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Existence and continuity for the ε -approximation equilibrium problems in Hadamard spaces

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Abstract

In this paper, the existence of ε -approximate equilibrium points for a bifunction is proved under suitable conditions in the framework of a Hadamard space. We also give the sufficient conditions for the continuity of ε -approximate solution maps to equilibrium problems. Then we apply our results to constrained minimization problems and Nash-equilibrium problems.

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1 Introduction

Let K be a nonempty subset of a linear space X and $f : K \times K \rightarrow \mathbb{R}$ a bifunction. The equilibrium problem, for short (**EP**), is defined as follows:

$$\text{Find } x \in K \text{ such that } f(x, y) \geq 0, \quad \forall y \in K. \quad (1.1)$$

(**EP**) was formulated by Blum and Oettli [1] in 1994. It is shown in the paper that (**EP**) includes many classical problems in optimization, such as variational inequality, constrained minimization, fixed point, Nash-equilibrium problems, *etc.* Since that time, there has been extensive study in various directions; see [2–4] and references therein. Almost all extended and generalized results are studied in linear spaces. Actually, some applications in optimization problems cannot be considered in linear space. Recently, in the past decade, many optimization problems were investigated on nonlinear space, such as Riemannian manifolds, hyperbolic space and geodesic spaces, for example in [5–7]. In 2012, Colao *et al.* [8] first established the existence for an equilibrium point in the setting of Hadamard manifolds. By using an analogous KKM lemma, the authors developed an equilibrium theory and gave applications to optimization-related problems in Hadamard manifolds. This paper is an inspiration for studying the generalized and/or application of (**EP**) in Hadamard manifolds; see [5, 9, 10]. The KKM lemma is the main tool for proving an existence result for (**EP**). The proof of the KKM lemma relies on the fact that the closed convex hull of any finite set is compact. It was mentioned by Kopecká and Reich [11] that, in any Hadamard

spaces, the closed convex hull of a finite number of points might not be compact. In order to extend [8]’s results to Hadamard spaces, Niculescu and Roventa [12] first introduced the famous KKM lemma and proved the existence of a minimax point in a Hadamard space which has the convex hull finite property. However, there are few papers, *e.g.* [13, 14], which deal with the existence for the solution of (EP) in Hadamard spaces.

On the other hand, when we have a mathematical problem, not only the existence of the solution for that problem but also the stability for the solution set of the problem is investigated. Roughly speaking this is about how a slight change in a parameter of a given mathematical problem could affect the solution or solution set of the problem. Hence, stability may be understood as lower (upper) semicontinuity, continuity, and Lipschitz or Hölder continuity. Currently, such a study is, in fact, very important as many useful mathematical problems are usually approximately solved by the problem. Since data of mathematical models of practical problems are obtained by measuring devices or statistical records, the models are also approximations, and hence their exact solutions are the acceptability of the approximate solution, based on a certain allowed error on the parameters of the problem. However, concerning continuity, results of approximate solutions to the parametric equilibrium problems, *e.g.* [15–17], are established in linear spaces.

Motivated and inspired by the above literature, the aim of this paper is to establish the existence result for (EP) and continuity of the solution mapping to parametric ε -approximate parametric equilibrium problem, for short (PEP), in the setting of Hadamard spaces. We also give applications to constrained minimization problems and Nash-equilibrium problems.

2 Preliminaries

Let (X, d) be a metric space. X is called a Hadamard space if it complete and if for each pair of points $x, y \in X$ there exists a point $w \in X$ such that for all $z \in X$

$$d^2(z, w) \leq \frac{1}{2}d^2(z, x) + \frac{1}{2}d^2(z, y) - \frac{1}{4}d^2(x, y).$$

A Hadamard space is sometimes called a global nonpositive curvature space or complete CAT(0) space. In the rest of the paper, we denote by X the Hadamard space. It is well known that, in Hadamard spaces, for each $x, y \in X$ there exists a unique point z in the geodesic segment joining x to y with

$$d(z, x) = td(x, y) \quad \text{and} \quad d(z, y) = (1 - t)d(x, y).$$

We also denote by $[[x, y]]$ the geodesic segment joining x to y , that is, $[[x, y]] = \{(1 - t)x \oplus ty : t \in [0, 1]\}$. A subset C of X is convex if $[[x, y]] \subseteq C$ for all $x, y \in C$.

Let $A \subseteq X$ be a nonempty subset. Then the convex hull of A , denoted $co(A)$, is defined as the intersection of all convex subset containing A . It is easy to see that $co(A) = \bigcup_{n=0}^{\infty} A_n$, where $A_0 = A$, and for $n \geq 1$, the set A_n is the union of all geodesics with start and end in A_{n-1} . The convex hull of a finite subset is not necessarily closed, and likewise the closed convex hull of A , denoted $\overline{co}(A)$, is the intersection of all closed convex subsets containing A . It is shown in [18] that if A_1, \dots, A_n are compact convex subsets in a locally convex Hausdorff space, then the convex hull of their union is also compact.

Definition 1 A function $g : C \rightarrow \mathbb{R}$ is called geodesic convex if the function $g \circ c : [0, 1] \rightarrow \mathbb{R}$ is convex for each geodesic $c : [0, 1] \rightarrow X, c(t) = c_t$, i.e.,

$$g(c_t) \leq tg(x_1) + (1 - t)g(x_2)$$

for all $t \in (0, 1)$. The function g is called geodesic concave if $-g$ is geodesic convex.

We recall the KKM mapping in the setting of Hadamard spaces which is used for the existence result for (EP).

Definition 2 [12] Let C be a nonempty subset of a Hadamard space X . A multivalued mapping $G : C \rightrightarrows E$ is said to be a KKM mapping if

$$co(F) \subseteq \bigcup_{x \in F} G(x),$$

for every nonempty finite subset F of C .

The following concept of the convex hull finite property was first introduced by [12] and used to prove the analog of the KKM lemma in Hadamard spaces.

Definition 3 [12] We say that a Hadamard space X has the convex hull finite property if the closed convex hull of every nonempty finite family of points of X has the fixed point property.

Lemma 4 [12] Suppose that X is a Hadamard space with the convex hull finite property and C is a nonempty subset of X . Furthermore, suppose $T : X \rightrightarrows X$ is a KKM mapping with closed values. Then, if $T(z)$ is compact for some $z \in C$, then

$$\bigcap_{x \in C} T(x) \neq \emptyset.$$

Throughout this paper, if not otherwise specified, let X be a Hadamard space, M be metric space, and $A \subseteq X$ be nonempty set.

Let $N(\mu_0) \subset M$ be a neighborhood of the considered point μ_0 . Let $K : M \rightrightarrows A$ be a nonempty set-valued mapping and $f : A \times A \times M \rightarrow \mathbb{R}$.

For each $\mu \in N(\mu_0)$, we consider the following parametric equilibrium problem (PEP): Find $x \in K(\mu)$ such that

$$f(x, y, \mu) \geq 0, \quad \forall y \in K(\mu). \tag{2.1}$$

Let $S(\mu)$ be the efficient solution set of (2.1), that is,

$$S(\mu) := \{x \in K(\mu) : f(x, y, \mu) \geq 0, \forall y \in K(\mu)\}.$$

For each $\mu \in N(\mu_0)$ and $\varepsilon \geq 0$, let $\tilde{S}(\varepsilon, \mu)$ denote the set of ε -approximate solution set to (2.1), that is,

$$\tilde{S}(\varepsilon, \mu) := \{x \in K(\mu) : f(x, y, \mu) + \varepsilon \geq 0, \forall y \in K(\mu)\}.$$

We collect some concepts and properties of semicontinuity for set-valued mappings in metric spaces.

Definition 5 [19, 20] Let X and Y be topological spaces, $T : X \rightrightarrows Y$ be a set-valued mapping.

- (i) T is said to be upper semicontinuous (u.s.c., for short) at $x_0 \in X$ iff for any open set V containing $T(x_0)$, there exists an open set U containing x_0 such that $T(x) \subseteq V$ for all $x \in U$.
- (ii) T is said to be lower semicontinuous (l.s.c., for short) at $x_0 \in X$ iff for any open set V with $T(x_0) \cap V \neq \emptyset$, there exists an open set U containing x_0 such that $T(x) \cap V \neq \emptyset$ for all $x \in U$.
- (iii) T is said to be continuous at $x_0 \in X$ iff it is both l.s.c. and u.s.c.

Proposition 6 [19, 21] Let X and Y be metric spaces and $T : X \rightrightarrows Y$ and $x_0 \in X$. Then the following are equivalent:

- (i) $T(\cdot)$ is l.s.c. at x_0 ;
- (ii) if $\{x_n\}$ is any sequence such that $x_n \rightarrow x_0$ and $G \subseteq Y$ an open subset such that $T(x_0) \cap G \neq \emptyset$, then

$$\exists N \geq 1 : T(x_n) \cap G \neq \emptyset, \quad \forall n \geq N;$$

- (iii) if $\{x_n\}$ is any sequence such that $x_n \rightarrow x_0$ and $y_0 \in T(x_0)$ arbitrary, then there is a sequence $\{y_n\}$ with $y_n \in T(x_n)$ such that $y_n \rightarrow y_0$.

Proposition 7 [19, 21] Let X and Y be metric spaces and $T : X \rightrightarrows Y$ and $x_0 \in X$. Then the following are equivalent:

- (i) $T(\cdot)$ is u.s.c. at x_0 ;
- (ii) if $x_0 \in X$ and $\{x_n\}$ is any sequence such that $x_n \rightarrow x_0$ and $V \subseteq Y$ an open subset such that $T(x_0) \subseteq V$, then

$$\exists N \geq 1 : T(x_n) \subseteq V, \quad \forall n \geq N.$$

Let $A, \{A_n\}_{n \in \mathbb{N}}$ be nonempty sets in a metric space X . We shall use the following notations:

$$\liminf A_n = \{x \in X : \exists \{x_n\} \text{ such that } x_n \in A_n, \forall n \in \mathbb{N}, x_n \rightarrow x\};$$

$$\limsup A_n = \{x \in X : \exists \{n_k\} \text{ and } \exists \{x_{n_k}\} \text{ such that } x_{n_k} \in A_{n_k}, \forall k \in \mathbb{N}, x_{n_k} \rightarrow x\}.$$

Remark 8 It follows from (iii) in Proposition 6 that $T(\cdot)$ is l.s.c. at x_0 iff, for every sequence $x_n \rightarrow x_0$, we have $T(x_0) \subseteq \liminf T(x_n)$.

3 Existence and continuity results

In this section, we first present the existence result for an ε -approximate solution for **(EP)**. We also study the continuity of ε -approximate solution maps for **(PEP)**.

Theorem 9 Let K be a nonempty closed convex subset of a Hadamard space X which has the convex hull finite property. For any given $\varepsilon > 0$, let $f : K \times K \rightarrow \mathbb{R}$ be a bifunction such that

- (i) for any $x \in K, f(x, x) \geq 0$;
- (ii) for every $x \in K$, the set $\{y \in K : f(x, y) + \varepsilon < 0\}$ is convex set;
- (iii) for every $y \in K, f(\cdot, y)$ is upper semicontinuous;
- (iv) there exist a compact set $L \subseteq X$ and a point $y_0 \in L \cap K$ such that

$$f(x, y_0) + \varepsilon < 0, \quad \forall x \in K \setminus L.$$

Then there exists a point $x_0 \in L \cap K$ such that

$$f(x_0, y) + \varepsilon \geq 0, \quad \forall y \in K.$$

Proof Define the mapping $G : K \rightrightarrows K$ as follows:

$$G(y) = \{x \in K : f(x, y) + \varepsilon \geq 0\} \quad \text{for all } y \in K.$$

We first prove that $G(y)$ is closed for all $y \in K$. Let $\{x_m\}$ be a sequence in $G(y)$ such that $x_m \rightarrow x'$ for some $x' \in X$. Then we have $f(x_m, y) + \varepsilon \geq 0$ for all m . It follows from (iii) that

$$f(x', y) + \varepsilon \geq \limsup_{m \rightarrow \infty} f(x_m, y) + \varepsilon \geq 0.$$

This implies that $G(y)$ is closed for all $y \in K$. By condition (iv) there exists a point $y_0 \in K$ for which $G(y_0) \subseteq L$. Since L is compact, we see that $G(y_0)$ is also. We want to prove that $G(\cdot)$ is a KKM mapping. We prove this by contradiction, suppose that G is not a KKM mapping. Then there exist a finite subset $A = \{x_1, x_2, \dots, x_n\}$ of K and a point $\hat{x} \in \text{co}(A)$ such that $\hat{x} \notin G(y_i)$ for all $i = 1, \dots, n$. Thus

$$f(\hat{x}, y_i) + \varepsilon < 0, \quad \forall i \in \{1, 2, \dots, n\}.$$

That is, for any $i \in \{1, \dots, n\}$,

$$y_i \in \{y \in K : f(\hat{x}, y) + \varepsilon < 0\}.$$

Since the set $\{y \in K : f(\hat{x}, y) + \varepsilon < 0\}$ is convex, we get

$$\hat{x} \in \text{co}(\{y_1, y_2, \dots, y_n\}) \subseteq \{y \in K : f(\hat{x}, y) + \varepsilon < 0\},$$

which is a contradiction to the assumption (i). Then by Lemma 4, there exists a point $x_0 \in K$ such that

$$x_0 \in \bigcap_{y \in K} G(y).$$

Hence, there exists $x_0 \in L \cap K$ such that

$$f(x_0, y) + \varepsilon \geq 0, \quad \forall y \in K. \quad \square$$

Remark 10

- (i) Our Theorem 9 is closely related to Theorem 3.3 in [13], which is presented in the sense of a minimax inequality.
- (ii) If $\varepsilon = 0$, then we have the existence result for an exact solution for **(EP)** which encloses the result in [12].

By setting $L = K$ in Theorem 9, the following corollary is immediately obtained.

Corollary 11 *Let K be a nonempty compact convex subset of a Hadamard space X which has the convex hull finite property. For any given $\varepsilon > 0$, let $f : K \times K \rightarrow \mathbb{R}$ be a bifunction such that*

- (i) *for any $x \in K, f(x, x) \geq 0$;*
- (ii) *for every $x \in K$, the set $\{y \in K : f(x, y) + \varepsilon < 0\}$ is convex set;*
- (iii) *for every $y \in K, f(\cdot, y)$ is upper semicontinuous.*

Then there exists a point $x_0 \in K$ such that

$$f(x_0, y) + \varepsilon \geq 0, \quad \forall y \in K.$$

The following example shows that the ε -approximate solution for **(EP)** depends on ε . Consider \mathbb{R}^2 with the usual Euclidean meter $d(\cdot, \cdot)$ and $\|\cdot\|$, which are defined by

$$d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We define the radial metric d_r by

$$d_r(x, y) = \begin{cases} d(x, y), & \text{if } y = tx \text{ for some } t \in \mathbb{R}; \\ d(x, 0) + d(y, 0), & \text{otherwise.} \end{cases}$$

Then $X := (\mathbb{R}^2, d_r)$ is an \mathbb{R} -tree with radial meter d_r .

Example 12 Let X be an \mathbb{R} -tree with radial meter d_r . We put $K = [(1, 0), (0, 1)]$ and $f(x, y) = (x_1 + x_2) - 0.3$. After calculating we easily get

$$\tilde{S}(\varepsilon, \mu) = \begin{cases} [(0, 0.3), (0, 1)] \cup [(0.3, 0), (1, 0)], & \text{if } 0 \leq \varepsilon < 0.3, \\ [(1, 0), (0, 1)], & \text{if } \varepsilon \geq 0.3. \end{cases}$$

The following corollary is a sufficient condition for the existence of the parametric ε -approximate solution **(PEP)**.

Corollary 13 *For any given $\varepsilon > 0$, assume that there exists a neighborhood $N(\mu_0)$ of μ_0 satisfying the following conditions:*

- (i) *for each $\mu \in N(\mu_0), K(\mu)$ is a nonempty, compact and convex valued;*
- (ii) *for each $\mu \in N(\mu_0)$ and each $x \in K(N(\mu_0)), f(x, x, \mu) \geq 0$;*
- (iii) *for each $x \in K(N(\mu_0)),$ the set $\{y \in K(N(\mu_0)) : f(x, y, \mu) + \varepsilon < 0\}$ is convex set;*
- (iv) *for each $\mu \in N(\mu_0)$ and each $y \in K(N(\mu_0)), f(\cdot, y, \mu)$ is upper semicontinuous on $K(N(\mu_0))$.*

Then for each $\mu \in N(\mu_0), \tilde{S}(\varepsilon, \mu)$ is nonempty and compact.

Now, we give the sufficient conditions for continuity of an approximate solution \tilde{S} at (ε_0, μ_0) . Relying on the existence theorem for ε -approximation (EP), we assume that the solution of the ε -approximation exists for all $(\varepsilon, \mu) \in \mathbb{R}_+ \cup \{0\} \times M$.

Theorem 14 Consider the ε -approximate (PEP). We assume that the following conditions hold:

- (C₁) K is continuous at μ_0 and $K(\mu_0)$ has compact and convex valued;
- (C₂) there exists a neighborhood $N(\mu_0)$ of μ_0 such that $f(\cdot, \cdot, \cdot)$ is continuous on $K(N(\mu_0)) \times K(N(\mu_0)) \times \{\mu_0\}$;
- (C₃) for each $y \in K(\mu_0)$, $f(\cdot, y, \mu_0)$ is a geodesic concave function on $K(N(\mu_0))$.

Then $\tilde{S}(\cdot, \cdot)$ is l.s.c. at (ε_0, μ_0) .

Proof We first prove that $\tilde{S}(\cdot, \mu_0)$ is l.s.c. at ε_0 . Suppose not, then there exist an open set $G \subseteq X$ and a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \rightarrow \varepsilon_0$ such that

$$\tilde{S}(\varepsilon_0, \mu_0) \cap G \neq \emptyset \quad \text{but} \quad \tilde{S}(\varepsilon_n, \mu_0) \cap G = \emptyset \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

Note that $\tilde{S}(\varepsilon_1, \mu_0) \subseteq \tilde{S}(\varepsilon_2, \mu_0)$, if $\varepsilon_1 \leq \varepsilon_2$.

We claim that $\varepsilon_0 > \varepsilon_n$ for all $n \in \mathbb{N}$. If not, there exists $\bar{N} \in \mathbb{N}$ such that $\varepsilon_0 \leq \varepsilon_{\bar{N}}$. That is,

$$\tilde{S}(\varepsilon_0, \mu_0) \subseteq \tilde{S}(\varepsilon_{\bar{N}}, \mu_0), \quad \text{and so} \quad \emptyset \neq \tilde{S}(\varepsilon_0, \mu_0) \cap G \subseteq \tilde{S}(\varepsilon_{\bar{N}}, \mu_0) \cap G,$$

which is a contradiction to (3.1). So we have the claim.

Let $x_0 \in \tilde{S}(0, \mu_0)$ and $\bar{x} \in \tilde{S}(\varepsilon_0, \mu_0) \cap G$. We set $x_n = \frac{\varepsilon_0 - \varepsilon_n}{\varepsilon_0} \bar{x} \oplus \frac{\varepsilon_n}{\varepsilon_0} x_0$, then $x_n \rightarrow \bar{x}$ as $n \rightarrow \infty$. Since $\bar{x} \in G$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\varepsilon_0 - \varepsilon_{n_0}}{\varepsilon_0} \bar{x} \oplus \frac{\varepsilon_{n_0}}{\varepsilon_0} x_0 = x_{n_0} \in G. \tag{3.2}$$

We claim that $x_{n_0} \in \tilde{S}(\varepsilon_{n_0}, \mu_0)$. Since $x_0 \in \tilde{S}(0, \mu_0)$ and $\bar{x} \in \tilde{S}(\varepsilon_0, \mu_0)$, we have, for all $y \in K(\mu_0)$,

$$f(x_0, y, \mu_0) \geq 0 \quad \text{and} \quad f(\bar{x}, y, \mu_0) + \varepsilon_0 \geq 0.$$

Hence

$$\frac{\varepsilon_0 - \varepsilon_{n_0}}{\varepsilon_0} f(x_0, y, \mu_0) \geq 0 \quad \text{and} \quad \frac{\varepsilon_{n_0}}{\varepsilon_0} f(\bar{x}, y, \mu_0) + \frac{\varepsilon_{n_0}}{\varepsilon_0} \varepsilon_0 \geq 0.$$

It follows from (iii) that

$$f\left(\frac{\varepsilon_0 - \varepsilon_{n_0}}{\varepsilon_0} \bar{x} \oplus \frac{\varepsilon_{n_0}}{\varepsilon_0} x_0, y, \mu_0\right) \geq \frac{\varepsilon_0 - \varepsilon_{n_0}}{\varepsilon_0} f(x_0, y, \mu_0) + \frac{\varepsilon_{n_0}}{\varepsilon_0} f(\bar{x}, y, \mu_0) + \varepsilon_{n_0} \geq 0.$$

Therefore, $x_{n_0} \in \tilde{S}(\varepsilon_{n_0}, \mu_0)$, which leads to a contradiction to (3.1). We can conclude that $\tilde{S}(\cdot, \mu_0)$ is l.s.c. at ε_0 .

Next, we prove that $\tilde{S}(\varepsilon, \cdot)$ is l.s.c. at μ_0 for all $\varepsilon > 0$. By the assumption of nonemptiness of the exact solution, we set the following strict ε -approximate solution maps:

$$\tilde{S}_S(\varepsilon, \mu) := \{x \in K(\mu_0) : f(x, y, \mu_0) + \varepsilon > 0, \forall y \in K(\mu_0)\}.$$

It is clear that, for any $\mu \in N(\mu_0)$,

$$\emptyset \neq \tilde{S}(0, \mu) \subseteq \tilde{S}_S(\varepsilon, \mu) \subseteq \tilde{S}(\varepsilon, \mu). \tag{3.3}$$

We show that $\tilde{S}_S(\varepsilon, \cdot)$ is l.s.c. at μ_0 for all $\varepsilon > 0$. If not, there are $\bar{x} \in \tilde{S}_S(\varepsilon, \mu_0)$ and a sequence $\mu_n \rightarrow \mu_0$, but for all $x_n \in \tilde{S}_S(\varepsilon, \mu_n)$, $x_n \not\rightarrow \bar{x}$. Since $K(\mu_0)$ is l.s.c. at μ_0 , there exists $\bar{x}_n \in K(\mu_n)$ with $\bar{x}_n \rightarrow \bar{x}$. By the above contradiction assumption, there must be a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ such that $\bar{x}_{n_k} \notin \tilde{S}_S(\varepsilon, \mu_n)$, i.e., there exists $y_{n_k} \in K(\mu_{n_k})$ such that

$$f(\bar{x}_{n_k}, y_{n_k}, \mu_{n_k}) + \varepsilon \leq 0. \tag{3.4}$$

Since $K(\cdot)$ is u.s.c. at μ_0 and $K(\mu_0)$ is compact valued, there exists $\bar{y} \in K(\mu_0)$ such that $y_{n_k} \rightarrow \bar{y}$. From (3.4) and continuity of $f(\cdot, \cdot, \cdot)$ on $K(N(\mu_0)) \times K(N(\mu_0)) \times \{\mu_0\}$, we get

$$f(\bar{x}, \bar{y}, \mu_0) + \varepsilon \leq 0,$$

which leads to a contradiction. Hence $\tilde{S}_S(\varepsilon, \cdot)$ is l.s.c. at μ_0 .

Next, we claim that

$$\tilde{S}(\varepsilon, \mu_0) \subseteq cl(\tilde{S}_S(\varepsilon, \mu_0)). \tag{3.5}$$

For any $x_0 \in \tilde{S}(\varepsilon, \mu_0)$ and $x_1 \in \tilde{S}_S(\varepsilon, \mu_0)$, we put $x_t = (1 - t)x_0 \oplus tx_1$, $t \in (0, 1)$. Then $x_t \rightarrow x_0$ as $t \rightarrow +0$. Since $f(\cdot, y, \mu_0)$ is geodesic concave, we have

$$\begin{aligned} f(x_t, y, \mu_0) + \varepsilon &\geq (1 - t)f(x_0, y, \mu_0) + tf(x_1, y, \mu_0) + \varepsilon \\ &= (1 - t)(f(x_0, y, \mu_0) + \varepsilon) + t(f(x_1, y, \mu_0) + \varepsilon) \\ &\geq 0, \quad \forall y \in K(\mu_0). \end{aligned}$$

Hence, $x_t \in \tilde{S}_S(\varepsilon, \mu_0)$. This implies that $x_0 \in cl(\tilde{S}_S(\varepsilon, \mu_0))$. So, we have the claim. Hence, for all $\varepsilon > 0$, $\tilde{S}_S(\varepsilon, \cdot)$ is l.s.c. at μ_0 . Thus, for any $\mu_m \rightarrow \mu_0$

$$cl(\tilde{S}_S(\varepsilon, \mu_0)) \subseteq \liminf \tilde{S}_S(\varepsilon, \mu_m).$$

Combining the last inequality with (3.3) and (3.5), we have

$$\tilde{S}(\varepsilon, \mu_0) \subseteq cl(\tilde{S}_S(\varepsilon, \mu_0)) \subseteq \liminf \tilde{S}_S(\varepsilon, \mu_m) \subseteq \liminf \tilde{S}(\varepsilon, \mu_m).$$

Therefore $\tilde{S}(\varepsilon, \cdot)$ is l.s.c. at μ_0 . We can conclude from Step 1 and 2 that $\tilde{S}(\cdot, \cdot)$ is l.s.c. at (ε_0, μ_0) . □

Theorem 15 Assume that the conditions (C_1) - (C_2) hold. Then $\tilde{S}(\cdot, \cdot)$ is u.s.c. at (ε_0, μ_0) .

Proof We prove by contradiction, suppose that $\tilde{S}(\cdot, \cdot)$ is not u.s.c. at (ε_0, μ_0) . Then there exist an open neighborhood U of $\tilde{S}(\varepsilon_0, \mu_0)$ and a sequence $\{(\varepsilon_n, \mu_n)\}$ converging to (ε_0, μ_0) such that

$$\tilde{S}(\varepsilon_n, \mu_n) \not\subseteq U, \quad \forall n \in \mathbb{N}.$$

Then there exists some $x_n \in \tilde{S}(\varepsilon_n, \mu_n)$ such that

$$x_n \notin U, \quad \forall n \in \mathbb{N}. \tag{3.6}$$

Since $x_n \in \tilde{S}(\varepsilon_n, \mu_n)$, we have $x_n \in K(\mu_n)$. By the assumption, $K(\cdot)$ is u.s.c. being compact valued at μ_0 , then we see that there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightarrow x^*$.

We will show that $x^* \in \tilde{S}(\varepsilon_0, \mu_0)$. Suppose to the contrary that $x^* \notin \tilde{S}(\varepsilon_0, \mu_0)$. Then there exists $y^* \in A(\mu_0)$ such that

$$f(x^*, y^*, \mu_0) + \varepsilon < 0. \tag{3.7}$$

Since $K(\cdot)$ is l.s.c. at μ_0 and $y^* \in K(\mu_0)$, $\varepsilon_n \rightarrow \varepsilon_0$ and $\mu_n \rightarrow \mu_0$, we see that there exists $y_n \in \tilde{S}(\varepsilon_n, \mu_n)$ such that $y_n \rightarrow y^*$. It follows from $y_n \in K(\mu_n)$ that

$$f(x_n, y_n, \mu_n) + \varepsilon \geq 0, \quad \forall n \in \mathbb{N}.$$

Condition (C_2) gives $f(x^*, y^*, \mu_0) + \varepsilon \geq 0$. This is a contradiction with (3.7). Hence, we have $x^* \in \tilde{S}(\varepsilon_0, \mu_0) \subseteq U$. Since $x_n \rightarrow x^*$ and U is an open set, there exists some $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$, which is a contradiction with (3.6). Therefore, $\tilde{S}(\cdot, \cdot)$ is u.s.c. at (ε_0, μ_0) . □

Theorem 16 *Consider the ε -approximate (PEP). We assume that the conditions (C_1) - (C_3) hold. Then $\tilde{S}(\cdot, \cdot)$ is continuous at (ε_0, μ_0) .*

The following example illustrates that Theorem 16 cannot apply with exact solution maps to (PEP).

Example 17 Let X be an \mathbb{R} -tree with radial meter d_r . We put $M = [0, 1]$ and $K(\mu) = [(1, 0), (0, 1)]$ and $f(x, y, \mu) = -\mu(x_1 + x_2)$. Then the assumptions of Theorem 9 are satisfied. Direct computations give us the approximate solution set, for positive $\varepsilon > 0$ and $\mu \in M$,

$$\tilde{S}(\varepsilon, \mu) = \begin{cases} [(1, 0), (0, 1)], & \text{if } \mu = 0, \\ [(\min\{\frac{\varepsilon}{\mu}, 1\}, 0), (0, \min\{\frac{\varepsilon}{\mu}, 1\})], & \text{if } \mu \neq 0. \end{cases}$$

We see that $\tilde{S}(\cdot, \cdot)$ is not l.s.c. at $(0, 0)$. Indeed, for $(\varepsilon_n, \mu_n) = (\frac{1}{2n}, \frac{1}{n}) \rightarrow (0, 0)$ and $(1, 0) \in \tilde{S}(0, 0)$, it is clear that there is no sequence in $\tilde{S}(\frac{1}{2n}, \frac{1}{n}) = [(\frac{1}{2}, 0), (0, \frac{1}{2})]$ which converges to $(1, 0)$. Hence, $\tilde{S}(\cdot, \cdot)$ is not l.s.c. at $(0, 0)$.

4 Applications

As mentioned in Section 1, the (EP) contains many optimization related problems as special cases. Therefore, we derive the continuity of the results of Section 3 for such special cases. In this section, we give applications to constrained minimization problems and Nash-equilibrium problems.

4.1 Constrained minimization problems

Let X, M, A , and K be as in Section 3. Let $g : A \times M \rightarrow \mathbb{R}$. For $\mu \in M$, we consider the constrained minimization problem, for short (CMP), as follows:

$$\text{Finding } x \in K(\mu) \text{ such that } g(x, \mu) \leq g(y, \mu), \quad \forall y \in K(\mu). \tag{4.1}$$

If we set $f(x, y, \mu) = g(y, \mu) - g(x, \mu)$, we see that (CMP) becomes an example of (EP). For $(\varepsilon, \mu) \in \mathbb{R}_+ \times M$, we denote the parametric ε -approximate solution set of (CMP) by $\tilde{S}_{CMP}(\varepsilon, \mu)$, that is,

$$\tilde{S}_{CMP}(\varepsilon, \mu) = \{x \in K(\mu) \text{ such that } g(y, \mu) - g(x, \mu) + \varepsilon \geq 0, \forall y \in K(\mu)\}. \tag{4.2}$$

Now, we present the continuity result for (CMP); we assume that $\tilde{S}_{CMP}(\varepsilon, \mu)$ exists for all $(\varepsilon, \mu) \in \mathbb{R}_+ \cup \{0\} \times M$.

Corollary 18 *Assume that the following conditions hold:*

- (M₁) K is continuous at μ_0 and $K(\mu_0)$ is compact and convex;
- (M₂) there exists a neighborhood $N(\mu_0)$ of μ_0 such that g is continuous in $K(N(\mu_0)) \times \{\mu_0\}$;
- (M₃) $g(\cdot, \mu_0)$ is geodesic convex in $K(N(\mu_0))$.

Then $\tilde{S}_{CMP}(\cdot, \cdot)$ is continuous at (ε_0, μ_0) .

Proof It suffices to show that all assumptions in Theorem 16 hold. It is easy to see that (C₁) and (C₂) hold. We only check that (C₃) holds. For any $x_1, x_2 \in K(N(\mu_0))$, $t \in [0, 1]$ and $y \in K(N(\mu_0))$, we put $x_t = tx_1 \oplus (1 - t)x_2$. Then

$$\begin{aligned} & f(x_t, y, \mu_0) - tf(x_1, y, \mu_0) - (1 - t)f(x_2, y, \mu_0) \\ &= g(y, \mu_0) - g(x_t, \mu_0) - t(g(y, \mu_0) - g(x_1, \mu_0)) - (1 - t)(g(y, \mu_0) - g(x_2, \mu_0)) \\ &= g(y, \mu_0) - tg(x_1, \mu_0) - (1 - t)g(x_2, \mu_0) \\ &\quad - t(g(y, \mu_0) - g(x_1, \mu_0)) - (1 - t)(g(y, \mu_0) - g(x_2, \mu_0)) \\ &= tg(x_1, \mu_0) + (1 - t)g(x_2, \mu_0) - g(x_t, \mu_0) \geq 0. \end{aligned}$$

This implies that (C₃) holds. This completes the proof. □

4.2 Nash-equilibrium problems

Let $I = \{1, 2, \dots, N\}$ be a finite index set which denotes the set of players. For any $i \in I$, consider X_i to be a Hadamard space where the strategy set $K_i \subseteq X_i$ of the i th player will be given. Let $K = \prod_{i=1}^N K_i$ belong to the product Hadamard space $X = \prod_{i=1}^N X_i$. For arbitrary $x = (x_1, \dots, x_N) \in X$ we define $x^i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$. Suppose that for every $i \in I$ there

exists a payoff function $f_i : K \rightarrow \mathbb{R}$ representing the loss of each player, depending on the strategies of all player. A Nash equilibrium [22] is a strategy profile with the property that the strategy of each player is an optimal choice given the strategy of the other players. Thus, a strategy $x = (x_1, x_2, \dots, x_N)$ is a Nash equilibrium if

$$f_i(x) \leq f_i(x^i, y_i), \quad \forall y_i \in K_i.$$

This problem can be formulated as an equilibrium problem by defining the bifunction $F : K \times K \rightarrow \mathbb{R}$ as

$$F(x, y) = \sum_{i \in I} (f_i(x^i, y_i) - f_i(x)), \quad \forall (x, y) \in K \times K. \tag{4.3}$$

It can be shown that x is a Nash-equilibrium point if and only if it is an a solution of the equilibrium problem (4.3). Our aim is to consider the continuity for Nash-equilibrium points, we assume further that the strategy set K_i and payoff function f_i are perturbed, that is, they depend on a perturbation parameter μ of a metric space M for all i . More precisely, $f_i : K \times M \rightarrow \mathbb{R}$ and $K_i : M \rightrightarrows X$ for all i .

For $(\varepsilon, \mu) \in \mathbb{R}_+ \times M$, we denote the ε -approximate solution set of Nash-equilibrium points by $\tilde{S}_{NEP}(\varepsilon, \mu)$, that is,

$$\tilde{S}_{NEP}(\varepsilon, \mu) = \{x \in K(\mu) \text{ such that } F(x, y) + \varepsilon \geq 0, \forall y \in K(\mu)\}. \tag{4.4}$$

Corollary 19 *We assume that the following conditions hold.*

- (N₁) $K_i(\cdot)$ is continuous at μ_0 and $K_i(\mu_0)$ is compact and convex valued for all i ;
- (N₂) there exists a neighborhood $N(\mu_0)$ of μ_0 such that $f_i(\cdot, \cdot, \cdot)$ is continuous on $K_i(N(\mu_0)) \times K_i(N(\mu_0)) \times \{\mu_0\}$ for all i ;
- (N₃) for each $y \in K_i(N(\mu_0))$, $f_i(\cdot, y, \mu_0)$ is a geodetically concave function on $K_i(N(\mu_0))$ for all i .

Then $\tilde{S}_{NEP}(\cdot, \cdot)$ is continuous at (ε_0, μ_0) .

Proof If we set $K(\mu) = \prod_{i=1}^N K_i(\mu)$ and F as in (4.3), we see that all assumptions in Theorem 16 are satisfied. Hence, we have the desired conclusion. □

5 Conclusions

In this paper, the classical existence result for ε -approximate equilibrium problems is proved in the framework of the Hadamard space (nonlinear space). We also establish the sufficient conditions for the continuity of ε -approximate solution maps to equilibrium problems. However, the conclusion of Theorem 16 is not true for an exact solution, in the case $\varepsilon = 0$. As applications, we apply the continuity results to constrained minimization problems and Nash-equilibrium problems.

Competing interests

The author declares that they have no competing interests.

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