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Boundedness of positive operators on weighted amalgams

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Abstract

In this article, we characterize the pairs (u, v) of positive measurable functions such that T maps the weighted amalgam $(L^{\bar{p}}(v), \ell^{\bar{q}})$ in $(L^p(u), \ell^q)$ for all $1 < p, q, \bar{p}, \bar{q} < \infty$, where T belongs to a class of positive operators which includes Hardy operators, maximal operators, and fractional integrals.

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1. Introduction

Let u be a positive function of one real variable and let $p, q > 1$. The amalgam $(L^p(u), \ell^q)$ is the space of one variable real functions which are locally in $L^p(u)$ and globally in ℓ^q. More precisely,

$$(L^p(u), \ell^q) = \{f : ||f||_{p,u,q} < \infty\},$$

where

$$||f||_{p,u,q} = \left\{ \sum_{n\in Z} \left(\int_n^{n+1} |f|^p u \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}}.$$

These spaces were introduced by Wiener in [1]. The article [2] describes the role played by amalgams in Harmonic Analysis.

Carton-Lebrun, Heinig, and Hoffmann studied in [3] the boundedness of the Hardy operator $Pf(x) = \int_{-\infty}^x |f|$ in weighted amalgam spaces. They characterized the pairs of weights (u, v) such that the inequality

$$||Pf||_{p,u,q} \le C||f||_{\bar{p},v,\bar{q}} \tag{1.1}$$

holds for all f, with a constant C independent of f, whenever $1 < \bar{q} \le q < \infty$. The characterization of the pairs (u, v) for (1.1) to hold in the case $1 < q < \bar{q} < \infty$ has been recently completed by Ortega and Ramírez ([4]), who have also characterized the weak type inequality

$$\|Pf\|_{p,\infty;u,q} \le C\|f\|_{\bar{p},v,\bar{q}},$$

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where $||g||_{p,\infty;u,q} = \left\{ \sum_{n\in\mathbb{Z}} ||g\mathcal{X}_{(n,n+1)}||_{p,\infty,u}^q \right\}^{\frac{1}{q}}$.

There are several articles dealing with the boundedness in weighted amalgams of other operators different from Hardy's one. Specifically, Carton-Lebrun, Heinig, and Hoffmann studied in [3] weighted inequalities in amalgams for the Hardy-Littlewood maximal operator as well as for some integral operators with kernel $K(x, y)$ increasing in the second variable and decreasing in the first one. On the other hand, Rakotondratsimba ([5]) characterized some weighted inequalities in amalgams (corresponding to the cases $1 < \bar{p} \le p < \infty$ and $1 < \bar{q} \le q < \infty$) for the fractional maximal operators and the fractional integrals. Finally, the authors characterized in [6] the weighted inequalities for some generalized Hardy operators, including the fractional integrals of order greater than one, in all cases $1 < p, \bar{p}, q, \bar{q} < \infty$, extending also results due to Heinig and Kufner [7].

Analyzing the results in the articles cited above, one can see some common features that lead to explore the possibility of giving a general theorem characterizing the boundedness in weighted amalgams of a wide family of positive operators, and providing, in such a way, a unified approach to the subject. This is the purpose of this article.

2. The results

We consider an operator T acting on real measurable functions f of one real variable and define a sequence $\{T_n\}_{n\in\mathbb{Z}}$ of local operators by

$$T_n f(x) = T(f\mathcal{X}_{(n-1,n+2)})(x) \quad x \in (n-1, n+2).$$

We assume that there exists a discrete operator T^{d}, i.e., which transforms sequences of real numbers in sequences of real numbers, verifying the following conditions:

(i) There exists $C > 0$ such that for all non-negative functions f, all $n \in \mathbb{Z}$ and all $x \in (n, n+1)$, the inequality

$$T\left(f\mathcal{X}_{(-\infty,n-1)} + f\mathcal{X}_{(n+2,\infty)}\right)(x) \le CT^{\mathrm{d}}\left(\left\{\int_{m-1}^{m} f\right\}\right)(n) \tag{2.1}$$

holds.

(ii) There exists $C > 0$ such that for all sequences $\{a_k\}$ of non-negative real numbers and $n \in \mathbb{Z}$, the inequality

$$T^{\mathrm{d}}(\{a_k\})(n) \le CTf(y), \tag{2.2}$$

holds for all $y \in (n, n+1)$ and all non-negative f such that $\int_{m-1}^{m} f = a_m$ for all m.

We also assume that T verifies $Tf = T|f|$, $T(\lambda f) = |\lambda|\, Tf$, $T(f+g)(x) \le Tf(x) + Tg(x)$ and $Tf(x) \le Tg(x)$ if $0 \le f(x) \le g(x)$.

We will say that an operator T verifying all the above conditions is admissible.

There is a number of important admissible operators in Analysis. For instance: Hardy operators, Hardy-Littlewood maximal operators, Riemann-Liouville, and Weyl fractional integral operators, maximal fractional operators, etc.

Our main result is the following one:

Theorem 1. *Let $1 < p, q, \bar{p}, \bar{q} < \infty$. Let u and v be positive locally integrable functions on $\mathbb{R}$ and let T be an admissible operator. Then there exists a constant $C > 0$ such that the inequality*

$$||Tf||_{p,u,q} \leq C||f||_{\bar{p},v,\bar{q}}. \tag{2.3}$$

holds for all measurable functions f if and only if the following conditions hold:

(i) T^{d} *is bounded from* $\ell^{\bar{q}}(\{v_n\})$*to* $\ell^{q}(\{u_n\})$, *where* $v_n = \left(\int_{n-1}^{n} v^{1-\bar{p}'}\right)^{-\frac{\bar{q}}{\bar{p}'}}$ *and* $u_n = \left(\int_{n}^{n+1} u\right)^{\frac{q}{p}}.$

(ii) (a) $\sup_{n\in\mathbb{Z}} ||T_n||_{(L^{\bar{p}}(v),L^{p}(u))} < \infty$*in the case* $1 < \bar{q} \leq q < \infty$.

(b) $\{||T_n||_{(L^{\bar{p}}(v),L^{p}(u))}\} \in \ell^{s}$, *with* $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$, *in the case* $1 < q < \bar{q} < \infty$.

The proof of Theorem 1 is contained in Sect. 3.

Working as in Theorem 1, we can also prove the following weak type result:

Theorem 2. *Let $1 < p, q, \bar{p}, \bar{q} < \infty$. Let u and v be positive locally integrable functions on $\mathbb{R}$ and let T be an admissible operator. Then there exists a constant $C > 0$ such that the inequality*

$$||Tf||_{p,\infty,u,q} \leq C||f||_{\bar{p},v,\bar{q}} \tag{2.4}$$

holds for all measurable functions f if and only if the following conditions hold:

(i) T^{d} *is bounded from* $\ell^{\bar{q}}(\{v_n\})$*to* ℓ^{q} $(\{u_n\}),$*), with* v_n *and un defined as in Theorem 1.*

(ii) (a) $\sup_{n\in\mathbb{Z}} ||T_n||_{(L^{\bar{p}}(v),L^{p,\infty}(u))} < \infty$*in the case* $1 < \bar{q} \leq q < \infty$.

(b) $\{||T_n||_{(L^{\bar{p}}(v),L^{p,\infty}(u))}\} \in \ell^{s}$, *with* $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$, *in the case* $1 < q < \bar{q} < \infty$.

If conditions on the weights u, v, and $\{u_n\}$, $\{v_n\}$ characterizing the boundedness of the operators T_n and T^{d}, respectively, are available in the literature, we immediately obtain, by applying Theorems 1 and 2, conditions guaranteeing the boundedness of T between the weighted amalgams. In this sense, our result includes, as particular cases, most of the results cited above from the papers [3-7], as well as other corresponding to operators whose behavior on weighted amalgams has not been studied yet.

Thus, if M^- is the one-sided Hardy-Littlewood maximal operator defined by

$$M^{-}f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f|,$$

we have:

(i) The discrete operator $(M^-)^{\mathrm{d}}$, defined by

$$(M^-)^{\mathrm{d}}(\{a_n\})(j) = \sup_{k\le j-1} \frac{1}{j-k} \sum_{i=k}^{j-1} |a_i|,$$

verifies conditions (2.1) and (2.2).

(ii) The local operators M_n^- are defined by

$$M_n^- f(x) = \sup_{0<h\le x-n+1} \frac{1}{h} \int_{x-h}^{x} |f|, \; x \in (n-1, n+2).$$

(iii) If $p = \bar{p}$ and $q = \bar{q}$, there are well-known conditions on the weights u, v, and $\{u_n\}$, $\{v_n\}$ that characterize the boundedness of M_n^- and $(M^-)^{\mathrm{d}}$ (see, for instance [8-10]).

Therefore, we obtain the following result:

Theorem 3. *The following statements are equivalent:*

(i) M^- *is bounded from* $(L^p(w), \ell^q)$ *to* $(L^p(w), \ell^q)$.

(ii) M^- *is bounded from* $(L^p(w), \ell^q)$ *to* $(L^{p,\infty}(w), \ell^q)$.

(iii) *The next conditions hold simultaneously:*

(a) $w \in A^-_{p,(n-1,n+2)}$ *for all n, uniformly, and*

(b) *the pair* $(\{u_n\}, \{v_n\})$ *verifies the discrete Sawyer's condition* S_q^-, *i.e., there exists* $C > 0$ *such that*

$$\sum_{j=r}^{k} ((M^-)^{\mathrm{d}}(\{v_n^{1-q'}\}))^q(j) u_j \le C \sum_{j=r}^{k} v_j^{1-q'},$$

for all $r, k \in \mathbb{Z}$ *with* $r \le k$.

We can state a similar result for the one-sided maximal operator M^+. In this case, the operator $(M^+)^{\mathrm{d}}$ defined by

$$(M^+)^{\mathrm{d}}(\{a_n\})(j) = \sup_{k\ge j+3} \frac{1}{k-j-2} \sum_{i=j+3}^{k} |a_i|,$$

verifies conditions (2.1) and (2.2). The theorem is the next one:

Theorem 4. *The following statements are equivalent:*

(i) M^+ *is bounded from* $(L^p(w), \ell^q)$ *to* $(L^p(w), \ell^q)$.

(ii) M^+ *is bounded from* $(L^p(w), \ell^q)$ *to* $(L^{p,\infty}(w), \ell^q)$.

(iii) *The next conditions hold simultaneously:*

(a) $w \in A^+_{p,(n-1,n+2)}$ *for all n, uniformly, and*

(b) *the pair* $(\{u_n\}, \{v_{n-3}\})$ *verifies the discrete Sawyer's condition* S_q^+, *i.e., there exists* $C > 0$ *such that*

$$\sum_{j=r}^{k} ((M^+)^{\mathrm{d}}(\{v_n^{1-q'}\}))^q(j) u_j \le C \sum_{j=r}^{k} v_j^{1-q'},$$

for all $r, k \in \mathbb{Z}$ *with* $r \leq k$.

If M is the Hardy-Littlewood maximal operator, defined by

$$Mf(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f|,$$

then M is admissible, with $M^{\mathrm{d}}(\{a_n\})(j) = \sup_{r \leq j \leq k} \frac{1}{k-r+1} \sum_{i=r}^{k} |a_i|$, and there are well-known results, due to Muckenhoupt ([11]) and Sawyer ([12]), which characterize the boundedness of M in weighted Lebesgue spaces. Applying Theorems 1 and 2, we get the following result:

Theorem 5. *The following statements are equivalent:*

(i) *M is bounded from* $(L^p(w), \ell^q)$ *to* $(L^p(w), \ell^q)$.
(ii) *M is bounded from* $(L^p(w), \ell^q)$ *to* $(L^{p,\infty}(w), \ell^q)$.
(iii) *The next conditions hold simultaneously:*
(a) $w \in A_{p,(n-1,n+2)}$ *for all n, uniformly, and*
(b) *the pair* $(\{u_n\}, \{v_n\})$ *verifies the discrete two-sided Sawyer's condition* S_q, *i.e., there exists* $C > 0$ *such that*

$$\sum_{j=r}^{k} (M^{\mathrm{d}}(\{v_n^{1-q'}\})^q(j) u_j \leq C \sum_{j=r}^{k} v_j^{1-q'}$$

for all $r, k \in \mathbb{Z}$ *with* $r \leq k$.

This result improves the one obtained by Carton-Lebrun, Heinig and Hofmann in [3], in the sense that the conditions we give are necessary and sufficient for the boundedness of the maximal operator in the amalgam $(L^p(w), \ell^q)$, while in [3] only sufficient conditons were given. We also prove the equivalence between the strong type inequality and the weak type inequality. The equivalence (i) ⇔ (iii) in Theorem 5 is included in Rakotondratsimba's paper [5], where the proof of the admissibility of M can also be found.

Finally, we will apply our results to the fractional maximal operator M_α, $0 < \alpha < 1$, defined by

$$M_\alpha f(x) = \sup_{c<x<d} \frac{1}{(d-c)^{1-\alpha}} \int_c^d |f|.$$

The proof of the admissibility of M_α, with the obvious M_α^{d}, is implied in Rakotondratsimba's paper ([5]).

Verbitsky ([13]) in the case $1 < q < p < \infty$ and Sawyer ([12]) in the case $1 < p \leq q < \infty$ characterized the boundedness of M_α from L^p to $L^q(w)$. These results allow us to give necessary and sufficient conditions on the weight u for M_α to be bounded from $(L^{\bar{p}}, \ell^{\bar{q}})$ to $(L^p(u), \ell^q)$.

Before stating the theorem, we introduce the notation:

(i) If $1 < \bar{q} < \infty$, we define $H : \mathbb{Z} \to \mathbb{R}$ by

$$H(i) = \sup_{r \le i \le k} \frac{1}{(k - r + 1)^{1-\alpha\bar{q}}} \sum_{j=r}^{k} u_j.$$

(ii) If $1 < \bar{q} \le q$, we define

$$J = \sup_{r \le k} \frac{\|\mathcal{X}_{[r,k]} M_\alpha^{\mathrm{d}}(\mathcal{X}_{[r,k]})\|_{\ell^q(\{u_j\})}}{(k - r + 1)^{\frac{1}{\bar{q}}}}.$$

(iii) If $1 < \bar{p} < \infty$ and $n \in \mathbb{Z}$, we define for $x \in (n - 1, n + 2)$

$$H_n(x) = \sup_{x \in I \subset (n-1,n+2)} \frac{1}{|I|^{1-\alpha\bar{p}}} \int_I u.$$

(iv) If $1 < \bar{p} < p$ and $n \in \mathbb{Z}$, we define

$$J_n = \sup_{I \subset (n-1,n+2)} \frac{\|\mathcal{X}_I M_\alpha(\mathcal{X}_I)\|_{L^p(u)}}{|I|^{\frac{1}{\bar{p}}}}.$$

The result reads as follows.

Theorem 6. *M_α is bounded from $(L^{\bar{p}}, \ell^{\bar{q}})$ to $(L^p(u), \ell^q)$ if and only if*

(i) *in the case* $1 < \bar{p} \le p < \infty$ *and* $1 < \bar{q} \le q < \infty$, $\sup_{n \in \mathbb{Z}} J_n < \infty$ *and* $J < \infty$;

(ii) *in the case* $1 < p < \bar{p} < \infty$ *and* $1 < \bar{q} \le q < \infty$, $\sup_{n \in \mathbb{Z}} \|H_n\|_{L^{\frac{p}{\bar{p}-p}}(u)} < \infty$ *and* $J < \infty$;

(iii) *in the case* $1 < \bar{p} \le p < \infty$ *and* $1 < q < \bar{q} < \infty$, $\{J_n\}_n \in \ell^s$, *where* $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$, *and* $H \in \ell^{\frac{q}{\bar{q}-q}}(\{u_j\})$;

(iv) *in the case* $1 < p < \bar{p} < \infty$ *and* $1 < q < \bar{q} < \infty$, $\|H_n\|_{L^{\frac{p}{\bar{p}-p}}(u)} \in \ell^s$ *and* $H \in \ell^{\frac{q}{\bar{q}-q}}(\{u_j\})$.

3. Proof of Theorem 1

Let us suppose that the inequality (2.3) holds. Let $n \in \mathbb{Z}$ and let f be a non-negative function supported in $(n - 1, n + 2)$. Then, on one hand,

$$\|f\|_{\bar{p},v,\bar{q}} = \left\{ \left(\int_{n-1}^{n} f^{\bar{p}} v \right)^{\frac{\bar{q}}{\bar{p}}} + \left(\int_{n}^{n+1} f^{\bar{p}} v \right)^{\frac{\bar{q}}{\bar{p}}} + \left(\int_{n+1}^{n+2} f^{\bar{p}} v \right)^{\frac{\bar{q}}{\bar{p}}} \right\}^{\frac{1}{\bar{q}}} \le C_{\bar{p},\bar{q}} \left(\int_{n-1}^{n+2} f^{\bar{p}} v \right)^{\frac{1}{\bar{p}}},$$

and, on the other hand,

$$
\begin{aligned}
||Tf||_{p,u,q} &\geq \left\{ \left(\int_{n-1}^{n} (Tf)^p u \right)^{\frac{q}{p}} + \left(\int_{n}^{n+1} (Tf)^p u \right)^{\frac{q}{p}} + \left(\int_{n+1}^{n+2} (Tf)^p u \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} \\
&\geq C_{p,q} \left(\int_{n-1}^{n+2} (Tf)^p u \right)^{\frac{1}{p}} \\
&\geq C_{p,q} \left(\int_{n-1}^{n+2} (T_n f)^p u \right)^{\frac{1}{p}} \\
&= C_{p,q} ||T_n f||_{p,u}.
\end{aligned}
$$

Therefore, by (2.3), T_n is bounded and $||T_n||_{(L^{\bar{p}}(v),L^p(u))} \leq C$, where C is a positive constant independent of n. Then (ii)a holds independently of the relationship between q and $\bar{q}$. Let us prove that if $1 < q < \bar{q} < \infty$, then (ii)b also holds.

It is well known that $||T_n||_{(L^{\bar{p}}(v),L^p(u))} = \sup_{\{f:||f||_{L^{\bar{p}}(v)}=1\}} ||T_n f||_{L^p(u)}$. Therefore, for each n there exists a non-negative measurable function f_n, with support in $(n - 1, n + 2)$ and with $||f_n||_{(L^{\bar{p}}(v),(n-1,n+2))} = 1$, such that $||T_n||_{(L^{\bar{p}}(v),L^p(u))} < ||T_n f_n||_{L^p(u)} + \frac{1}{2^{|n|}}$.

Since $\left\{ \frac{1}{2^{|n|}} \right\} \in \ell^s$, to prove that $\{||T_n||_{(L^{\bar{p}}(v),L^p(u))}\} \in \ell^s$ it suffices to see that $\{||T_n f_n||_{L^p(u)}\} \in \ell^s$.

Let $\{a_n\}$ be a sequence of non-negative real numbers and $f = \sum_n a_n f_n$. For each $n \in \mathbb{Z}$, $f(x) \geq a_n f_n(x)$ and then $Tf(x) \geq a_n T_n f_n(x)$ for all $x \in (n - 1, n + 2)$. Thus,

$$
||Tf||_{p,u,q} \geq C \left\{ \sum_{n\in\mathbb{Z}} \left(\int_{n-1}^{n+2} a_n^p (T_n f_n)^p u \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} = C \left\{ \sum_{n\in\mathbb{Z}} a_n^q ||T_n f_n||_{L^p(u)}^q \right\}^{\frac{1}{q}}.
$$

Then, from (2.3) we deduce

$$
\begin{aligned}
\left\{ \sum_{n\in\mathbb{Z}} a_n^q ||T_n f_n||_{L^p(u)}^q \right\}^{\frac{1}{q}} &\leq C \left\{ \sum_{n\in\mathbb{Z}} \left(\int_{n-1}^{n+2} f^{\bar{p}} v \right)^{\frac{\bar{q}}{\bar{p}}} \right\}^{\frac{1}{\bar{q}}} \\
&\leq C \left\{ \sum_{n\in\mathbb{Z}} a_n^{\bar{q}} \left(\int_{n-1}^{n+2} f_n^{\bar{p}} v \right)^{\frac{\bar{q}}{\bar{p}}} \right\}^{\frac{1}{\bar{q}}} \\
&= C \left\{ \sum_{n\in\mathbb{Z}} a_n^{\bar{q}} \right\}.
\end{aligned}
$$

This means that the identity operator is bounded from $\ell^{\bar{q}}$ to $\ell^{q}\left(\left\{||T_n f_n||^{q}_{L^p(u)}\right\}\right)$. Then $\{||T_n f_n||_{L^p(u)}\} \in \ell^s$, by applying the following lemma (see [4]).

Lemma 1. *Let* $1 < q < \bar{q} < \infty$*and* $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$. *Suppose that* $\{u_n\}$ *and* $\{v_n\}$ *are sequences of positive real numbers. The following statements are equivalent:*

(i) *There exists* $C > 0$ *such that the inequality*

$$\left\{\sum_{n\in\mathbf{Z}} (|a_n|u_n)^q\right\}^{\frac{1}{q}} \leq C\left\{\sum_{n\in\mathbf{Z}} (|a_n|v_n)^{\bar{q}}\right\}^{\frac{1}{\bar{q}}}$$

holds for all sequences $\{a_n\}$ *of real numbers.*

(ii) *The sequence* $\{u_n v_n^{-1}\}$*belongs to the space* l^s.

On the other hand, let us prove that (i) holds. If $\{a_m\}$ is a a sequence of non-negative real numbers and

$$f = \sum_{m\in\mathbb{Z}} a_m \chi_{(m-1,m)} \left(\int_{m-1}^{m} v^{1-\bar{p}'}\right)^{-1} v^{1-\bar{p}'},$$

then $\int_{m-1}^{m} f = a_m$, $\int_{m-1}^{m} f^{\bar{p}} v = a_m^{\bar{p}}\left(\int_{m-1}^{m} v^{1-\bar{p}'}\right)^{1-\bar{p}}$ and by the properties of the operator T we have

$$\begin{aligned}
||Tf||_{p,u,q} &= \left\{\sum_{n\in\mathbb{Z}} \left(\int_{n}^{n+1} (Tf)^p(x)u(x)\,\mathrm{d}x\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}} \\
&\geq C\left\{\sum_{n\in\mathbb{Z}} \left(\int_{n}^{n+1} T^{\mathrm{d}}\left(\left\{\int_{m-1}^{m} f\right\}\right)^{p}(n)u(x)\,\mathrm{d}x\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}} \\
&= C\left\{\sum_{n\in\mathbb{Z}} T^{\mathrm{d}}(\{a_m\})^q(n)\left(\int_{n}^{n+1} u(x)\,\mathrm{d}x\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}} \\
&= ||T^{\mathrm{d}}\{a_m\}||_{\ell^q\{u_n\}}.
\end{aligned}$$

Applying (2.3) we obtain

$$\begin{aligned}
||T^{\mathrm{d}}\{a_m\}||_{\ell^q(\{u_n\})} &\le C\left\{\sum_{n\in\mathbb{Z}}\left(\int_n^{n+1} f^{\bar{p}}v\right)^{\frac{\bar{q}}{\bar{p}}}\right\}^{\frac{1}{\bar{q}}}\\
&= C\left\{\sum_{n\in\mathbb{Z}} a_n^{\bar{q}}\left(\int_{n-1}^{n} v^{1-\bar{p}'}\right)^{-\frac{\bar{q}}{\bar{p}'}}\right\}^{\frac{1}{\bar{q}}}\\
&= \ ||a_n||_{\ell^{\bar{q}}(\{v_n\})},
\end{aligned}$$

which means that the discrete operator T^{d} is bounded from $\ell^{\bar{q}}(\{v_n\})$ to ℓ^q $(\{u_n\})$, as we wished to prove.

Conversely, let us suppose that (i) and (ii) hold. Then, we have

$$\begin{aligned}
||Tf||_{p,u,q} &\le C\left\{\sum_{n\in\mathbb{Z}}\left(\int_n^{n+1}(T(f\chi_{(-\infty,n-1)}+f\chi_{(n+2,\infty)}))^p u\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\
&\quad + C\left\{\sum_{n\in\mathbb{Z}}\left(\int_n^{n+1}(Tf\chi_{(n-1,n+2)})^p u\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\
&\le C\left\{\sum_{n\in\mathbb{Z}}(T^{\mathrm{d}}(\{a_m\})(n))^q\left(\int_n^{n+1} u\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\
&\quad + C\left\{\sum_{n\in\mathbb{Z}}\left(\int_n^{n+1}(T_n f)^p u\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\
&= C(I_1+I_2),
\end{aligned}$$

where $a_m = \int_{m-1}^m f$.

Applying that T^{d} is bounded from $\ell^{\bar{q}}(\{v_n\})$ to $\ell^{q}(\{u_n\})$ and Hölder inequality, we obtain

$$
\begin{aligned}
I_1 &\le C\left\{\sum_{n\in\mathbb{Z}} a_n^{\bar{q}}\left(\int_{n-1}^{n} v^{1-\bar{p}'}\right)^{-\frac{\bar{q}}{\bar{p}'}}\right\}^{\frac{1}{\bar{q}}}\\
&= C\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n} f\right)^{\bar{q}}\left(\int_{n-1}^{n} v^{1-\bar{p}'}\right)^{-\frac{\bar{q}}{\bar{p}'}}\right\}^{\frac{1}{\bar{q}}}\\
&\le C\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n} f^{\bar{p}}v\right)^{\frac{\bar{q}}{\bar{p}}}\left(\int_{n-1}^{n} v^{1-\bar{p}'}\right)^{\frac{\bar{q}}{\bar{p}'}}\left(\int_{n-1}^{n} v^{1-\bar{p}'}\right)^{-\frac{\bar{q}}{\bar{p}'}}\right\}^{\frac{1}{\bar{q}}}\\
&= C\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n} f^{\bar{p}}v\right)^{\frac{\bar{q}}{\bar{p}}}\right\}^{\frac{1}{\bar{q}}}\\
&= C\|f\|_{\bar{p},v,\bar{q}}.
\end{aligned}
$$

Now we estimate I_2. If $1 < \bar{q} \le q < \infty$, since (ii)a holds, we know that the operators T_n are uniformly bounded from $L^p(u, (n - 1, n + 2))$ to $L^{\bar{p}}(v, (n-1, n+2))$ and then

$$
\begin{aligned}
I_2 &\le \left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+2} (T_nf)^{p}u\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\
&\le C\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+2} f^{\bar{p}}v\right)^{\frac{q}{\bar{p}}}\right\}^{\frac{1}{q}}\\
&\le C\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+2} f^{\bar{p}}v\right)^{\frac{\bar{q}}{\bar{p}}}\right\}^{\frac{1}{\bar{q}}}\\
&\le C\|f\|_{\bar{p},v,\bar{q}}.
\end{aligned}
$$

Let us suppose, finally, that $1 < q < \bar{q} < \infty$. Then (ii)b holds and, therefore,

$$\begin{aligned}
I_2 &\leq C\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+2} T_n f^p u\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}} \\
&\leq C\left\{\sum_{n\in\mathbb{Z}}\left(\|T_n\|_{(L^{\bar{p}}(v),L^p(u))}\right)^q\left(\int_{n-1}^{n+2} f^{\bar{p}} v\right)^{\frac{q}{\bar{p}}}\right\}^{\frac{1}{q}} \\
&\leq C\left\{\left(\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+2} f^{\bar{p}} v\right)^{\frac{\bar{q}}{\bar{p}}}\right)^{\frac{q}{\bar{q}}}\left(\sum_{n\in\mathbb{Z}}\left(\|T_n\|_{(L^{\bar{p}}(v),L^p(u))}\right)^{\frac{q\bar{q}}{\bar{q}-q}}\right)^{\frac{\bar{q}-q}{\bar{q}}}\right\}^{\frac{1}{q}} \\
&= C\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+2} f^{\bar{p}} v\right)^{\frac{\bar{q}}{\bar{p}}}\right\}^{\frac{1}{\bar{q}}}\left(\sum_{n\in\mathbb{Z}}\left(\|T_n\|_{(L^{\bar{p}}(v),L^p(u))}\right)^s\right)^{\frac{1}{s}} \\
&\leq C\|f\|_{\bar{p},v,\bar{q}}.
\end{aligned}$$

This finishes the proof of the theorem.

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Authors' contributions
Both authors participated similarly in the conception and proofs of the results. Both authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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