^\alpha$ and $\mathcal{Z}^\alpha$ are Banach spaces under the norms

$$\|f\|_{\mathcal{B}^\alpha} = \left|f(0)\right| + \sup_{z\in\mathcal{D}}\left(1-|z|^2\right)^\alpha \left|f'(z)\right|,$$

$$\|f\|_{\mathcal{Z}^\alpha} = \left|f(0)\right| + \left|f'(0)\right| + \sup_{z\in\mathcal{D}}\left(1-|z|^2\right)^\alpha \left|f''(z)\right|,$$

respectively. For some results on the Zygmund-type spaces on various domains in the complex plane and $\mathcal{C}^n$ and operators on them, see, for example, [1–18]. The α-Bloch space is introduced and studied by numerous authors. For the general theory of α-Bloch or Bloch-type spaces and operators of them, see, *e.g.*, [4, 19–41]. Recently, many authors studied different classes of Bloch-type spaces, where the typical weight function, $\omega(z) = 1 - |z|^2$, $z \in \mathcal{D}$, is replaced by a bounded continuous positive function μ defined on $\mathcal{D}$. More precisely, a function $f \in \mathcal{H}(\mathcal{D})$ is called a μ-Bloch function, denoted by $f \in \mathcal{B}^\mu$, if $\|f\|_\mu = \sup_{z\in\mathcal{D}} \mu(z)|f'(z)| < \infty$. If $\mu(z) = \omega(z)^\alpha$, $\alpha > 0$, $\mathcal{B}^\mu$ is just the α-Bloch space $\mathcal{B}^\alpha$. It is readily seen that $\mathcal{B}^\mu$ is a Banach space with the norm $\|f\|_{\mathcal{B}^\mu} = |f(0)| + \|f\|_\mu$.

Recently, Ramos Fernández in [42] used Young's functions to define the Bloch-Orlicz space. More precisely, let $\varphi : [0,+\infty) \to [0,+\infty)$ be a strictly increasing convex function such that $\varphi(0) = 0$ and note that from these conditions it follows that $\lim_{t\to+\infty} \varphi(t) = +\infty$. The Bloch-Orlicz space associated with the function φ, denoted by $\mathcal{B}^\varphi$, is the class of all analytic functions f in $\mathcal{D}$ such that

$$\sup_{z\in\mathcal{D}}\bigl(1-|z|^2\bigr)\varphi\bigl(\lambda\bigl|f'(z)\bigr|\bigr) < \infty$$

for some $\lambda > 0$ depending on f. Also, since φ is convex, it is not hard to see that Minkowski's functional

$$\|f\|_\varphi = \inf\left\{k > 0 : S_\varphi\left(\frac{f'}{k}\right) \le 1\right\}$$

defines a seminorm for $\mathcal{B}^\varphi$, which, in this case, is known as Luxemburg's seminorm, where

$$S_\varphi(f) = \sup_{z\in\mathcal{D}}\bigl(1-|z|^2\bigr)\varphi\bigl(\bigl|f(z)\bigr|\bigr).$$

Moreover, it can be shown that $\mathcal{B}^\varphi$ is a Banach space with the norm $\|f\|_{\mathcal{B}^\varphi} = |f(0)| + \|f\|_\varphi$. We also have that the Bloch-Orlicz space is isometrically equal to a particular μ-Bloch space, where $\mu(z) = \frac{1}{\varphi^{-1}(\frac{1}{1-|z|^2})}$ with $z \in \mathcal{D}$. Thus, for any $f \in \mathcal{B}^\varphi$, we have

$$\|f\|_{\mathcal{B}^\varphi} = \bigl|f(0)\bigr| + \sup_{z\in\mathcal{D}} \mu(z)\bigl|f'(z)\bigr|.$$

When φ is the identity map on $[0,+\infty)$, $\mathcal{B}^\varphi$ is the so-called Bloch space $\mathcal{B}$.

Let $u \in \mathcal{H}(\mathcal{D})$ and ϕ be an analytic self-map of $\mathcal{D}$. The differentiation operator D, the multiplication operator M_u and the composition operator C_ϕ are defined by

$$(Df)(z) = f'(z), \qquad (M_uf)(z) = u(z)f(z), \qquad (C_\phi f)(z) = f\bigl(\phi(z)\bigr), \quad f \in \mathcal{H}(\mathcal{D}).$$

There is a considerable interest in studying the above mentioned operators as well as their products (see, *e.g.*, [1–38, 41–56] and the related references therein).

A product-type operator DM_uC_ϕ is defined as follows:

$$(DM_uC_\phi f)(z) = u'(z)f\bigl(\phi(z)\bigr) + u(z)\phi'(z)f'\bigl(\phi(z)\bigr), \quad u,f \in \mathcal{H}(\mathcal{D}).$$

For $0 < \alpha < \infty$ and $\frac{1}{2} < |a| < 1$, we define the test functions (see [1])

$$f_a(z) = \frac{1}{\overline{a}^2}\left[\frac{(1-|a|^2)^2}{(1-\overline{a}z)^\alpha} - \frac{1-|a|^2}{(1-\overline{a}z)^{\alpha-1}}\right],$$
$$h_a(z) = \frac{1}{\overline{a}}\int_0^z \frac{1-|a|^2}{(1-\overline{a}\lambda)^\alpha}\,d\lambda, \quad z \in \mathcal{D}.$$

It is easy to show that $f_a, h_a \in \mathcal{Z}^\alpha$ and $f_a(a) = 0$,

$$f_a'(a) = h_a'(a) = \frac{1}{\bar{a}}\big(1 - |a|^2\big)^{1-\alpha}, \qquad f_a''(a) = \frac{2\alpha}{(1 - |a|^2)^\alpha}, \qquad h_a''(a) = \frac{\alpha}{(1 - |a|^2)^\alpha}.$$

Esmaeili and Lindström in [1] investigated weighted composition operators between Zygmund-type spaces. Ramos Fernández in [42] studied the boundedness and compactness of composition operators on Bloch-Orlicz spaces. Li and Stević in [5] investigated products of Volterra-type operator and composition operator from H^∞ and Bloch spaces to Zygmund spaces, and they in [8] studied products of composition and differentiation operators from Zygmund spaces to Bloch spaces and Bers spaces. Liu and Yu in [25] characterized the boundedness and compactness of products of composition, multiplication and radial derivative operators from logarithmic Bloch spaces to weighted-type spaces on the unit ball. Sharma in [27] studied the boundedness and compactness of products of composition multiplication and differentiation between Bergman and Bloch-type spaces. In [52], Stević investigated the properties of weighted differentiation composition operators from mixed-norm spaces to weighted-type spaces. Stević in [13] studied weighted radial operators from the mixed-norm space to the nth weighted-type space on the unit ball. Stević *et al.* in [54] characterized the boundedness and compactness of products of multiplication composition and differentiation operators on weighted Bergman spaces. Zhu in [18] studied extended Cesàro operators from mixed-norm spaces to Zygmund-type spaces.

Motivated by the above papers, in this paper, we investigate the boundedness and compactness of the product-type operator DM_uC_ϕ from Zygmund-type spaces to the Bloch-Orlicz space. The paper is organized as follows. In Section 2, we give some necessary and sufficient conditions for the boundedness of the operator $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$. In Section 3, we give some necessary and sufficient conditions for the compactness of the operator $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$.

Throughout this paper,

$$\mu(z) = \frac{1}{\varphi^{-1}(\frac{1}{1-|z|^2})},$$

and we use letter C to denote a positive constant whose value may change at each occurrence.

2 The boundedness of $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$

The following lemma was essentially proved in [3] and [11] (see also [1]).

Lemma 1 *For $f \in \mathcal{Z}^\alpha$ and $\alpha > 0$. Then*:

(i) *For $0 < \alpha < 1$, $|f'(z)| \leq \frac{2}{1-\alpha}\|f\|_{\mathcal{Z}^\alpha}$ and $|f(z)| \leq \frac{2}{1-\alpha}\|f\|_{\mathcal{Z}^\alpha}$.*

(ii) *For $\alpha = 1$, $|f'(z)| \leq \log\frac{e}{1-|z|^2}\|f\|_{\mathcal{Z}}$ and $|f(z)| \leq \|f\|_{\mathcal{Z}}$.*

(iii) *For $\alpha > 1$, $|f'(z)| \leq \frac{2}{\alpha-1}\frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha-1}}$. For $\alpha = 2$, $|f'(z)| \leq \frac{e}{1-|z|^2}\|f\|_{\mathcal{Z}^2}$.*

(iv) *For $1 < \alpha < 2$, $|f(z)| \leq \frac{2}{(\alpha-1)(2-\alpha)}\|f\|_{\mathcal{Z}^\alpha}$.*

(v) *For $\alpha = 2$, $|f(z)| \leq 2\log\frac{e}{1-|z|^2}\|f\|_{\mathcal{Z}^2}$.*

(vi) *For $\alpha > 2$, $|f(z)| \leq \frac{2}{(\alpha-1)(\alpha-2)}\frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha-2}}$.*

Lemma 2 *If $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded and $0 < \alpha < \infty$, then the following conditions hold*:

$$k_1 = \sup_{z\in\mathcal{D}} \mu(z)\left|u''(z)\right| < \infty, \tag{1}$$

$$k_2 = \sup_{z\in\mathcal{D}} \mu(z)\left|2u'(z)\phi'(z) + u(z)\phi''(z)\right| < \infty, \tag{2}$$

$$k_3 = \sup_{z\in\mathcal{D}} \mu(z)\left|u(z)\right|\left|\phi'(z)\right|^2 < \infty. \tag{3}$$

Proof Suppose that $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded. Taking the function $f(z) = 1 \in \mathcal{Z}^\alpha$ and using the obvious fact that $\|f\|_{\mathcal{Z}^\alpha} = 1$, we have that

$$S_\varphi\left(\frac{(DM_uC_\phi f)'(z)}{C\|f\|_{\mathcal{Z}^\alpha}}\right) = S_\varphi\left(\frac{u''(z)}{C}\right) = \sup_{z\in\mathcal{D}}\left(1-|z|^2\right)\varphi\left(\frac{|u''(z)|}{C}\right) \le 1,$$

from which it follows that (1) holds. Taking the function $f(z) = z \in \mathcal{Z}^\alpha$ and using the fact that $\|f\|_{\mathcal{Z}^\alpha} = 1$, we obtain

$$\begin{aligned} &S_\varphi\left(\frac{(DM_uC_\phi f)'(z)}{C\|f\|_{\mathcal{Z}^\alpha}}\right) \\ &\quad = S_\varphi\left(\frac{u''(z)\phi(z) + 2u'(z)\phi'(z) + u(z)\phi''(z)}{C}\right) \\ &\quad = \sup_{z\in\mathcal{D}}\left(1-|z|^2\right)\varphi\left(\frac{|u''(z)\phi(z) + 2u'(z)\phi'(z) + u(z)\phi''(z)|}{C}\right) \le 1. \end{aligned}$$

Hence

$$\sup_{z\in\mathcal{D}} \mu(z)\left|2u'(z)\phi'(z) + u(z)\phi''(z) + u''(z)\phi(z)\right| < \infty.$$

From this, (1) and by the boundedness of $\phi(z)$, condition (2) easily follows. Now taking the function $f(z) = z^2 \in \mathcal{Z}^\alpha$ and using the fact that $\|f\|_{\mathcal{Z}^\alpha} = 2$, we get

$$S_\varphi\left(\frac{u''(z)(\phi(z))^2 + 2\phi(z)(2u'(z)\phi'(z) + u(z)\phi''(z)) + 2u(z)\phi'(z)^2}{2C}\right) \le 1.$$

Hence

$$\sup_{z\in\mathcal{D}} \mu(z)\left|u''(z)\left(\phi(z)\right)^2 + 2\phi(z)\left(2u'(z)\phi'(z) + u(z)\phi''(z)\right) + 2u(z)\phi'(z)^2\right| < \infty.$$

From this, (1), (2) and the boundedness of $\phi(z)$, we obtain (3). □

Now, we are ready to characterize the boundedness of the product-type operator $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$. For this purpose we need to break the problem into five different cases: $0 < \alpha < 1$, $\alpha = 1$, $1 < \alpha < 2$, $\alpha = 2$ and $\alpha > 2$.

Theorem 3 *Let $u \in \mathcal{H}(\mathcal{D})$, ϕ be an analytic self-map of $\mathcal{D}$ and $0 < \alpha < 1$. Then $DM_uC_\phi :$ $\mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded if and only if $k_1 < \infty$, $k_2 < \infty$ and*

$$k_4 = \sup_{z\in\mathcal{D}} \frac{\mu(z)|u(z)||\phi'(z)|^2}{(1-|\phi(z)|^2)^\alpha} < \infty. \tag{4}$$

Proof Suppose that $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded, by Lemma 2 we know that $k_1, k_2, k_3 < \infty$. Now we will prove (4). Let

$$g_{\phi(\omega)}(z) = f_{\phi(\omega)}(z) - h_{\phi(\omega)}(z) + h_{\phi(\omega)}\big(\phi(\omega)\big)$$

for all $z \in \mathcal{D}$ and $\omega \in \mathcal{D}$ such that $\frac{1}{2} < |\phi(\omega)| < 1$, then $g_{\phi(\omega)} \in \mathcal{Z}^\alpha$, and

$$g_{\phi(\omega)}\big(\phi(\omega)\big) = g'_{\phi(\omega)}\big(\phi(\omega)\big) = 0, \qquad g''_{\phi(\omega)}\big(\phi(\omega)\big) = \frac{\alpha}{(1 - |\phi(\omega)|^2)^\alpha}.$$

By the boundedness of $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$, we have $\|DM_uC_\phi g_{\phi(\omega)}\|_{\mathcal{B}^\varphi} \le C$, then

$$1 \ge S_\varphi\left(\frac{(DM_uC_\phi g_{\phi(\omega)})'(z)}{C}\right) \ge \sup_{\frac{1}{2}<|\phi(\omega)|<1} \big(1 - |\omega|^2\big)\varphi\left(\frac{\alpha|u(\omega)||\phi'(\omega)|^2}{C(1 - |\phi(\omega)|^2)^\alpha}\right).$$

It follows that

$$\sup_{\frac{1}{2}<|\phi(\omega)|<1} \frac{\mu(\omega)|u(\omega)||\phi'(\omega)|^2}{(1 - |\phi(\omega)|^2)^\alpha} < \infty. \tag{5}$$

By $k_3 < \infty$, we see that

$$\sup_{|\phi(\omega)|\le\frac{1}{2}} \frac{\mu(\omega)|u(\omega)||\phi'(\omega)|^2}{(1 - |\phi(\omega)|^2)^\alpha} \le C \sup_{|\phi(\omega)|\le\frac{1}{2}} \mu(\omega)\big|u(\omega)\big|\big|\phi'(\omega)\big|^2 < \infty. \tag{6}$$

From (5) and (6), we obtain (4).

Suppose that $k_1, k_2, k_4 < \infty$. For each $f \in \mathcal{Z}^\alpha \setminus \{0\}$, by Lemma 1(i) we have

$$\begin{aligned} &S_\varphi\left(\frac{(DM_uC_\phi f)'(z)}{C\|f\|_{\mathcal{Z}^\alpha}}\right) \\ &\quad\le \sup_{z\in\mathcal{D}}\big(1 - |z|^2\big)\varphi\left[\frac{(k_1|f(\phi(z))| + k_2|f'(\phi(z))| + k_4(1 - |\phi(z)|^2)^\alpha|f''(\phi(z))|)}{C\mu(z)\|f\|_{\mathcal{Z}^\alpha}}\right] \\ &\quad\le \sup_{z\in\mathcal{D}}\big(1 - |z|^2\big)\varphi\left[\frac{k_1\frac{2}{1-\alpha} + k_2\frac{2}{1-\alpha} + k_4}{C\mu(z)}\right] \le 1, \end{aligned}$$

where C is a constant such that $C \ge k_1\frac{2}{1-\alpha} + k_2\frac{2}{1-\alpha} + k_4$. Here we use the fact that

$$\sup_{z\in\mathcal{D}}\big(1 - \big|\phi(z)\big|^2\big)^\alpha\big|f''\big(\phi(z)\big)\big| \le \|f\|_{\mathcal{Z}^\alpha}.$$

Now, we can conclude that there exists a constant C such that $\|DM_uC_\phi f\|_{\mathcal{B}^\varphi} \le C\|f\|_{\mathcal{Z}^\alpha}$ for all $f \in \mathcal{Z}^\alpha$, so the product-type operator $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded. □

Theorem 4 *Let $u \in \mathcal{H}(\mathcal{D})$ and ϕ be an analytic self-map of $\mathcal{D}$. Then $DM_uC_\phi : \mathcal{Z} \to \mathcal{B}^\varphi$ is bounded if and only if $k_1 < \infty$,*

$$k_5 = \sup_{z\in\mathcal{D}} \mu(z)\big|2u'(z)\phi'(z) + u(z)\phi''(z)\big| \log\frac{e}{1 - |\phi(z)|^2} < \infty, \tag{7}$$

$$k_6 = \sup_{z\in\mathcal{D}} \frac{\mu(z)|u(z)||\phi'(z)|^2}{1 - |\phi(z)|^2} < \infty. \tag{8}$$

Proof Suppose that $DM_uC_\phi : \mathcal{Z} \to \mathcal{B}^\varphi$ is bounded, by Lemma 2 we know that $k_1, k_2, k_3 < \infty$. Let

$$r(z) = (z-1)\left[\left(1 + \log\frac{e}{1-z}\right)^2 + 1\right],$$
$$s_a(z) = \frac{r(\bar{a}z)}{\bar{a}}\left(\log\frac{e}{1-|a|^2}\right)^{-1} - \int_0^z \log\frac{e}{1-\bar{a}\lambda}\,d\lambda - c_1 + c_2,$$

where

$$c_1 = \frac{r(|a|^2)}{\bar{a}}\left(\log\frac{e}{1-|a|^2}\right)^{-1}, \qquad c_2 = \int_0^a \log\frac{e}{1-\bar{a}\lambda}\,d\lambda$$

for any $a \in \mathcal{D}$ such that $\frac{1}{2} < |a| < 1$. Then we have

$$\left|s_a''(z)\right| = \frac{2}{1-|z|}\left(C + \log\frac{e}{1-|a|}\right)\left(\log\frac{e}{1-|a|^2}\right)^{-1} + \frac{1}{1-|z|} \le \frac{C}{1-|z|}$$

for $\frac{1}{2} < |a| < 1$ and $\sup_{\frac{1}{2}<|a|<1} \|s_a\|_{\mathcal{Z}} < \infty$.

Now let $a = \phi(\omega)$, $\omega \in \mathcal{D}$ such that $\frac{1}{2} < |\phi(\omega)| < 1$, then

$$s_{\phi(\omega)}\big(\phi(\omega)\big) = s'_{\phi(\omega)}\big(\phi(\omega)\big) = 0, \qquad s''_{\phi(\omega)}\big(\phi(\omega)\big) = \frac{\overline{\phi(\omega)}}{1-|\phi(\omega)|^2}.$$

By the boundedness of $DM_uC_\phi : \mathcal{Z} \to \mathcal{B}^\varphi$, we have $\|DM_uC_\phi s_{\phi(\omega)}\|_{\mathcal{B}^\varphi} \le C$, then

$$1 \ge S_\varphi\left(\frac{(DM_uC_\phi s_{\phi(\omega)})'(z)}{C}\right) \ge \sup_{\frac{1}{2}<|\phi(\omega)|<1}\big(1-|\omega|^2\big)\varphi\left(\frac{|u(\omega)||\phi'(\omega)|^2|\phi(\omega)|}{C(1-|\phi(\omega)|^2)}\right).$$

From this it follows that

$$\frac{1}{2}\sup_{\frac{1}{2}<|\phi(\omega)|<1}\frac{\mu(\omega)|u(\omega)||\phi'(\omega)|^2}{1-|\phi(\omega)|^2} \le \sup_{\frac{1}{2}<|\phi(\omega)|<1}\frac{\mu(\omega)|u(\omega)||\phi'(\omega)|^2|\phi(\omega)|}{1-|\phi(\omega)|^2} < \infty. \tag{9}$$

By $k_3 < \infty$ we see that

$$\sup_{|\phi(\omega)|\le\frac{1}{2}}\frac{\mu(\omega)|u(\omega)||\phi'(\omega)|^2}{1-|\phi(\omega)|^2} \le \frac{4}{3}\sup_{|\phi(\omega)|\le\frac{1}{2}}\mu(\omega)\big|u(\omega)\big|\big|\phi'(\omega)\big|^2 < \infty. \tag{10}$$

From (9) and (10) we obtain $k_6 < \infty$.

Let

$$t_{\phi(\omega)}(z) = \frac{r(\overline{\phi(\omega)}z)}{\overline{\phi(\omega)}}\left(\log\frac{e}{1-|\phi(\omega)|^2}\right)^{-1} - c_1$$

for $\omega \in \mathcal{D}$ such that $\frac{1}{2} < |\phi(\omega)| < 1$, then, as above, we can get that $t_{\phi(\omega)} \in \mathcal{Z}$ and

$$t_{\phi(\omega)}\big(\phi(\omega)\big) = 0, \qquad t'_{\phi(\omega)}\big(\phi(\omega)\big) = \log\frac{e}{1-|\phi(\omega)|^2}, \qquad t''_{\phi(\omega)}\big(\phi(\omega)\big) = \frac{2\overline{\phi(\omega)}}{1-|\phi(\omega)|^2}.$$

By the boundedness of $DM_uC_\phi : \mathcal{Z} \to \mathcal{B}^\varphi$, we have $\|DM_uC_\phi t_{\phi(\omega)}\|_{\mathcal{B}^\varphi} \le C$, then

$$
\begin{aligned}
1 &\ge S_\varphi\left(\frac{(DM_uC_\phi t_{\phi(\omega)})'(z)}{C}\right) \\
&\ge \sup_{\frac{1}{2}<|\phi(\omega)|<1} \bigl(1-|\omega|^2\bigr)\varphi\left(\frac{|(DM_uC_\phi t_{\phi(\omega)})'(\omega)|}{C}\right) \\
&\ge \sup_{\frac{1}{2}<|\phi(\omega)|<1} \bigl(1-|\omega|^2\bigr) \\
&\quad\cdot \varphi\left(\frac{|(2u'(\omega)\phi'(\omega)+u(\omega)\phi''(\omega))\log\frac{e}{1-|\phi(\omega)|^2}+u(\omega)(\phi'(\omega))^2\frac{2\overline{\phi(\omega)}}{1-|\phi(\omega)|^2}|}{C}\right).
\end{aligned}
$$

From this and by $k_6 < \infty$, we get

$$
\begin{aligned}
&\sup_{\frac{1}{2}<|\phi(\omega)|<1} \mu(\omega)\bigl|2u'(\omega)\phi'(\omega)+u(\omega)\phi''(\omega)\bigr|\log\frac{e}{1-|\phi(\omega)|^2} \\
&\quad\le C + 2Ck_6 < \infty.
\end{aligned}
\tag{11}
$$

By $k_2 < \infty$ we see that

$$
\begin{aligned}
&\sup_{|\phi(\omega)|\le\frac{1}{2}} \mu(\omega)\bigl|2u'(\omega)\phi'(\omega)+u(\omega)\phi''(\omega)\bigr|\log\frac{e}{1-|\phi(\omega)|^2} \\
&\quad\le C\sup_{|\phi(\omega)|\le\frac{1}{2}} \mu(\omega)\bigl|2u'(\omega)\phi'(\omega)+u(\omega)\phi''(\omega)\bigr| < \infty.
\end{aligned}
\tag{12}
$$

From (11) and (12) we obtain (7).

Suppose that $k_1, k_5, k_6 < \infty$. Then, by Lemma 1(ii) and similar to the proof of Theorem 3, we get that $DM_uC_\phi : \mathcal{Z} \to \mathcal{B}^\varphi$ is bounded. □

Theorem 5 *Let $u \in \mathcal{H}(\mathcal{D})$, ϕ be an analytic self-map of $\mathcal{D}$ and $1 < \alpha < 2$. Then $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded if and only if $k_1 < \infty$,*

$$
k_7 = \sup_{z\in\mathcal{D}} \frac{\mu(z)|2u'(z)\phi'(z)+u(z)\phi''(z)|}{(1-|\phi(z)|^2)^{\alpha-1}} < \infty,
\tag{13}
$$

$$
k_8 = \sup_{z\in\mathcal{D}} \frac{\mu(z)|u(z)||\phi'(z)|^2}{(1-|\phi(z)|^2)^{\alpha}} < \infty.
\tag{14}
$$

Proof Suppose that $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded, by Lemma 2 we know that $k_1, k_2, k_3 < \infty$. Inequality (14) can be proved as in Theorem 3. Using the test function $f_{\phi(\omega)}(z)$ in Section 1, where $z \in \mathcal{D}$, $\omega \in \mathcal{D}$ such that $\frac{1}{2} < |\phi(\omega)| < 1$, then we have that $f_{\phi(\omega)} \in \mathcal{Z}^\alpha$, and

$$
\begin{aligned}
&f_{\phi(\omega)}\bigl(\phi(\omega)\bigr) = 0, \\
&f'_{\phi(\omega)}\bigl(\phi(\omega)\bigr) = \frac{1}{\overline{\phi(\omega)}(1-|\phi(\omega)|^2)^{\alpha-1}}, \\
&f''_{\phi(\omega)}\bigl(\phi(\omega)\bigr) = \frac{2\alpha}{(1-|\phi(\omega)|^2)^{\alpha}}.
\end{aligned}
$$

By the boundedness of $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$, we have $\|DM_uC_\phi f_{\phi(\omega)}\|_{\mathcal{B}^\varphi} \leq C$, then

$$\begin{aligned}
1 &\geq S_\varphi\left(\frac{(DM_uC_\phi f_{\phi(\omega)})'(z)}{C}\right) \\
&\geq \sup_{\frac{1}{2}<|\phi(\omega)|<1} \left(1-|\omega|^2\right)\varphi\left(\frac{|(DM_uC_\phi f_{\phi(\omega)})'(\omega)|}{C}\right) \\
&\geq \sup_{\frac{1}{2}<|\phi(\omega)|<1} \left(1-|\omega|^2\right)\varphi\left(\frac{|\frac{2u'(\omega)\phi'(\omega)+u(\omega)\phi''(\omega)}{\overline{\phi(\omega)}(1-|\phi(\omega)|^2)^{\alpha-1}} + \frac{2\alpha u(\omega)\phi'(\omega)^2}{(1-|\phi(\omega)|^2)^\alpha}|}{C}\right).
\end{aligned}$$

From this and by $k_8 < \infty$, we get

$$\sup_{\frac{1}{2}<|\phi(\omega)|<1} \frac{\mu(\omega)|2u'(\omega)\phi'(\omega)+u(\omega)\phi''(\omega)|}{(1-|\phi(\omega)|^2)^{\alpha-1}} \leq C + 2C\alpha k_8 < \infty.$$

Then, according to the former proof with $k_2 < \infty$, we can get (13).

Suppose that $k_1, k_7, k_8 < \infty$. Then, by Lemma 1(iii) and (iv) and similar to the proof of Theorem 3, we get that $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded. □

Theorem 6 *Let $u \in \mathcal{H}(\mathcal{D})$ and ϕ be an analytic self-map of $\mathcal{D}$. Then $DM_uC_\phi : \mathcal{Z}^2 \to \mathcal{B}^\varphi$ is bounded if and only if*

$$k_9 = \sup_{z\in\mathcal{D}} \mu(z)\left|u''(z)\right| \log\frac{e}{1-|\phi(z)|^2} < \infty, \tag{15}$$

$$k_{10} = \sup_{z\in\mathcal{D}} \frac{\mu(z)|2u'(z)\phi'(z)+u(z)\phi''(z)|}{1-|\phi(z)|^2} < \infty, \tag{16}$$

$$k_{11} = \sup_{z\in\mathcal{D}} \frac{\mu(z)|u(z)||\phi'(z)|^2}{(1-|\phi(z)|^2)^2} < \infty. \tag{17}$$

Proof Suppose that $DM_uC_\phi : \mathcal{Z}^2 \to \mathcal{B}^\varphi$ is bounded, from Lemma 2 we know that $k_1, k_2, k_3 < \infty$. By repeating the arguments in the proof of Theorem 3 and Theorem 5, (16) and (17) can be proved similarly. Hence we only need to show $k_9 < \infty$. For every $z \in \mathcal{D}$ and $\omega \in \mathcal{D}$ such that $\frac{1}{2} < |\phi(\omega)| < 1$, let $p_{\phi(\omega)}(z) = \log\frac{e}{1-\overline{\phi(\omega)}z}$. Clearly $p_{\phi(\omega)} \in \mathcal{Z}^2$, and $p_{\phi(\omega)}(\phi(\omega)) = \log\frac{e}{1-|\phi(\omega)|^2}$,

$$\begin{aligned}
p'_{\phi(\omega)}\left(\phi(\omega)\right) &= \frac{\overline{\phi(\omega)}}{1-|\phi(\omega)|^2}, \\
p''_{\phi(\omega)}\left(\phi(\omega)\right) &= \frac{\overline{\phi(\omega)}^2}{(1-|\phi(\omega)|^2)^2}.
\end{aligned}$$

By the boundedness of $DM_uC_\phi : \mathcal{Z}^2 \to \mathcal{B}^\varphi$, we have $\|DM_uC_\phi p_{\phi(\omega)}\|_{\mathcal{B}^\varphi} \leq C$, then

$$\begin{aligned}
1 &\geq S_\varphi\left(\frac{(DM_uC_\phi p_{\phi(\omega)})'(z)}{C}\right) \\
&\geq \sup_{\frac{1}{2}<|\phi(\omega)|<1} \left(1-|\omega|^2\right)\varphi\left(\frac{|(DM_uC_\phi p_{\phi(\omega)})'(\omega)|}{C}\right)
\end{aligned}$$

$$\geq \sup_{\frac{1}{2}<|\phi(\omega)|<1} \left(1-|\omega|^2\right)$$

$$\cdot \varphi\left(\frac{|u''(\omega)\log\frac{e}{1-|\phi(\omega)|^2} + \frac{(2u'(\omega)\phi'(\omega)+u(\omega)\phi''(\omega))\overline{\phi(\omega)}}{1-|\phi(\omega)|^2} + \frac{u(\omega)(\phi'(\omega))^2\overline{\phi(\omega)}^2}{(1-|\phi(\omega)|^2)^2}|}{C}\right).$$

By $k_{10}, k_{11} < \infty$ we get

$$\sup_{\frac{1}{2}<|\phi(\omega)|<1} \mu(\omega)\left|u''(\omega)\right| \log\frac{e}{1-|\phi(\omega)|^2} \leq C + Ck_{10} + Ck_{11} < \infty. \tag{18}$$

By $k_1 < \infty$ we see that

$$\sup_{|\phi(\omega)|\leq\frac{1}{2}} \mu(\omega)\left|u''(\omega)\right| \log\frac{e}{1-|\phi(\omega)|^2} \leq C \sup_{|\phi(\omega)|\leq\frac{1}{2}} \mu(\omega)\left|u''(\omega)\right| < \infty. \tag{19}$$

From (18) and (19) we obtain (15).

Suppose that $k_9, k_{10}, k_{11} < \infty$. Then, by Lemma 1(iii) and (v) and similar to the proof of Theorem 3, we get that $DM_uC_\phi : \mathcal{Z}^2 \to \mathcal{B}^\varphi$ is bounded. □

Theorem 7 *Let $u \in \mathcal{H}(\mathcal{D})$, ϕ be an analytic self-map of $\mathcal{D}$ and $\alpha > 2$. Then $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded if and only if*

$$k_{12} = \sup_{z\in\mathcal{D}} \frac{\mu(z)|u''(z)|}{(1-|\phi(z)|^2)^{\alpha-2}} < \infty, \tag{20}$$

$$k_{13} = \sup_{z\in\mathcal{D}} \frac{\mu(z)|2u'(z)\phi'(z) + u(z)\phi''(z)|}{(1-|\phi(z)|^2)^{\alpha-1}} < \infty, \tag{21}$$

$$k_{14} = \sup_{z\in\mathcal{D}} \frac{\mu(z)|u(z)||\phi'(z)|^2}{(1-|\phi(z)|^2)^{\alpha}} < \infty. \tag{22}$$

Proof Suppose that $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded, by Lemma 2 we know that $k_1, k_2, k_3 < \infty$. With the same argument as in Theorem 5 one can show that (21) and (22) hold.

Now we prove that $k_{12} < \infty$. For every $a, z \in \mathcal{D}$, define $q_a(z) = \frac{(1-|a|^2)^2}{(1-\bar{a}z)^\alpha}$. Then $\sup_{z\in\mathcal{D}}(1-|z|^2)^\alpha|q_a''(z)| \leq 4\alpha \cdot 2^\alpha \cdot (\alpha+1)$, which shows that $q_a \in \mathcal{Z}^\alpha$. Now we let $a = \phi(\omega)$ for every $\omega \in \mathcal{D}$ such that $\frac{1}{2} < |\phi(\omega)| < 1$, and we have

$$q_{\phi(\omega)}\left(\phi(\omega)\right) = \frac{1}{(1-|\phi(\omega)|^2)^{\alpha-2}},$$

$$q'_{\phi(\omega)}\left(\phi(\omega)\right) = \frac{\alpha\overline{\phi(\omega)}}{(1-|\phi(\omega)|^2)^{\alpha-1}}, \qquad q''_{\phi(\omega)}\left(\phi(\omega)\right) = \frac{\alpha(\alpha+1)\overline{\phi(\omega)}^2}{(1-|\phi(\omega)|^2)^{\alpha}}.$$

By the boundedness of $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$, we have $\|DM_uC_\phi q_{\phi(\omega)}\|_{\mathcal{B}^\varphi} \leq C$, then

$$1 \geq S_\varphi\left(\frac{(DM_uC_\phi q_{\phi(\omega)})'(z)}{C}\right)$$

$$\geq \sup_{\frac{1}{2}<|\phi(\omega)|<1} \left(1-|\omega|^2\right)$$

$$\cdot \varphi\left(\frac{|\frac{u''(\omega)}{(1-|\phi(\omega)|^2)^{\alpha-2}} + \frac{\alpha\overline{\phi(\omega)}(2u'(\omega)\phi'(\omega)+u(\omega)\phi''(\omega))}{(1-|\phi(\omega)|^2)^{\alpha-1}} + \frac{\alpha(\alpha+1)\overline{\phi(\omega)}^2u(\omega)\phi'(\omega)^2}{(1-|\phi(\omega)|^2)^{\alpha}}|}{C}\right).$$

Then we have

$$\begin{aligned}
&\sup_{\frac{1}{2}<|\phi(\omega)|<1} \frac{\mu(\omega)|u''(\omega)|}{(1-|\phi(\omega)|^2)^{\alpha-2}} \\
&\leq C + \sup_{\frac{1}{2}<|\phi(\omega)|<1} \alpha\mu(\omega)\left|\frac{2u'(\omega)\phi'(\omega)+u(\omega)\phi''(\omega)}{\overline{\phi(\omega)}(1-|\phi(\omega)|^2)^{\alpha-1}} + \frac{\alpha+1}{2\alpha}\frac{2\alpha u(\omega)\phi'(\omega)^2}{(1-|\phi(\omega)|^2)^{\alpha}}\right| \\
&\leq C + \sup_{\frac{1}{2}<|\phi(\omega)|<1} \alpha\mu(\omega)\left\{\left|\frac{2u'(\omega)\phi'(\omega)+u(\omega)\phi''(\omega)}{\overline{\phi(\omega)}(1-|\phi(\omega)|^2)^{\alpha-1}}\right| + \frac{\alpha+1}{2\alpha}\left|\frac{2\alpha u(\omega)\phi'(\omega)^2}{(1-|\phi(\omega)|^2)^{\alpha}}\right|\right\}. \qquad (23)
\end{aligned}$$

Since $\frac{\alpha+1}{2\alpha} < 1$, then by (21), (22), (23) and according to the former proof with $k_1 < \infty$ for $|\phi(\omega)| \leq \frac{1}{2}$, then $k_{12} < \infty$. Suppose that $k_{12}, k_{13}, k_{14} < \infty$. Then, by Lemma 1(iii) and (vi) and similar to the proof of Theorem 3, we get that $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded. □

3 The compactness of $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$

In order to prove the compactness of the product-type operator $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$, we need the following lemmas. The proof of the following lemma is similar to that of Proposition 3.11 in [43]. The details are omitted.

Lemma 8 *Let $u \in \mathcal{H}(\mathcal{D})$, ϕ be an analytic self-map of $\mathcal{D}$ and $0 < \alpha < \infty$. Then $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is compact if and only if $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded and for any bounded sequence $\{f_n\}_{n\in\mathcal{N}}$ in $\mathcal{Z}^\alpha$ which converges to zero uniformly on compact subsets of $\mathcal{D}$ as $n \to \infty$, we have $\|DM_uC_\phi f_n\|_{\mathcal{B}^\varphi} \to 0$ as $n \to \infty$.*

The following lemma was essentially proved in paper [11] in Lemma 2.5.

Lemma 9 *Fix $0 < \alpha < 2$ and let $\{f_n\}_{n\in\mathcal{N}}$ be a bounded sequence in $\mathcal{Z}^\alpha$ which converges to zero uniformly on compact subsets of $\mathcal{D}$ as $n \to \infty$. Then $\lim_{n\to\infty}\sup_{z\in\mathcal{D}}|f_n(z)| = 0$. Moreover, for $0 < \alpha < 1$, if $\{f_n\}_{n\in\mathcal{N}}$ is a bounded sequence in $\mathcal{Z}^\alpha$ which converges to zero uniformly on compact subsets of $\mathcal{D}$ as $n \to \infty$, then $\lim_{n\to\infty}\sup_{z\in\mathcal{D}}|f_n'(z)| = 0$.*

Theorem 10 *Let $u \in \mathcal{H}(\mathcal{D})$, ϕ be an analytic self-map of $\mathcal{D}$ and $0 < \alpha < 1$. Then $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is compact if and only if $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded,*

$$\lim_{|\phi(z)|\to 1} \frac{\mu(z)|u(z)||\phi'(z)|^2}{(1-|\phi(z)|^2)^\alpha} = 0. \qquad (24)$$

Proof Suppose that $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is compact. It is clear that $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded. By Lemma 2, we have that $k_1, k_2, k_3 < \infty$. Let $\{z_n\}_{n\in\mathcal{N}}$ be a sequence in $\mathcal{D}$ such that $|\phi(z_n)| \to 1$ as $n \to \infty$. Without loss of generality, suppose that $|\phi(z_n)| > \frac{1}{2}$ for all n. Taking the function

$$g_n(z) = \frac{1}{\overline{\phi(z_n)}^2}\left[\frac{(1-|\phi(z_n)|^2)^2}{(1-\overline{\phi(z_n)}z)^\alpha} - \frac{1-|\phi(z_n)|^2}{(1-\overline{\phi(z_n)}z)^{\alpha-1}}\right] - \frac{1}{\overline{\phi(z_n)}}\int_0^z \frac{1-|\phi(z_n)|^2}{(1-\overline{\phi(z_n)}\lambda)^\alpha}\,d\lambda.$$

Then $\sup_{n\in\mathcal{N}}\|g_n\|_{\mathcal{Z}^\alpha} < \infty$, and $g_n \to 0$ uniformly on compact subsets of $\mathcal{D}$. Since $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is compact, then $\lim_{n\to\infty}\|DM_uC_\phi g_n\|_{\mathcal{B}^\varphi} = 0$. Since $\lim_{n\to\infty}|\phi(z_n)| = 1$, then

$\lim_{n\to\infty}\sup_{z\in\mathcal{D}}|g_n(z)| = 0$. Moreover, we have

$$g_n'\big(\phi(z_n)\big) = 0, \qquad g_n''\big(\phi(z_n)\big) = \frac{\alpha}{(1-|\phi(z_n)|^2)^\alpha}.$$

Then

$$1 \geq S_\varphi\left(\frac{(DM_uC_\phi g_n)'(z_n)}{\|DM_uC_\phi g_n\|_{\mathcal{B}^\varphi}}\right) \geq \big(1-|z_n|^2\big)\varphi\left(\frac{|u''(z_n)g_n(\phi(z_n)) + \frac{\alpha u(z_n)\phi'(z_n)^2}{(1-|\phi(z_n)|^2)^\alpha}|}{\|DM_uC_\phi g_n\|_{\mathcal{B}^\varphi}}\right).$$

Hence

$$\left|\frac{\alpha\mu(z_n)|u(z_n)||\phi'(z_n)|^2}{(1-|\phi(z_n)|^2)^\alpha} - \mu(z_n)\big|u''(z_n)\big|\big|g_n\big(\phi(z_n)\big)\big|\right| \leq \|DM_uC_\phi g_n\|_{\mathcal{B}^\varphi}.$$

Therefore

$$\lim_{|\phi(z_n)|\to 1}\frac{\mu(z_n)|u(z_n)||\phi'(z_n)|^2}{(1-|\phi(z_n)|^2)^\alpha} = \lim_{n\to\infty}\frac{\alpha\mu(z_n)|u(z_n)||\phi'(z_n)|^2}{(1-|\phi(z_n)|^2)^\alpha} = 0.$$

Suppose that $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded and (24) holds. Then $k_1, k_2, k_3 < \infty$ by Lemma 2 and for every $\epsilon > 0$, there is $\delta \in (0,1)$ such that

$$\frac{\mu(z)|u(z)||\phi'(z)|^2}{(1-|\phi(z)|^2)^\alpha} < \epsilon \tag{25}$$

whenever $\delta < |\phi(z)| < 1$. Assume that $\{f_n\}_{n\in\mathcal{N}}$ is a sequence in $\mathcal{Z}^\alpha$ such that $\sup_{n\in\mathcal{N}}\|f_n\|_{\mathcal{Z}^\alpha} \leq L$, and f_n converges to 0 uniformly on compact subsets of $\mathcal{D}$ as $n\to\infty$. Let $K = \{z \in \mathcal{D} : |\phi(z)| \leq \delta\}$. Then by $k_1, k_2, k_3 < \infty$ and (25) it follows that

$$\begin{aligned}
&\sup_{z\in\mathcal{D}}\mu(z)\big|(DM_uC_\phi f_n)'(z)\big| \\
&\quad\leq \sup_{z\in\mathcal{D}}\mu(z)\big|u''(z)\big|\big|f_n\big(\phi(z)\big)\big| + \sup_{z\in\mathcal{D}}\mu(z)\big|2u'(z)\phi'(z) + u(z)\phi''(z)\big|\big|f_n'\big(\phi(z)\big)\big| \\
&\qquad + \sup_{z\in K}\mu(z)\big|u(z)\big|\big|\phi'(z)\big|^2\big|f_n''\big(\phi(z)\big)\big| + \sup_{z\in\mathcal{D}\setminus K}\mu(z)\big|u(z)\big|\big|\phi'(z)\big|^2\big|f_n''\big(\phi(z)\big)\big| \\
&\quad\leq k_1\sup_{z\in\mathcal{D}}\big|f_n\big(\phi(z)\big)\big| + k_2\sup_{z\in\mathcal{D}}\big|f_n'\big(\phi(z)\big)\big| + k_3\sup_{z\in K}\big|f_n''\big(\phi(z)\big)\big| \\
&\qquad + \sup_{z\in\mathcal{D}\setminus K}\frac{\mu(z)|u(z)||\phi'(z)|^2(1-|\phi(z)|^2)^\alpha|f_n''(\phi(z))|}{(1-|\phi(z)|^2)^\alpha} \\
&\quad\leq k_1\sup_{\omega\in\mathcal{D}}\big|f_n(\omega)\big| + k_2\sup_{\omega\in\mathcal{D}}\big|f_n'(\omega)\big| + k_3\sup_{|\omega|\leq\delta}\big|f_n''(\omega)\big| + L\epsilon.
\end{aligned}$$

Here we use the fact that $\sup_{z\in\mathcal{D}}(1-|\phi(z)|^2)^\alpha|f_n''(\phi(z))| \leq \|f_n\|_{\mathcal{Z}^\alpha} \leq L$. So we obtain

$$\begin{aligned}
&\|DM_uC_\phi f_n\|_{B^\varphi} \\
&\quad= \big|u'(0)f_n\big(\phi(0)\big) + u(0)\phi'(0)f_n'\big(\phi(0)\big)\big| + \sup_{z\in\mathcal{D}}\mu(z)\big|(DM_uC_\phi f_n)'(z)\big| \\
&\quad\leq \big|u'(0)\big|\big|f_n\big(\phi(0)\big)\big| + \big|u(0)\big|\big|\phi'(0)\big|\big|f_n'\big(\phi(0)\big)\big| \\
&\qquad + k_1\sup_{\omega\in\mathcal{D}}\big|f_n(\omega)\big| + k_2\sup_{\omega\in\mathcal{D}}\big|f_n'(\omega)\big| + k_3\sup_{|\omega|\leq\delta}\big|f_n''(\omega)\big| + L\epsilon.
\end{aligned} \tag{26}$$

Since f_n converges to 0 uniformly on compact subsets of $\mathcal{D}$ as $n \to \infty$, Cauchy's estimation gives that f_n', f_n'' also do as $n \to \infty$. In particular, since $\{\omega : |\omega| \leq \delta\}$ and $\{\phi(0)\}$ are compact, it follows that

$$\lim_{n\to\infty} \{|u'(0)||f_n(\phi(0))| + |u(0)||\phi'(0)||f_n'(\phi(0))|\} = 0 \quad \text{and} \quad \lim_{n\to\infty} k_3 \sup_{|\omega|\leq\delta} |f_n''(\omega)| = 0.$$

Moreover, since $0 < \alpha < 1$, by Lemma 9 we have

$$\lim_{n\to\infty} \sup_{\omega\in\mathcal{D}} |f_n(\omega)| = 0, \qquad \lim_{n\to\infty} \sup_{\omega\in\mathcal{D}} |f_n'(\omega)| = 0.$$

Hence, letting $n \to \infty$ in (26), we get

$$\lim_{n\to\infty} \|DM_u C_\phi f_n\|_{\mathcal{B}^\varphi} = 0.$$

Employing Lemma 8 the implication follows. □

Theorem 11 *Let $u \in \mathcal{H}(\mathcal{D})$ and ϕ be an analytic self-map of $\mathcal{D}$. Then $DM_uC_\phi : \mathcal{Z} \to \mathcal{B}^\varphi$ is compact if and only if $DM_uC_\phi : \mathcal{Z} \to \mathcal{B}^\varphi$ is bounded,*

$$\lim_{|\phi(z)|\to 1} \mu(z)|2u'(z)\phi'(z) + u(z)\phi''(z)| \log \frac{e}{1 - |\phi(z)|^2} = 0, \tag{27}$$

$$\lim_{|\phi(z)|\to 1} \frac{\mu(z)|u(z)||\phi'(z)|^2}{1 - |\phi(z)|^2} = 0. \tag{28}$$

Proof Suppose that $DM_uC_\phi : \mathcal{Z} \to \mathcal{B}^\varphi$ is compact. It is clear that $DM_uC_\phi : \mathcal{Z} \to \mathcal{B}^\varphi$ is bounded. By Lemma 2, we have that $k_1, k_2, k_3 < \infty$. Let $\{z_n\}_{n\in\mathcal{N}}$ be a sequence in $\mathcal{D}$ such that $|\phi(z_n)| \to 1$ as $n \to \infty$. Without loss of generality, we may suppose that $|\phi(z_n)| > \frac{1}{2}$ for all n. Taking the function

$$s_n(z) = \frac{r(\overline{\phi(z_n)}z)}{\overline{\phi(z_n)}} \left(\log \frac{e}{1 - |\phi(z_n)|^2}\right)^{-1} - \left(\log \frac{e}{1 - |\phi(z_n)|^2}\right)^{-2} \int_0^z \log^3 \frac{e}{1 - \overline{\phi(z_n)}\lambda}\, d\lambda.$$

Then $\sup_{n\in\mathcal{N}} \|s_n\|_{\mathcal{Z}} < \infty$ by the proof of Theorem 4, and $s_n \to 0$ uniformly on compact subsets of $\mathcal{D}$ by a direct calculation. Consequently, $\lim_{n\to\infty} \sup_{z\in\mathcal{D}} |s_n(z)| = 0$ by Lemma 9. Since $DM_uC_\phi : \mathcal{Z} \to \mathcal{B}^\varphi$ is compact, then $\lim_{n\to\infty} \|DM_uC_\phi s_n\|_{\mathcal{B}^\varphi} = 0$. Moreover, we have

$$s_n'(\phi(z_n)) = 0, \qquad s_n''(\phi(z_n)) = -\frac{\overline{\phi(z_n)}}{1 - |\phi(z_n)|^2}.$$

Then

$$\begin{aligned} 1 &\geq S_\varphi\left(\frac{(DM_uC_\phi s_n)'(z_n)}{\|DM_uC_\phi s_n\|_{\mathcal{B}^\varphi}}\right) \\ &\geq (1 - |z_n|^2)\varphi\left(\frac{|u''(z_n)s_n(\phi(z_n)) + \frac{-\overline{\phi(z_n)}u(z_n)\phi'(z_n)^2}{1-|\phi(z_n)|^2}|}{\|DM_uC_\phi g_n\|_{\mathcal{B}^\varphi}}\right). \end{aligned}$$

It follows that

$$\left|\frac{\mu(z_n)|\phi(z_n)||u(z_n)||\phi'(z_n)|^2}{1 - |\phi(z_n)|^2} - \mu(z_n)|u''(z_n)||s_n(\phi(z_n))|\right| \leq \|DM_uC_\phi s_n\|_{\mathcal{B}^\varphi}.$$

Therefore

$$\lim_{|\phi(z_n)|\to 1}\frac{\mu(z_n)|u(z_n)||\phi'(z_n)|^2}{1-|\phi(z_n)|^2}=\lim_{n\to\infty}\frac{\mu(z_n)|\phi(z_n)||u(z_n)||\phi'(z_n)|^2}{1-|\phi(z_n)|^2}=0. \tag{29}$$

On the other hand, let

$$t_n(z)=\frac{\overline{\phi(z_n)}z-1}{\overline{\phi(z_n)}}\left[\left(1+\log\frac{e}{1-\overline{\phi(z_n)}z}\right)^2+1\right]\left(\log\frac{e}{1-|\phi(z_n)|^2}\right)^{-1}-c_n,$$

where

$$c_n=\frac{|\phi(z_n)|^2-1}{\overline{\phi(z_n)}}\left[\left(1+\log\frac{e}{1-|\phi(z_n)|^2}\right)^2+1\right]\left(\log\frac{e}{1-|\phi(z_n)|^2}\right)^{-1}$$

such that $\lim_{n\to\infty}c_n=0$. By a direct calculation, we may easily prove that $t_n\to 0$ uniformly on compact subsets of $\mathcal{D}$, and $\sup_{n\in\mathcal{N}}\|t_n\|_{\mathcal{Z}}<\infty$ by the proof of Theorem 4. Since $DM_uC_\phi:\mathcal{Z}\to\mathcal{B}^\varphi$ is compact, then $\lim_{n\to\infty}\|DM_uC_\phi t_n\|_{\mathcal{B}^\varphi}=0$. Moreover, we have

$$t_n\big(\phi(z_n)\big)=0,\qquad t_n'\big(\phi(z_n)\big)=\log\frac{e}{1-|\phi(z_n)|^2},\qquad t_n''\big(\phi(z_n)\big)=\frac{2\overline{\phi(z_n)}}{1-|\phi(z_n)|^2}.$$

Then

$$\begin{aligned}1&\geq S_\varphi\left(\frac{(DM_uC_\phi t_n)'(z_n)}{\|DM_uC_\phi t_n\|_{\mathcal{B}^\varphi}}\right)\\&\geq\big(1-|z_n|^2\big)\\&\quad\cdot\varphi\left(\frac{|(2u'(z_n)\phi'(z_n)+u(z_n)\phi''(z_n))\log\frac{e}{1-|\phi(z_n)|^2}+\frac{2\overline{\phi(z_n)}u(z_n)\phi'(z_n)^2}{1-|\phi(z_n)|^2}|}{\|DM_uC_\phi t_n\|_{\mathcal{B}^\varphi}}\right).\end{aligned}$$

It follows that

$$\begin{aligned}&\mu(z_n)\big|2u'(z_n)\phi'(z_n)+u(z_n)\phi''(z_n)\big|\log\frac{e}{1-|\phi(z_n)|^2}\\&\quad\leq\|DM_uC_\phi t_n\|_{\mathcal{B}^\varphi}+\frac{2\mu(z_n)|\phi(z_n)||u(z_n)||\phi'(z_n)|^2}{1-|\phi(z_n)|^2}.\end{aligned}\tag{30}$$

Letting $n\to\infty$ in (30) and combining with (29), we can get

$$\lim_{|\phi(z_n)|\to 1}\mu(z_n)\big|2u'(z_n)\phi'(z_n)+u(z_n)\phi''(z_n)\big|\log\frac{e}{1-|\phi(z_n)|^2}=0. \tag{31}$$

The implication follows from (29) and (31).

Conversely, by Lemma 1(ii), Lemma 2, Lemma 8 and Lemma 9, we can prove the converse implication similar to Theorem 10, so we omit the details. □

Theorem 12 *Let $u\in\mathcal{H}(\mathcal{D})$, ϕ be an analytic self-map of $\mathcal{D}$ and $1<\alpha<2$. Then $DM_uC_\phi:\mathcal{Z}^\alpha\to\mathcal{B}^\varphi$ is compact if and only if $DM_uC_\phi:\mathcal{Z}^\alpha\to\mathcal{B}^\varphi$ is bounded,*

$$\lim_{|\phi(z)|\to 1}\frac{\mu(z)|2u'(z)\phi'(z)+u(z)\phi''(z)|}{(1-|\phi(z)|^2)^{\alpha-1}}=0, \tag{32}$$

$$\lim_{|\phi(z)|\to 1} \frac{\mu(z)|u(z)||\phi'(z)|^2}{(1-|\phi(z)|^2)^\alpha} = 0. \tag{33}$$

Proof Suppose that $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is compact. It is clear that $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded. By Lemma 2, we have that $k_1, k_2, k_3 < \infty$. Let $\{z_n\}_{n\in\mathcal{N}}$ be a sequence in $\mathcal{D}$ such that $|\phi(z_n)| \to 1$ as $n \to \infty$. Without loss of generality, we may suppose that $|\phi(z_n)| > \frac{1}{2}$ for all n. Then (33) can be proved as the method of (24) in Theorem 10, so we only need to show that (32) holds. Taking the function

$$f_n(z) = \frac{1}{\overline{\phi(z_n)}^2}\left[\frac{(1-|\phi(z_n)|^2)^2}{(1-\overline{\phi(z_n)}z)^\alpha} - \frac{1-|\phi(z_n)|^2}{(1-\overline{\phi(z_n)}z)^{\alpha-1}}\right].$$

Then $\sup_{n\in\mathcal{N}} \|f_n\|_{\mathcal{Z}^\alpha} < \infty$, and $f_n \to 0$ uniformly on compact subsets of $\mathcal{D}$. Since $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is compact, it gives $\lim_{n\to\infty} \|DM_uC_\phi f_n\|_{\mathcal{B}^\varphi} = 0$. Moreover, we have

$$f_n\big(\phi(z_n)\big) = 0, \qquad f_n'\big(\phi(z_n)\big) = \frac{1}{\overline{\phi(z_n)}(1-|\phi(z_n)|^2)^{\alpha-1}}, \qquad f_n''\big(\phi(z_n)\big) = \frac{2\alpha}{(1-|\phi(z_n)|^2)^\alpha}.$$

Then

$$\begin{aligned} 1 &\ge S_\varphi\left(\frac{(DM_uC_\phi f_n)'(z_n)}{\|DM_uC_\phi f_n\|_{\mathcal{B}^\varphi}}\right) \\ &\ge \big(1-|z_n|^2\big)\varphi\left(\frac{|\frac{2u'(z_n)\phi'(z_n)+u(z_n)\phi''(z_n)}{\overline{\phi(z_n)}(1-|\phi(z_n)|^2)^{\alpha-1}} + \frac{2\alpha u(z_n)\phi'(z_n)^2}{(1-|\phi(z_n)|^2)^\alpha}|}{\|DM_uC_\phi f_n\|_{\mathcal{B}^\varphi}}\right). \end{aligned}$$

It follows that

$$\left|\frac{\mu(z_n)|2u'(z_n)\phi'(z_n)+u(z_n)\phi''(z_n)|}{|\phi(z_n)|(1-|\phi(z_n)|^2)^{\alpha-1}} - \frac{2\alpha\mu(z_n)|u(z_n)||\phi'(z_n)|^2}{(1-|\phi(z_n)|^2)^\alpha}\right| \le \|DM_uC_\phi f_n\|_{\mathcal{B}^\varphi}.$$

Therefore

$$\lim_{|\phi(z_n)|\to 1} \frac{\mu(z_n)|2u'(z_n)\phi'(z_n)+u(z_n)\phi''(z_n)|}{(1-|\phi(z_n)|^2)^{\alpha-1}} = 0.$$

By Lemma 1(iii), Lemma 2, Lemma 8 and Lemma 9, we can prove the converse implication similar to Theorem 10, so we omit the details. □

Theorem 13 *Let $u \in \mathcal{H}(\mathcal{D})$ and ϕ be an analytic self-map of $\mathcal{D}$. Then $DM_uC_\phi : \mathcal{Z}^2 \to \mathcal{B}^\varphi$ is compact if and only if $DM_uC_\phi : \mathcal{Z}^2 \to \mathcal{B}^\varphi$ is bounded,*

$$\lim_{|\phi(z)|\to 1} \mu(z)\big|u''(z)\big| \log\frac{e}{1-|\phi(z)|^2} = 0, \tag{34}$$

$$\lim_{|\phi(z)|\to 1} \frac{\mu(z)|2u'(z)\phi'(z)+u(z)\phi''(z)|}{1-|\phi(z)|^2} = 0, \tag{35}$$

$$\lim_{|\phi(z)|\to 1} \frac{\mu(z)|u(z)||\phi'(z)|^2}{(1-|\phi(z)|^2)^2} = 0. \tag{36}$$

Proof Suppose that $DM_uC_\phi : \mathcal{Z}^2 \to \mathcal{B}^\varphi$ is compact. It is clear that $DM_uC_\phi : \mathcal{Z}^2 \to \mathcal{B}^\varphi$ is bounded. By Lemma 2, we have that $k_1, k_2, k_3 < \infty$. Let $\{z_n\}_{n\in\mathcal{N}}$ be a sequence in $\mathcal{D}$ such that $|\phi(z_n)| \to 1$ as $n \to \infty$. Without loss of generality, we may suppose that $|\phi(z_n)| > \frac{1}{2}$ for all n. Then, by repeating the arguments in the proof of Theorem 10 and Theorem 12, (35) and (36) can be proved similarly, so we only need to show that (34) holds. Taking the function

$$p_n(z) = \left(1 + \left(\log \frac{e}{1 - \overline{\phi(z_n)}z}\right)^2\right)\left(\log \frac{e}{1 - |\phi(z_n)|^2}\right)^{-1}. \tag{37}$$

Then we have

$$p_n'(z) = \frac{2\overline{\phi(z_n)}}{1 - \overline{\phi(z_n)}z}\left(\log \frac{e}{1 - \overline{\phi(z_n)}z}\right)\left(\log \frac{e}{1 - |\phi(z_n)|^2}\right)^{-1}, \tag{38}$$

$$p_n''(z) = \frac{2\overline{\phi(z_n)}^2}{(1 - \overline{\phi(z_n)}z)^2}\left(\log \frac{e}{1 - \overline{\phi(z_n)}z} + 1\right)\left(\log \frac{e}{1 - |\phi(z_n)|^2}\right)^{-1}. \tag{39}$$

It is easy to show that $\{p_n\}_{n\in\mathcal{N}}$ is a bounded sequence in $\mathcal{Z}^2$, and $p_n \to 0$ uniformly on compact subsets of $\mathcal{D}$. Since $DM_uC_\phi : \mathcal{Z}^2 \to \mathcal{B}^\varphi$ is compact, then $\lim_{n\to\infty} \|DM_uC_\phi p_n\|_{\mathcal{B}^\varphi} = 0$.

From (37), (38) and (39), we can get that

$$\begin{aligned}
&\mu(z_n)\big|u''(z_n)\big|\left[\log \frac{e}{1 - |\phi(z_n)|^2} + \left(\log \frac{e}{1 - |\phi(z_n)|^2}\right)^{-1}\right] \\
&\quad - \frac{2\mu(z_n)|\phi(z_n)||2u'(z_n)\phi'(z_n) + u(z_n)\phi''(z_n)|}{1 - |\phi(z_n)|^2} \\
&\quad - \frac{2\mu(z_n)|\phi(z_n)|^2|u(z_n)||\phi'(z_n)|^2[1 + (\log \frac{e}{1-|\phi(z_n)|^2})^{-1}]}{(1 - |\phi(z_n)|^2)^2} \\
&\leq \|DM_uC_\phi p_n\|_{\mathcal{B}^\varphi}.
\end{aligned}$$

Since $\lim_{n\to\infty}(\log \frac{e}{1-|\phi(z_n)|^2})^{-1} = 0$, and by (35) and (36), we can get (34).

By Lemma 1(iii) and (v), Lemma 2 and Lemma 8, we can prove the converse implication similar to Theorem 10, so we omit the details. □

Theorem 14 *Let $u \in \mathcal{H}(\mathcal{D})$, ϕ be an analytic self-map of $\mathcal{D}$ and $\alpha > 2$. Then $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is compact if and only if $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded,*

$$\lim_{|\phi(z)|\to 1} \frac{\mu(z)|u''(z)|}{(1 - |\phi(z)|^2)^{\alpha-2}} = 0, \tag{40}$$

$$\lim_{|\phi(z)|\to 1} \frac{\mu(z)|2u'(z)\phi'(z) + u(z)\phi''(z)|}{(1 - |\phi(z)|^2)^{\alpha-1}} = 0, \tag{41}$$

$$\lim_{|\phi(z)|\to 1} \frac{\mu(z)|u(z)||\phi'(z)|^2}{(1 - |\phi(z)|^2)^{\alpha}} = 0. \tag{42}$$

Proof Suppose that $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is compact. It is clear that $DM_uC_\phi : \mathcal{Z}^\alpha \to \mathcal{B}^\varphi$ is bounded. By Lemma 2, we have that $k_1, k_2, k_3 < \infty$. Let $\{z_n\}_{n\in\mathcal{N}}$ be a sequence in $\mathcal{D}$ such that $|\phi(z_n)| \to 1$ as $n \to \infty$. Without loss of generality, we may suppose that $|\phi(z_n)| > \frac{1}{2}$ for all n. Then, by repeating the arguments in the proof of Theorem 10 and Theorem 12, (41) and (42) can be proved similarly, so we only need to show that (40) holds.

Now let $q_n(z) = \frac{(1-|\phi(z_n)|^2)^2}{(1-\overline{\phi(z_n)}z)^\alpha}$, then $\sup_{n\in\mathcal{N}} \|q_n\|_{\mathcal{Z}^\alpha} < \infty$, $q_n \to 0$ uniformly on compact subsets of $\mathcal{D}$, and

$$q_n\big(\phi(z_n)\big) = \frac{1}{(1-|\phi(z_n)|^2)^{\alpha-2}},$$

$$q_n'\big(\phi(z_n)\big) = \frac{\alpha\overline{\phi(z_n)}}{(1-|\phi(z_n)|^2)^{\alpha-1}}, \qquad q_n''\big(\phi(z_n)\big) = \frac{\alpha(\alpha+1)\overline{\phi(z_n)}^2}{(1-|\phi(z_n)|^2)^{\alpha}}.$$

Then we have

$$\begin{aligned}
&\frac{\mu(z_n)|u''(z_n)|}{(1-|\phi(z_n)|^2)^{\alpha-2}} \\
&\quad\le \|DM_uC_\phi q_n\|_{\mathcal{B}^\varphi} \\
&\qquad + \alpha\big|\phi(z_n)\big|\mu(z_n)\left|\frac{2u'(z_n)\phi'(z_n)+u(z_n)\phi''(z_n)}{(1-|\phi(z_n)|^2)^{\alpha-1}} + (\alpha+1)\overline{\phi(z_n)}\frac{u(z_n)\phi'(z_n)^2}{(1-|\phi(z_n)|^2)^{\alpha}}\right| \\
&\quad\le \|DM_uC_\phi q_n\|_{\mathcal{B}^\varphi} + \alpha\mu(z_n)\left|\frac{2u'(z_n)\phi'(z_n)+u(z_n)\phi''(z_n)}{\overline{\phi(z_n)}(1-|\phi(z_n)|^2)^{\alpha-1}} + \frac{\alpha+1}{2\alpha}\frac{2\alpha u(z_n)\phi'(z_n)^2}{(1-|\phi(z_n)|^2)^{\alpha}}\right| \\
&\quad\le \|DM_uC_\phi q_n\|_{\mathcal{B}^\varphi} \\
&\qquad + \alpha\mu(z_n)\left\{\left|\frac{2u'(z_n)\phi'(z_n)+u(z_n)\phi''(z_n)}{\overline{\phi(z_n)}(1-|\phi(z_n)|^2)^{\alpha-1}}\right| + \frac{\alpha+1}{2\alpha}\left|\frac{2\alpha u(z_n)\phi'(z_n)^2}{(1-|\phi(z_n)|^2)^{\alpha}}\right|\right\}.
\end{aligned} \tag{43}$$

Since $\frac{\alpha+1}{2\alpha} < 1$, then by (41), (42) and letting $n \to \infty$ in (43), we can get (40).

For the converse, by Lemma 1(iii) and (vi), Lemma 2 and Lemma 8, we can prove the converse implication similar to Theorem 10, so we omit the details. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper, read and approved the final manuscript.

Acknowledgements
This work is supported by the NNSF of China (Nos. 11201127; 11271112) and IRTSTHN (14IRTSTHN023).

Received: 26 February 2015 Accepted: 2 April 2015 Published online: 16 April 2015

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