Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some basic properties of certain subclasses of meromorphically starlike functions

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Abstract

In this paper, we introduce and investigate certain subclasses of meromorphically starlike functions. Such results as coefficient inequalities, neighborhoods, partial sums, and inclusion relationships are derived. Relevant connections of the results derived here with those in earlier works are also pointed out.

MSC: Primary 30C45; secondary 30C80

Keywords: meromorphic function; starlike function; Hadamard product (or convolution); neighborhood; partial sum

1 Introduction

Let Σ denote the class of functions f of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.1}$$

which are *analytic* in the *punctured* open unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}.$$

A function $f \in \Sigma$ is said to be in the class $\mathcal{MS}^*(\alpha)$ of *meromorphically starlike functions of order* α if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (z \in \mathbb{U}; 0 \leqq \alpha < 1).$$

Let $\mathcal{P}$ denote the class of functions p given by

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \quad (z \in \mathbb{U}), \tag{1.2}$$

which are analytic in $\mathbb{U}$ and satisfy the condition

$$\Re\big(p(z)\big) > 0 \quad (z \in \mathbb{U}).$$

For some recent investigations on analytic starlike functions, see (for example) the earlier works [1–14] and the references cited in each of these earlier investigations.

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Given two functions $f, g \in \Sigma$, where f is given by (1.1) and g is given by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) := \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k =: (g * f)(z).$$

A function $f \in \Sigma$ is said to be in the class $\mathcal{H}(\beta, \lambda)$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)} + \beta \frac{z^2 f''(z)}{f(z)}\right) < \beta\lambda\left(\lambda + \frac{1}{2}\right) + \frac{\beta}{2} - \lambda \quad (z \in \mathbb{U}), \tag{1.3}$$

where (and throughout this paper unless otherwise mentioned) the parameters β and λ are constrained as follows:

$$\beta \geqq 0 \quad \text{and} \quad \frac{1}{2} \leqq \lambda < 1. \tag{1.4}$$

Clearly, we have

$$\mathcal{H}(0, \lambda) = \mathcal{MS}^*(\lambda).$$

In a recent paper, Wang *et al.* [15] had proved that if $f \in \mathcal{H}(\beta, \lambda)$, then $f \in \mathcal{MS}^*(\lambda)$, which implies that the class $\mathcal{H}(\beta, \lambda)$ is a subclass of the class $\mathcal{MS}^*(\lambda)$ of meromorphically starlike functions of order λ.

Let $\mathcal{H}^+(\beta, \lambda)$ denote the subset of $\mathcal{H}(\beta, \lambda)$ such that all functions $f \in \mathcal{H}(\beta, \lambda)$ having the following form:

$$f(z) = \frac{1}{z} - \sum_{k=1}^{\infty} a_k z^k \quad (a_k \geqq 0). \tag{1.5}$$

In the present paper, we aim at proving some coefficient inequalities, neighborhoods, partial sums and inclusion relationships for the function classes $\mathcal{H}(\beta, \lambda)$ and $\mathcal{H}^+(\beta, \lambda)$.

2 Preliminary results

In order to prove our main results, we need the following lemmas.

Lemma 2.1 (See [16]) *If the function $p \in \mathcal{P}$ is given by* (1.2), *then*

$$|p_k| \leqq 2 \quad (k \in \mathbb{N}).$$

Lemma 2.2 *Let $\beta > 0$ and $1-\gamma-2\beta > 0$. Suppose also that the sequence $\{A_k\}_{k=1}^{\infty}$ is defined by*

$$A_1 = \frac{1-\gamma-2\beta}{1-\beta} \quad \text{and} \quad A_{k+1} = \frac{2(1-\gamma-2\beta)}{1-2\beta+(\beta k+1)(k+1)}\left(1+\sum_{l=1}^{k} A_l\right) \quad (k \in \mathbb{N}). \tag{2.1}$$

Then

$$A_k = \frac{1-\gamma-2\beta}{1-\beta}\prod_{j=1}^{k-1}\frac{3-6\beta-2\gamma+j(\beta j+1-\beta)}{1-2\beta+(\beta j+1)(j+1)} \quad \bigl(k \in \mathbb{N}\setminus\{1\}\bigr). \tag{2.2}$$

Proof By virtue of (2.1), we easily get

$$\bigl[1-2\beta+(\beta k+1)(k+1)\bigr]A_{k+1} = 2(1-\gamma-2\beta)\left(1+\sum_{l=1}^{k} A_l\right), \tag{2.3}$$

and

$$\bigl[1-2\beta+(\beta k+1-\beta)k\bigr]A_k = 2(1-\gamma-2\beta)\left(1+\sum_{l=1}^{k-1} A_l\right). \tag{2.4}$$

Combining (2.3) and (2.4), we obtain

$$\frac{A_{k+1}}{A_k} = \frac{3-6\beta-2\gamma+k(\beta k+1-\beta)}{1-2\beta+(\beta k+1)(k+1)}. \tag{2.5}$$

Thus, for $k \geqq 2$, we deduce from (2.5) that

$$A_k = \frac{A_k}{A_{k-1}}\cdot\cdots\cdot\frac{A_3}{A_2}\cdot\frac{A_2}{A_1}\cdot A_1 = \frac{1-\gamma-2\beta}{1-\beta}\prod_{j=1}^{k-1}\frac{3-6\beta-2\gamma+j(\beta j+1-\beta)}{1-2\beta+(\beta j+1)(j+1)}.$$

The proof of Lemma 2.2 is evidently completed. □

The following two lemmas can be derived from [17, Theorem 1] (see also [18]), we here choose to omit the details of proof.

Lemma 2.3 *Let*

$$1+\beta\lambda\left(\lambda+\frac{1}{2}\right)-\lambda-\frac{3}{2}\beta > 0. \tag{2.6}$$

Suppose also that $f \in \Sigma$ is given by (1.1). *If*

$$\sum_{k=1}^{\infty}\bigl[k+\beta k(k-1)+\gamma\bigr]|a_k| \leqq 1-\gamma-2\beta, \tag{2.7}$$

where (*and throughout this paper unless otherwise mentioned*) *the parameter* γ *is constrained as follows*:

$$\gamma := \lambda - \beta\lambda\left(\lambda + \frac{1}{2}\right) - \frac{\beta}{2}, \tag{2.8}$$

then $f \in \mathcal{H}(\beta,\lambda)$.

Lemma 2.4 *Let* $f \in \Sigma$ *be given by* (1.5). *Suppose also that* γ *is defined by* (2.8) *and the condition* (2.6) *holds true*. *Then* $f \in \mathcal{H}^+(\beta,\lambda)$ *if and only if*

$$\sum_{k=1}^{\infty}\left[k + \beta k(k-1) + \gamma\right]a_k \leqq 1 - \gamma - 2\beta. \tag{2.9}$$

3 Main results

We begin by proving the following coefficient estimates for functions belonging to the class $\mathcal{H}(\beta,\lambda)$.

Theorem 3.1 *Let* γ *be defined by* (2.8). *If* $f \in \mathcal{H}(\beta,\lambda)$ *with* $0 < \beta < 2/5$, *then*

$$|a_1| \leqq \frac{1-\gamma-2\beta}{1-\beta},$$

and

$$|a_k| \leqq \frac{1-\gamma-2\beta}{1-\beta}\prod_{j=1}^{k-1}\frac{3-6\beta-2\gamma+j(\beta j+1-\beta)}{1-2\beta+(\beta j+1)(j+1)} \quad \bigl(k \in \mathbb{N}\setminus\{1\}\bigr).$$

Proof Suppose that

$$q(z) := -\frac{zf'(z)}{f(z)} - \beta\frac{z^2f''(z)}{f(z)} + \beta\lambda\left(\lambda+\frac{1}{2}\right) + \frac{\beta}{2} - \lambda. \tag{3.1}$$

Then, by the definition of the function class $\mathcal{H}(\beta,\lambda)$, we know that q is analytic in $\mathbb{U}$ and

$$\Re\bigl(q(z)\bigr) > 0 \quad (z \in \mathbb{U})$$

with

$$q(0) = 1 - \gamma - 2\beta > 0.$$

It follows from (2.8) and (3.1) that

$$q(z)f(z) = -zf'(z) - \beta z^2f''(z) - \gamma f(z). \tag{3.2}$$

By noting that

$$h(z) = \frac{q(z)}{1-\gamma-2\beta} \in \mathcal{P},$$

if we put

$$q(z) = c_0 + \sum_{k=1}^{\infty} c_k z^k \quad (c_0 = 1 - \gamma - 2\beta),$$

by Lemma 2.1, we know that

$$|c_k| \leqq 2(1 - \gamma - 2\beta) \quad (k \in \mathbb{N}).$$

It follows from (3.2) that

$$\begin{aligned} &\left(c_0 + \sum_{k=1}^{\infty} c_k z^k\right)\left(\frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k\right) \\ &\quad = \left(\frac{1}{z} - \sum_{k=1}^{\infty} k a_k z^k\right) - \left(\frac{2\beta}{z} + \beta \sum_{k=1}^{\infty} k(k-1) a_k z^k\right) - \gamma \left(\frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k\right). \end{aligned} \tag{3.3}$$

In view of (3.3), we get

$$(1 - \gamma - 2\beta) a_1 + c_2 = -a_1 - \gamma a_1 \tag{3.4}$$

and

$$\begin{aligned} &c_{k+2} + (1 - \gamma - 2\beta) a_{k+1} + \sum_{l=1}^{k} a_l c_{k+1-l} \\ &\quad = -(k+1) a_{k+1} - \beta k(k+1) a_{k+1} - \gamma a_{k+1} \quad (k \in \mathbb{N}). \end{aligned} \tag{3.5}$$

From (3.4), we obtain

$$|a_1| \leqq \frac{1 - \gamma - 2\beta}{1 - \beta}. \tag{3.6}$$

Moreover, we deduce from (3.5) that

$$|a_{k+1}| \leqq \frac{2(1 - \gamma - 2\beta)}{1 - 2\beta + (\beta k + 1)(k + 1)} \left(1 + \sum_{l=1}^{k} |a_l|\right) \quad (k \in \mathbb{N}). \tag{3.7}$$

Next, we define the sequence $\{A_k\}_{k=1}^{\infty}$ as follows:

$$A_1 = \frac{1 - \gamma - 2\beta}{1 - \beta} \quad \text{and} \quad A_{k+1} = \frac{2(1 - \gamma - 2\beta)}{1 - 2\beta + (\beta k + 1)(k + 1)} \left(1 + \sum_{l=1}^{k} A_l\right) \quad (k \in \mathbb{N}). \tag{3.8}$$

In order to prove that

$$|a_k| \leqq A_k \quad (k \in \mathbb{N}),$$

we make use of the principle of mathematical induction. By noting that

$$|a_1| \leqq A_1 = \frac{1-\gamma-2\beta}{1-\beta}.$$

Therefore, assuming that

$$|a_l| \leqq A_l \quad (l = 1, 2, 3, \ldots, k; k \in \mathbb{N}).$$

Combining (3.7) and (3.8), we get

$$\begin{aligned} |a_{k+1}| &\leqq \frac{2(1-\gamma-2\beta)}{1-2\beta+(\beta k+1)(k+1)}\left(1+\sum_{l=1}^{k}|a_l|\right) \\ &\leqq \frac{2(1-\gamma-2\beta)}{1-2\beta+(\beta k+1)(k+1)}\left(1+\sum_{l=1}^{k}A_l\right) = A_{k+1} \quad (k \in \mathbb{N}). \end{aligned}$$

Hence, by the principle of mathematical induction, we have

$$|a_k| \leqq A_k \quad (k \in \mathbb{N}) \tag{3.9}$$

as desired.

By means of Lemma 2.2 and (3.8), we know that (2.2) holds true. Combining (3.9) and (2.2), we readily get the coefficient estimates asserted by Theorem 3.1. □

Following the earlier works (based upon the familiar concept of neighborhood of analytic functions) by Goodman [19] and Ruscheweyh [20], and (more recently) by Altintaş *et al.* [21–24], Cătaş [25], Cho *et al.* [26], Liu and Srivastava [27–29], Frasin [30], Keerthi *et al.* [31], Srivastava *et al.* [32] and Wang *et al.* [33]. Assuming that γ is given by (2.8) and the condition (2.6) of Lemma 2.3 holds true, we here introduce the δ-neighborhood of a function $f \in \Sigma$ of the form (1.1) by means of the following definition:

$$\begin{aligned} \mathcal{N}_\delta(f) := \Bigg\{ & g \in \Sigma : g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k \text{ and} \\ & \sum_{k=1}^{\infty} \frac{k+\beta k(k-1)+\gamma}{1-\gamma-2\beta}|a_k - b_k| \leqq \delta \ (\delta \geqq 0) \Bigg\}. \end{aligned} \tag{3.10}$$

By making use of the definition (3.10), we now derive the following result.

Theorem 3.2 *Let the condition* (2.6) *hold true*. *If* $f \in \Sigma$ *satisfies the condition*

$$\frac{f(z)+\varepsilon z^{-1}}{1+\varepsilon} \in \mathcal{H}(\beta,\lambda) \quad \big(\varepsilon \in \mathbb{C}; |\varepsilon| < \delta; \delta > 0\big), \tag{3.11}$$

then

$$\mathcal{N}_\delta(f) \subset \mathcal{H}(\beta,\lambda). \tag{3.12}$$

Proof By noting that the condition (1.3) can be written as

$$\left|\frac{\frac{zf'(z)}{f(z)}+\beta\frac{z^2f''(z)}{f(z)}+1}{\frac{zf'(z)}{f(z)}+\beta\frac{z^2f''(z)}{f(z)}+2\gamma-1}\right|<1 \quad (z\in\mathbb{U}), \tag{3.13}$$

we easily find from (3.13) that a function $g\in\mathcal{H}(\beta,\lambda)$ if and only if

$$\frac{zg'(z)+\beta z^2g''(z)+g(z)}{zg'(z)+\beta z^2g''(z)+(2\gamma-1)g(z)}\neq\sigma \quad \bigl(z\in\mathbb{U};\sigma\in\mathbb{C};|\sigma|=1\bigr),$$

which is equivalent to

$$\frac{(g*\mathfrak{h})(z)}{z^{-1}}\neq 0 \quad (z\in\mathbb{U}), \tag{3.14}$$

where

$$\mathfrak{h}(z)=\frac{1}{z}+\sum_{k=1}^{\infty}c_kz^k \quad \left(c_k:=\frac{k+\beta k(k-1)+1-[k+\beta k(k-1)+(2\gamma-1)]\sigma}{2[\beta+(1-\gamma-\beta)\sigma]}\right). \tag{3.15}$$

It follows from (3.15) that

$$\begin{aligned}|c_k| &= \left|\frac{k+\beta k(k-1)+1-[k+\beta k(k-1)+(2\gamma-1)]\sigma}{2[\beta+(1-\gamma-\beta)\sigma]}\right| \\ &\leqq \frac{k+\beta k(k-1)+1+[k+\beta k(k-1)+(2\gamma-1)]|\sigma|}{2(1-\gamma-2\beta)|\sigma|} \\ &= \frac{k+\beta k(k-1)+\gamma}{1-\gamma-2\beta} \quad \bigl(|\sigma|=1\bigr).\end{aligned}$$

If $f\in\Sigma$ given by (1.1) satisfies the condition (3.11), we deduce from (3.14) that

$$\frac{(f*\mathfrak{h})(z)}{z^{-1}}\neq-\varepsilon \quad \bigl(|\varepsilon|<\delta;\delta>0\bigr),$$

or equivalently,

$$\left|\frac{(f*\mathfrak{h})(z)}{z^{-1}}\right|\geqq\delta \quad (z\in\mathbb{U};\delta>0). \tag{3.16}$$

We now suppose that

$$q(z)=\frac{1}{z}+\sum_{k=1}^{\infty}d_kz^k\in\mathcal{N}_\delta(f).$$

It follows from (3.10) that

$$\left|\frac{((q-f)*\mathfrak{h})(z)}{z^{-1}}\right|=\left|\sum_{k=1}^{\infty}(d_k-a_k)c_kz^{k+1}\right|\leqq|z|\sum_{k=1}^{\infty}\frac{k+\beta k(k-1)+\gamma}{1-\gamma-2\beta}|d_k-a_k|<\delta. \tag{3.17}$$

Combining (3.16) and (3.17), we easily find that

$$\left|\frac{(q*\mathfrak{h})(z)}{z^{-1}}\right| = \left|\frac{([f+(q-f)]*\mathfrak{h})(z)}{z^{-1}}\right| \geqq \left|\frac{(f*\mathfrak{h})(z)}{z^{-1}}\right| - \left|\frac{((q-f)*\mathfrak{h})(z)}{z^{-1}}\right| > 0,$$

which implies that

$$\frac{(q*\mathfrak{h})(z)}{z^{-1}} \neq 0 \quad (z \in \mathbb{U}).$$

Therefore, we have

$$q(z) \in \mathcal{N}_\delta(f) \subset \mathcal{H}(\beta,\lambda).$$

The proof of Theorem 3.2 is thus completed. □

Next, we derive the partial sums of the class $\mathcal{H}(\beta,\lambda)$. For some recent investigations involving the partial sums in analytic function theory, one can find in [28, 29, 34, 35] and the references cited therein.

Theorem 3.3 *Let $f \in \Sigma$ be given by* (1.1) *and define the partial sums $f_n(z)$ of f by*

$$f_n(z) = \frac{1}{z} + \sum_{k=1}^{n} a_k z^k \quad (n \in \mathbb{N}). \tag{3.18}$$

If

$$\sum_{k=1}^{\infty} \frac{k+\beta k(k-1)+\gamma}{1-\gamma-2\beta}|a_k| \leqq 1, \tag{3.19}$$

where γ is given by (2.8) *and the condition* (2.6) *holds true, then*

1. $f \in \mathcal{H}(\beta,\lambda)$;
2. $$\Re\left(\frac{f(z)}{f_n(z)}\right) \geqq \frac{n+\beta n(n+1)+2\beta+2\gamma}{n+\beta n(n+1)+1+\gamma} \quad (n \in \mathbb{N}; z \in \mathbb{U}), \tag{3.20}$$

   and

   $$\Re\left(\frac{f_n(z)}{f(z)}\right) \geqq \frac{n+\beta n(n+1)+1+\gamma}{n+\beta n(n+1)+2-2\beta} \quad (n \in \mathbb{N}; z \in \mathbb{U}). \tag{3.21}$$

   The bounds in (3.20) *and* (3.21) *are sharp.*

Proof First of all, we suppose that

$$f_1(z) = \frac{1}{z}.$$

We know that

$$\frac{f_1(z)+\varepsilon z^{-1}}{1+\varepsilon} = \frac{1}{z} \in \mathcal{H}(\beta,\lambda).$$

From (3.19), we easily find that

$$\sum_{k=1}^{\infty} \frac{k+\beta k(k-1)+\gamma}{1-\gamma-2\beta}|a_k - 0| \leqq 1,$$

which implies that $f \in \mathcal{N}_1(z^{-1})$. By virtue of Theorem 3.2, we deduce that

$$f \in \mathcal{N}_1\left(z^{-1}\right) \subset \mathcal{H}(\beta,\lambda).$$

Next, it is easy to see that

$$\frac{n+1+\beta n(n+1)+\gamma}{1-\gamma-2\beta} > \frac{n+\beta n(n-1)+\gamma}{1-\gamma-2\beta} > 1 \quad (n \in \mathbb{N}).$$

Therefore, we have

$$\sum_{k=1}^{n}|a_k| + \frac{n+\beta n(n+1)+1+\gamma}{1-\gamma-2\beta}\sum_{k=n+1}^{\infty}|a_k| \leqq \sum_{k=1}^{\infty}\frac{k+\beta k(k-1)+\gamma}{1-\gamma-2\beta}|a_k| \leqq 1. \tag{3.22}$$

We now suppose that

$$\begin{aligned} h_1(z) &= \frac{n+\beta n(n+1)+1+\gamma}{1-\gamma-2\beta}\left(\frac{f(z)}{f_n(z)} - \frac{n+\beta n(n+1)+2\beta+2\gamma}{n+\beta n(n+1)+1+\gamma}\right) \\ &= 1 + \frac{\frac{n+\beta n(n+1)+1+\gamma}{1-\gamma-2\beta}\sum_{k=n+1}^{\infty} a_k z^{k+1}}{1+\sum_{k=1}^{n} a_k z^{k+1}}. \end{aligned} \tag{3.23}$$

It follows from (3.22) and (3.23) that

$$\left|\frac{h_1(z)-1}{h_1(z)+1}\right| \leqq \frac{\frac{n+\beta n(n+1)+1+\gamma}{1-\gamma-2\beta}\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=1}^{n}|a_k| - \frac{n+\beta n(n+1)+1+\gamma}{1-\gamma-2\beta}\sum_{k=n+1}^{\infty}|a_k|} \leqq 1 \quad (z \in \mathbb{U}),$$

which shows that

$$\Re\left(h_1(z)\right) \geqq 0 \quad (z \in \mathbb{U}). \tag{3.24}$$

Combining (3.23) and (3.24), we deduce that the assertion (3.20) holds true.

Furthermore, if we put

$$f(z) = \frac{1}{z} - \frac{1-\gamma-2\beta}{n+\beta n(n+1)+1+\gamma}z^{n+1}, \tag{3.25}$$

then

$$\frac{f(z)}{f_n(z)} = 1 - \frac{1-\gamma-2\beta}{n+\beta n(n+1)+1+\gamma}z^{n+2} \to \frac{n+\beta n(n+1)+2\beta+2\gamma}{n+\beta n(n+1)+1+\gamma} \quad \left(z \to 1^-\right),$$

which implies that the bound in (3.20) is the best possible for each $n \in \mathbb{N}$.

Similarly, we suppose that

$$h_2(z) = \frac{n+\beta n(n+1)+2-2\beta}{1-\gamma-2\beta}\left(\frac{f_n(z)}{f(z)} - \frac{n+\beta n(n+1)+1+\gamma}{n+\beta n(n+1)+2-2\beta}\right)$$
$$= 1 - \frac{\frac{n+\beta n(n+1)+2-2\beta}{1-\gamma-2\beta}\sum_{k=n+1}^{\infty} a_k z^{k+1}}{1+\sum_{k=1}^{\infty} a_k z^{k+1}}. \tag{3.26}$$

In view of (3.22) and (3.26), we conclude that

$$\left|\frac{h_2(z)-1}{h_2(z)+1}\right| \leqq \frac{\frac{n+\beta n(n+1)+2-2\beta}{1-\gamma-2\beta}\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=1}^{n}|a_k| - \frac{n+\beta n(n+1)+2\beta+2\gamma}{1-\gamma-2\beta}\sum_{k=n+1}^{\infty}|a_k|} \leqq 1 \quad (z \in \mathbb{U}),$$

which implies that

$$\Re\bigl(h_2(z)\bigr) \geqq 0 \quad (z \in \mathbb{U}). \tag{3.27}$$

Combining (3.26) and (3.27), we readily get the assertion (3.21) of Theorem 3.3. The bound in (3.21) is sharp with the extremal function f given by (3.25). We thus complete the proof of Theorem 3.3. □

In what follows, we turn to quotients involving derivatives. The proof of Theorem 3.4 below is similar to that of Theorem 3.3, we here choose to omit the analogous details.

Theorem 3.4 *Let $f \in \Sigma$ be given by* (1.1) *and define the partial sums $f_n(z)$ of f by* (3.18). *If the conditions* (2.6) *and* (3.19) *hold, where γ is given by* (2.8), *then*

$$\Re\left(\frac{f'(z)}{f_n'(z)}\right) \geqq \frac{(n+2)\gamma+(n+1)(n+2)\beta}{n+\beta n(n+1)+1+\gamma} \quad (n \in \mathbb{N}; z \in \mathbb{U}), \tag{3.28}$$

and

$$\Re\left(\frac{f_n'(z)}{f'(z)}\right) \geqq \frac{n+\beta n(n+1)+1+\gamma}{(n-2)(n+1)\beta+2(n+1)-n\gamma} \quad (n \in \mathbb{N}; z \in \mathbb{U}). \tag{3.29}$$

The bounds in (3.28) *and* (3.29) *are sharp with the extremal function given by* (3.25).

Finally, we prove the following inclusion relationship for the function class $\mathcal{H}(\beta,\lambda)$.

Theorem 3.5 *Let*

$$\beta_1 \geqq \beta_2 \geqq 1 \quad \text{and} \quad \frac{1}{2} \leqq \lambda_1 \leqq \lambda_2 < 1.$$

Then

$$\mathcal{H}(\beta_1,\lambda_1) \subset \mathcal{H}(\beta_2,\lambda_2). \tag{3.30}$$

Proof Suppose that $f \in \mathcal{H}(\beta_1,\lambda_1)$. Then

$$\Re\left(\frac{zf'(z)}{f(z)} + \beta_1\frac{z^2 f''(z)}{f(z)}\right) < \lambda_1\left[\beta_1\left(\lambda_1+\frac{1}{2}\right)-1\right] + \frac{\beta_1}{2} \quad (z \in \mathbb{U}). \tag{3.31}$$

Since $\beta_1 \geqq \beta_2 \geqq 1$ and $1/2 \leqq \lambda_1 \leqq \lambda_2 < 1$, we find that

$$\lambda_1\left[\beta_1\left(\lambda_1 + \frac{1}{2}\right) - 1\right] + \frac{\beta_1}{2} \leqq \lambda_2\left[\beta_1\left(\lambda_2 + \frac{1}{2}\right) - 1\right] + \frac{\beta_1}{2}. \tag{3.32}$$

It follows from (3.31) and (3.32) that

$$\Re\left(\frac{zf'(z)}{f(z)} + \beta_1\frac{z^2f''(z)}{f(z)}\right) < \lambda_2\left[\beta_1\left(\lambda_2 + \frac{1}{2}\right) - 1\right] + \frac{\beta_1}{2} \quad (z \in \mathbb{U}), \tag{3.33}$$

which shows that $f \in \mathcal{H}(\beta_1, \lambda_2)$, and subsequently, we see that $f \in \mathcal{MS}^*(\lambda_2)$, that is,

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < -\lambda_2 \quad (z \in \mathbb{U}). \tag{3.34}$$

Now, by setting

$$\mu = \frac{\beta_2}{\beta_1},$$

so that

$$0 < \mu \leqq 1,$$

we easily find from (3.33) and (3.34) that

$$\begin{aligned}
&\Re\left(\frac{zf'(z)}{f(z)} + \beta_2\frac{z^2f''(z)}{f(z)} - \lambda_2\left[\beta_2\left(\lambda_2 + \frac{1}{2}\right) - 1\right] - \frac{\beta_2}{2}\right) \\
&\quad = \mu\Re\left(\frac{zf'(z)}{f(z)} + \beta_1\frac{z^2f''(z)}{f(z)} - \lambda_2\left[\beta_1\left(\lambda_2 + \frac{1}{2}\right) - 1\right] - \frac{\beta_1}{2}\right) + (1-\mu)\Re\left(\frac{zf'(z)}{f(z)} + \lambda_2\right) \\
&\quad < 0 \quad (z \in \mathbb{U}),
\end{aligned}$$

that is,

$$f \in \mathcal{H}(\beta_2, \lambda_2).$$

Therefore, the assertion (3.30) of Theorem 3.5 holds true. □

From Theorem 3.5 and the definition of the function class $\mathcal{H}^+(\beta, \lambda)$, we easily get the following inclusion relationship.

Corollary 3.6 *Let*

$$\beta_1 \geqq \beta_2 \geqq 1 \quad \textit{and} \quad \frac{1}{2} \leqq \lambda_1 \leqq \lambda_2 < 1.$$

Then

$$\mathcal{H}^+(\beta_1, \lambda_1) \subset \mathcal{H}^+(\beta_2, \lambda_2) \subset \mathcal{MS}^*(\lambda_2).$$

By virtue of Lemma 2.4, we obtain the following result.

Corollary 3.7 *Let* $f \in \mathcal{H}^+(\beta, \lambda)$. *Suppose also that* γ *is defined by* (2.8) *and the condition* (2.6) *holds true*. *Then*

$$a_k \leqq \frac{1-\gamma-2\beta}{k+\beta k(k-1)+\gamma}.$$

Each of these inequalities is sharp, with the extremal function given by

$$f_k(z) = \frac{1}{z} - \frac{1-\gamma-2\beta}{k+\beta k(k-1)+\gamma} z^k.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper together. They also read and approved the final manuscript.

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Acknowledgements
The present investigation was supported by the *National Natural Science Foundation* under Grant nos. 11301008, 11226088, 11301041 and 11101053, the *Foundation for Excellent Youth Teachers of Colleges and Universities of Henan Province* under Grant no. 2013GGJS-146, and the *Natural Science Foundation of Educational Committee of Henan Province* under Grant no. 14B110012 of the People's Republic of China. The authors would like to thank the referees for their valuable comments and suggestions, which essentially improved the quality of this paper.

Received: 9 October 2013 Accepted: 5 January 2014 Published: 24 Jan 2014

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10.1186/1029-242X-2014-29
Cite this article as: Wang et al.: Some basic properties of certain subclasses of meromorphically starlike functions. *Journal of Inequalities and Applications* 2014, 2014:29

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