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Boundedness of localization operators on Lorentz mixed-normed modulation spaces

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Abstract
In this work we study certain boundedness properties for localization operators on Lorentz mixed-normed modulation spaces, when the operator symbols belong to appropriate modulation spaces, Wiener amalgam spaces, and Lorentz spaces with mixed norms.

Keywords: localization operator; Lorentz spaces; Lorentz mixed normed spaces; Lorentz mixed-normed modulation spaces; Wiener amalgam spaces

1 Introduction

In this paper we will work on $\mathbb{R}^d$ with Lebesgue measure dx. We denote by $\mathcal{S}(\mathbb{R}^d)$ the space of complex-valued continuous functions on $\mathbb{R}^d$ rapidly decreasing at infinity. For any function $f:\mathbb{R}^d \to \mathbb{C}$, the translation and modulation operator are defined as $T_x f(t) = f(t-x)$ and $M_w f(t) = e^{2\pi i wt} f(t)$ for $x, w \in \mathbb{R}^d$, respectively. For $1 \leq p \leq \infty$, we write the Lebesgue spaces $(L^p(\mathbb{R}^d), \|\cdot\|_p)$.

Let $\langle x,t\rangle = \sum_{i=1}^d x_i t_i$ be the usual scalar product on $\mathbb{R}^d$. The Fourier transform $\hat{f}$ (or $\mathcal{F}f$) of $f \in L^1(\mathbb{R}^d)$ is defined to be

$$\hat{f}(t) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i\langle x,t\rangle}\, dx.$$

For a fixed nonzero $g \in \mathcal{S}(\mathbb{R}^d)$ the short-time Fourier transform (STFT) of a function $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window g is defined as

$$V_g f(x,w) = \langle f, M_w T_x g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)} e^{-2\pi i tw}\, dt,$$

for $x, w \in \mathbb{R}^d$. Then the localization operator $A_a^{\varphi_1,\varphi_2}$ with symbol a and windows φ_1, φ_2 is defined to be

$$A_a^{\varphi_1,\varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x,w) V_{\varphi_1} f(x,w) M_w T_x \varphi_2\, dx\, dw.$$

If $a \in \mathcal{S}'(\mathbb{R}^d)$ and $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then the localization operator is a well-defined continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. Moreover, it is to be interpreted in a weak sense as

$$\langle A_a^{\varphi_1,\varphi_2} f, g\rangle = \langle a V_{\varphi_1} f, V_{\varphi_2} g\rangle = \langle a, \overline{V_{\varphi_1} f} V_{\varphi_2} g\rangle$$

for $f, g \in \mathcal{S}(\mathbb{R}^d)$, [1, 2].

Fix a nonzero window $g \in \mathcal{S}(\mathbb{R}^d)$ and $1 \leq p, q \leq \infty$. Then the modulation space $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the short-time Fourier transform $V_g f$ is in the mixed-norm space $L^{p,q}(\mathbb{R}^{2d})$. The norm on $M^{p,q}(\mathbb{R}^d)$ is $\|f\|_{M^{p,q}} = \|V_g f\|_{L^{p,q}}$. If $p = q$, then we write $M^p(\mathbb{R}^d)$ instead of $M^{p,p}(\mathbb{R}^d)$. Modulation spaces are Banach spaces whose definitions are independent of the choice of the window g (see [2, 3]).

$L(p,q)$ spaces are function spaces that are closely related to L^p spaces. We consider complex-valued measurable functions f defined on a measure space (X, μ). The measure μ is assumed to be nonnegative. We assume that the functions f are finite valued a.e. and some $y > 0$, $\mu(E_y) < \infty$, where $E_y = E_y[f] = \{x \in X \mid |f(x)| > y\}$. Then, for $y > 0$,

$$\lambda_f(y) = \mu(E_y) = \mu\left(\left\{x \in X \mid \left|f(x)\right| > y\right\}\right)$$

is the distribution function of f. The rearrangement of f is given by

$$f^*(t) = \inf\left\{y > 0 \mid \lambda_f(y) \leq t\right\} = \sup\left\{y > 0 \mid \lambda_f(y) > t\right\}$$

for $t > 0$. The average function of f is also defined by

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)\, dt.$$

Note that λ_f, f^*, and f^{**} are nonincreasing and right continuous functions on $(0, \infty)$. If $\lambda_f(y)$ is continuous and strictly decreasing then $f^*(t)$ is the inverse function of $\lambda_f(y)$. The most important property of f^* is that it has the same distribution function as f. It follows that

$$\left(\int_X \left|f(x)\right|^p d\mu(x)\right)^{\frac{1}{p}} = \left(\int_0^\infty \left[f^*(t)\right]^p dt\right)^{\frac{1}{p}}. \tag{1.1}$$

The Lorentz space denoted by $L(p,q)(X,\mu)$ (shortly $L(p,q)$) is defined to be vector space of all (equivalence classes) of measurable functions f such that $\|f\|_{pq}^* < \infty$, where

$$\|f\|_{pq}^* = \begin{cases} (\frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1}[f^*(t)]^q\, dt)^{\frac{1}{q}}, & 0 < p, q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t), & 0 < p \leq q = \infty. \end{cases}$$

By (1.1), it follows that $\|f\|_{pp}^* = \|f\|_p$ and so $L(p,p) = L^p$. Also, $L(p,q)(X,\mu)$ is a normed space with the norm

$$\|f\|_{pq} = \begin{cases} (\frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1}[f^{**}(t)]^q\, dt)^{\frac{1}{q}}, & 0 < p, q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t), & 0 < p \leq q = \infty. \end{cases}$$

For any one of the cases $p = q = 1$; $p = q = \infty$ or $1 < p < \infty$ and $1 \leq q \leq \infty$, the Lorentz space $L(p,q)(X,\mu)$ is a Banach space with respect to the norm $\|\cdot\|_{pq}$. It is also well known that if $1 < p < \infty$, $1 \leq q \leq \infty$ we have

$$\|\cdot\|_{pq}^{*} \leq \|\cdot\|_{pq} \leq \frac{p}{p-1}\|\cdot\|_{pq}^{*}$$

(see [4, 5]).

Let X and Y be two measure spaces with σ-finite measures μ and ν, respectively, and let f be a complex-valued measurable function on $(X \times Y, \mu \times \nu)$, $1 < P = (p_1, p_2) < \infty$, and $1 \leq Q = (q_1, q_2) \leq \infty$. The Lorentz mixed norm space $L(P,Q) = L(P,Q)(X \times Y)$ is defined by

$$L(P,Q) = L(p_2,q_2)\big[L(p_1,q_1)\big] = \big\{f : \|f\|_{PQ} = \|f\|_{L(p_2,q_2)(L(p_1,q_1))} = \big\| \|f\|_{p_1q_1} \big\|_{p_2q_2} < \infty\big\}.$$

Thus, $L(P,Q)$ occurs by taking an $L(p_1,q_1)$-norm with respect to the first variable and an $L(p_2,q_2)$-norm with respect to the second variable. The $L(P,Q)$ space is a Banach space under the norm $\|\cdot\|_{PQ}$ (see [6, 7]).

Fix a window function $g \in \mathcal{S}(\mathbb{R}^d)\backslash\{0\}$, $1 \leq P = (p_1,p_2) < \infty$, and $1 \leq Q = (q_1,q_2) \leq \infty$. We let $M(P,Q)(\mathbb{R}^d)$ denote the subspace of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ consisting of $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the Gabor transform $V_g f$ of f is in the Lorentz mixed norm space $L(P,Q)(\mathbb{R}^{2d})$. We endow it with the norm $\|f\|_{M(P,Q)} = \|V_g f\|_{PQ}$, where $\|\cdot\|_{PQ}$ is the norm of the Lorentz mixed norm space. It is well known that $M(P,Q)(\mathbb{R}^d)$ is a Banach space and different windows yield equivalent norms. If $p_1 = q_1 = p$ and $p_2 = q_2 = q$, then the space $M(P,Q)(\mathbb{R}^d)$ is the standard modulation space $M^{p,q}(\mathbb{R}^d)$, and if $P = p$ and $Q = q$, in this case $M(P,Q)(\mathbb{R}^d) = M(p,q)(\mathbb{R}^d)$ (see [8, 9]), where the space $M(p,q)(\mathbb{R}^d)$ is Lorentz type modulation space (see [10]). Furthermore, the space $M(p,q)(\mathbb{R}^d)$ was generalized to $M(p,q,w)(\mathbb{R}^d)$ by taking weighted Lorentz space rather than Lorentz space (see [11, 12]).

In this paper, we will denote the Lorentz space by $L(p,q)$, the Lorentz mixed norm space by $L(P,Q)$, the standard modulation space by $M^{p,q}$, the Lorentz type modulation space by $M(p,q)$, and the Lorentz mixed-normed modulation space by $M(P,Q)$.

Let $1 \leq r,s \leq \infty$. Fix a compact $Q \subset \mathbb{R}^d$ with nonempty interior. Then the Wiener amalgam space $W(L^r,L^s)(\mathbb{R}^d)$ with local component $L^r(\mathbb{R}^d)$ and global component $L^s(\mathbb{R}^d)$ is defined as the space of all measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that $f\chi_K \in L^r(\mathbb{R}^d)$ for each compact subset $K \subset \mathbb{R}^d$, for which the norm

$$\|f\|_{W(L^r,L^s)} = \|F_f\|_s = \big\| \|f\chi_{Q+x}\|_r \big\|_s$$

is finite, where χ_K is the characteristic function of K and

$$F_f(x) = \|f\chi_{Q+x}\|_r \in L^s\big(\mathbb{R}^d\big).$$

It is known that if $r_1 \geq r_2$ and $s_1 \leq s_2$ then $W(L^{r_1},L^{s_1})(\mathbb{R}^d) \subset W(L^{r_2},L^{s_2})(\mathbb{R}^d)$. If $r = s$ then $W(L^r,L^r)(\mathbb{R}^d) = L^r(\mathbb{R}^d)$ (see [13–15]).

In this paper, we consider boundedness properties for localization operators acting on Lorentz mixed-normed modulation spaces for the symbols in appropriate function spaces like modulation spaces, Wiener amalgam spaces, and Lorentz spaces with mixed norms. Our results extend some results in [1, 12] to the Lorentz mixed-normed modulation spaces.

2 Boundedness of localization operators on Lorentz mixed normed modulation spaces

We start with the following lemma, which will be used later on.

Lemma 2.1 *Let* $\frac{1}{P}+\frac{1}{P'}=1$, $\frac{1}{Q_1}+\frac{1}{Q_2}\geq 1$, $f\in L(P,Q_1)(\mathbb{R}^{2d})$, $h\in L(P',Q_2)(\mathbb{R}^{2d})$. *Then* $f*h\in L^\infty(\mathbb{R}^{2d})$ *and*

$$L(P,Q_1)\big(\mathbb{R}^{2d}\big)*L\big(P',Q_2\big)\big(\mathbb{R}^{2d}\big)\hookrightarrow L^\infty\big(\mathbb{R}^{2d}\big) \tag{2.1}$$

with the norm inequality

$$\|f*h\|_\infty\leq\|f\|_{PQ_1}\|h\|_{P'Q_2}, \tag{2.2}$$

where $P=(p_1,p_2)$, $Q_1=(Q_1^1,Q_1^2)$, $Q_2=(Q_2^1,Q_2^2)$.

Proof It is well known that there are $L(p,q_1)*L(p',q_2)\hookrightarrow L^\infty$ convolution relations between Lorentz spaces and

$$\|f*h\|_\infty\leq\|f\|_{pq_1}\|h\|_{p'q_2},$$

where $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q_1}+\frac{1}{q_2}\geq 1$, by Theorem 3.6 in [5]. Then (2.1) and (2.2) can easily be verified by using iteration and the one variable proofs given in [5]. □

Let $g\in\mathcal{D}(\mathbb{R}^{2d})$ be a test function such that $\sum_{x\in\mathbb{Z}^{2d}}T_xg\equiv 1$. Let $X(\mathbb{R}^{2d})$ be a translation invariant Banach space of functions with the property that $\mathcal{D}\cdot X\subset X$. In the spirit of [13, 16], the Wiener amalgam space $W(X,L(P,Q))$ with local component X and global component $L(P,Q)$ is defined as the space of all functions or distributions for which the norm

$$\|f\|_{W(X,L(P,Q))}=\big\|\|f\cdot T_{(z_1,z_2)}\overline{g}\|_X\big\|_{PQ}$$

is finite, where $1\leq P<\infty$, $1\leq Q\leq\infty$. Moreover, different choices of $g\in\mathcal{D}$ yield equivalent norms and give the same space.

The boundedness of $A^{\varphi_1,\varphi_2}_{M_\zeta a}$ for $a\in M^\infty$ is established by our next theorem. The proof is similar to Lemma 4.1 in [1] but let us provide the details anyway, for completeness' sake.

Theorem 2.1

(i) *Let* $1<P<\infty$, $1\leq Q<\infty$. *If* $f\in M(P,Q)(\mathbb{R}^d)$ *and* $g\in M^1(\mathbb{R}^d)$, *then* $V_gf\in W(\mathcal{F}L^1,L(P,Q))(\mathbb{R}^{2d})$ *with*

$$\|V_gf\|_{W(\mathcal{F}L^1,L(P,Q))}\leq\|f\|_{M(P,Q)}\|g\|_{M^1}.$$

(ii) *Let* $\frac{1}{P}+\frac{1}{P'}=1$, $\frac{1}{Q_1}+\frac{1}{Q_2}\geq 1$. *If* $f\in M(P,Q_1)(\mathbb{R}^d)$ *and* $g\in M(P',Q_2)(\mathbb{R}^d)$, *then* $V_gf\in W(\mathcal{F}L^1,L^\infty)(\mathbb{R}^{2d})$ *with*

$$\|V_gf\|_{W(\mathcal{F}L^1,L^\infty)}\leq\|f\|_{M(P,Q_1)}\|g\|_{M(P',Q_2)}.$$

Proof (i) Let $\varphi \in \mathcal{S}(\mathbb{R}^d)\backslash\{0\}$ and set $\Phi = V_\varphi\varphi \in \mathcal{S}(\mathbb{R}^{2d})$. By using the equality $V_gf(x,w) = (f \cdot T_x\overline{g})^\wedge(w)$, we write

$$\begin{aligned}\|V_gf \cdot T_{(z_1,z_2)}\overline{\Phi}\|_{\mathcal{F}L^1} &= \int_{\mathbb{R}^{2d}} \left|(V_gf \cdot T_{(z_1,z_2)}\overline{\Phi})^\wedge(t)\right| dt \\ &= \int_{\mathbb{R}^{2d}} \left|V_\Phi V_gf(z_1,z_2,t_1,t_2)\right| dt_1\, dt_2 \\ &= \int_{\mathbb{R}^{2d}} \left|V_\varphi g(-z_1-t_2,t_1)V_\varphi f(-t_2,z_2+t_1)\right| dt_1\, dt_2 \\ &= \int_{\mathbb{R}^{2d}} \left|V_\varphi f(u_1,u_2)\right|\left|V_\varphi g(u_1-z_1,u_2-z_2)\right| du_1\, du_2 \\ &= |V_\varphi f| * |V_\varphi g|^\sim(z_1,z_2), \end{aligned} \tag{2.3}$$

for $f,g \in \mathcal{S}(\mathbb{R}^d)$, where $(V_\varphi g)^\sim(z) = (\overline{V_\varphi g})(-z)$, $z \in \mathbb{R}^{2d}$. Since $f,g \in \mathcal{S}(\mathbb{R}^d)$, then $f \in M(P,Q)(\mathbb{R}^d)$ and $g \in M^1(\mathbb{R}^d)$ by Proposition 2 in [8]. So $V_\varphi f \in L(P,Q)(\mathbb{R}^{2d})$ and $V_\varphi g \in L^1(\mathbb{R}^{2d})$. Then, by Proposition 4 in [8], we obtain

$$\begin{aligned}\|V_gf\|_{W(\mathcal{F}L^1,L(P,Q))} &= \left\| \|V_gf \cdot T_{(z_1,z_2)}\overline{\Phi}\|_{\mathcal{F}L^1} \right\|_{PQ} \\ &= \left\| |V_\varphi f| * |V_\varphi g|^\sim \right\|_{PQ} \\ &\le \|V_\varphi f\|_{PQ}\|V_\varphi g\|_1 \\ &= \|f\|_{M(P,Q)}\|g\|_{M^1}. \end{aligned} \tag{2.4}$$

This completes the proof.

(ii) Using Lemma 2.1 and (2.3), we have

$$\|V_gf\|_{W(\mathcal{F}L^1,L^\infty)} = \left\| |V_\varphi f| * |V_\varphi g|^\sim \right\|_\infty \le \|V_\varphi f\|_{PQ_1}\|V_\varphi g\|_{P'Q_2} = \|f\|_{M(P,Q_1)}\|g\|_{M(P',Q_2)}.$$

□

Theorem 2.2 *Let $1 < P < \infty$, $1 \le Q < \infty$. If $a \in M^\infty(\mathbb{R}^{2d})$, $\varphi_1,\varphi_2 \in M^1(\mathbb{R}^d)$, then $A^{\varphi_1,\varphi_2}_{M_\zeta a}$ is bounded on $M(P,Q)(\mathbb{R}^d)$ for every $\zeta \in \mathbb{R}^{2d}$ with*

$$\left\|A^{\varphi_1,\varphi_2}_{M_\zeta a}\right\|_{B(M(P,Q))} \le \|a\|_{M^\infty}\|\varphi_1\|_{M^1}\|\varphi_2\|_{M^1}.$$

Proof Let $f \in M(P,Q)(\mathbb{R}^d)$ and $g \in M(P',Q')(\mathbb{R}^d)$, where $\frac{1}{P} + \frac{1}{P'} = 1$, $\frac{1}{Q} + \frac{1}{Q'} = 1$. Then we write $\overline{V_{\varphi_1}f} \in W(\mathcal{F}L^1,L(P,Q))(\mathbb{R}^{2d})$ and $V_{\varphi_2}g \in W(\mathcal{F}L^1,L(P',Q'))(\mathbb{R}^{2d})$ by above theorem. Moreover, since $M(1,1)(\mathbb{R}^d) = M^1(\mathbb{R}^d)$, we have $W(\mathcal{F}L^1,L^1) = M^1 = M(1,1)$ by [16]. Hence using the Hölder inequalities for Wiener amalgam spaces [13] and (2.4) we obtain

$$\begin{aligned}\|\overline{V_{\varphi_1}f} \cdot V_{\varphi_2}g\|_{M^1} &= \|\overline{V_{\varphi_1}f} \cdot V_{\varphi_2}g\|_{W(\mathcal{F}L^1,L^1)} \\ &\le \|V_{\varphi_1}f\|_{W(\mathcal{F}L^1,L(P,Q))}\|V_{\varphi_2}g\|_{W(\mathcal{F}L^1,L(P',Q'))} \\ &\preceq \|\varphi_1\|_{M^1}\|\varphi_2\|_{M^1}\|f\|_{M(P,Q)}\|g\|_{M(P',Q')}. \end{aligned} \tag{2.5}$$

Thus by using (2.5) we have

$$\begin{aligned}\left|\langle A^{\varphi_1,\varphi_2}_{M_\zeta a}f,g\rangle\right| &= \left|\langle M_\zeta a, \overline{V_{\varphi_1}f}\cdot V_{\varphi_2}g\rangle\right| \le \|M_\zeta a\|_{M(\infty,\infty)}\|\overline{V_{\varphi_1}f}\cdot V_{\varphi_2}g\|_{M(1,1)}\\ &\le \|a\|_{M^\infty}\|\varphi_1\|_{M^1}\|\varphi_2\|_{M^1}\|f\|_{M(P,Q)}\|g\|_{M(P',Q')}.\end{aligned}$$

Hence we get

$$\left\|A^{\varphi_1,\varphi_2}_{M_\zeta a}\right\|_{B(M(P,Q))} \le \|a\|_{M^\infty}\|\varphi_1\|_{M^1}\|\varphi_2\|_{M^1}. \qquad \square$$

Theorem 2.3 *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)\backslash\{0\}$ be a window function. If $1 < P, Q < \infty$, $t' \in (1,\infty)$, $s \le t' \le r$ and $a \in W(L^r, L^s)$, then*

$$A^{\varphi,\varphi}_{M_\zeta a} : M(tP,tQ)(\mathbb{R}^d) \to M\big((tP')', (tQ')'\big)(\mathbb{R}^d)$$

is bounded for every $\zeta \in \mathbb{R}^{2d}$, where $\frac{1}{P}+\frac{1}{P'}=1$, $\frac{1}{Q}+\frac{1}{Q'}=1$, and $\frac{1}{t}+\frac{1}{t'}=1$, and the operator norm satisfies the estimate

$$\left\|A^{\varphi,\varphi}_{M_\zeta a}\right\| \le \|a\|_{W(L^r,L^s)}.$$

Proof Let $t < \infty$, $f \in M(tP,tQ)(\mathbb{R}^d)$, and $h \in M(tP',tQ')(\mathbb{R}^d)$. Then we have $V_\varphi f \in L(tP,tQ)(\mathbb{R}^{2d})$ and $V_\varphi h \in L(tP',tQ')(\mathbb{R}^{2d})$. Since $V_\varphi f \in L(tP,tQ)(\mathbb{R}^{2d})$, then $\|V_\varphi f\|^*_{(tP)(tQ)} < \infty$. By using the equality (3.6) in [12], we get

$$\begin{aligned}\|V_\varphi f\|^*_{(tP)(tQ)} &= \big\|\|V_\varphi f\|^*_{(tp_1)(tq_1)}\big\|^*_{(tp_2)(tq_2)} = \Big\|\big(\big\||V_\varphi f|^t\big\|^*_{p_1q_1}\big)^{\frac{1}{t}}\Big\|^*_{(tp_2)(tq_2)}\\ &= \Big(\Big\|\Big|\big(\big\||V_\varphi f|^t\big\|^*_{p_1q_1}\big)^{\frac{1}{t}}\Big|^t\Big\|^*_{p_2q_2}\Big)^{\frac{1}{t}} = \Big(\Big\|\big\||V_\varphi f|^t\big\|^*_{p_1q_1}\Big\|^*_{p_2q_2}\Big)^{\frac{1}{t}}\\ &= \big(\big\||V_\varphi f|^t\big\|^*_{PQ}\big)^{\frac{1}{t}}.\end{aligned} \tag{2.6}$$

Hence we have $|V_\varphi f|^t \in L(P,Q)(\mathbb{R}^{2d})$. Similarly, $|V_\varphi h|^t \in L(P',Q')(\mathbb{R}^{2d})$. By the Hölder inequality for Lorentz spaces with mixed norm and (2.6) we have

$$\begin{aligned}\|V_\varphi f\cdot V_\varphi h\|^t_t &= \big\||V_\varphi f|^t|V_\varphi h|^t\big\|_1 \le \big\||V_\varphi f|^t\big\|_{PQ}\big\||V_\varphi h|^t\big\|_{P'Q'}\\ &= \|V_\varphi f\|^t_{(tP)(tQ)}\|V_\varphi h\|^t_{(tP')(tQ')}.\end{aligned} \tag{2.7}$$

Since $a \in W(L^r,L^s)$, then $M_\zeta a \in W(L^r,L^s)$ for every $\zeta \in \mathbb{R}^{2d}$. Also since $W(L^r,L^s) \subset W(L^{t'},L^{t'}) = L^{t'}(\mathbb{R}^{2d})$, then we have

$$\|a\|_{t'} = \|M_\zeta a\|_{t'} \le \|M_\zeta a\|_{W(L^r,L^s)} = \|a\|_{W(L^r,L^s)}. \tag{2.8}$$

By using (2.7), (2.8), and applying again the Hölder inequality, we get

$$\begin{aligned}\left|\langle A^{\varphi,\varphi}_{M_\zeta a}f,h\rangle\right| &= \left|\langle M_\zeta a V_\varphi f, V_\varphi h\rangle\right|\\ &\le \iint_{\mathbb{R}^{2d}}\left|M_\zeta a(x,w)\right|\left|(V_\varphi f\cdot V_\varphi h)(x,w)\right|\,dx\,dw\\ &\le \|M_\zeta a\|_{t'}\|V_\varphi f\cdot V_\varphi h\|_t\end{aligned}$$

$$\begin{aligned}
&\le \|a\|_{t'} \|V_\varphi f\|_{(tP)(tQ)} \|V_\varphi h\|_{(tP')(tQ')} \\
&\le \|a\|_{W(L^r,L^s)} \|f\|_{M(tP,tQ)} \|h\|_{M(tP',tQ')}. \qquad (2.9)
\end{aligned}$$

If $(tp')', (tq')' \neq \infty$, then $(M((tP')',(tQ')')(\mathbb{R}^d))^* = M(tP',tQ')(\mathbb{R}^d)$ by Theorem 8 in [8]. Thus we have from (2.9) that

$$\|A^{\varphi,\varphi}_{M_\zeta a} f\|_{M((tP')',(tQ')')} = \sup_{0\neq h\in M(tP',tQ')} \frac{|\langle A^{\varphi,\varphi}_{M_\zeta a} f, h\rangle|}{\|h\|_{M(tP',tQ')}} \le \|a\|_{W(L^r,L^s)} \|f\|_{M(tP,tQ)}.$$

Hence $A^{\varphi,\varphi}_{M_\zeta a}$ is bounded. Also we have

$$\|A^{\varphi,\varphi}_{M_\zeta a}\| = \sup_{0\neq f\in M(tP,tQ)} \frac{\|A^{\varphi,\varphi}_{M_\zeta a} f\|_{M((tP')',(tQ')')}}{\|f\|_{M(tP,tQ)}} \le \|a\|_{W(L^r,L^s)}. \qquad \square$$

Theorem 2.4 *Let* $\varphi \in \bigcap_{1\le R,S<\infty} M(R,S)(\mathbb{R}^d)$, *where* $R = (r_1,r_2)$, $S = (s_1,s_2)$. *If* $1 \le s \le r \le \infty$ *and* $a \in W(L^r,L^s)$ *then*

$$A^{\varphi,\varphi}_{M_\zeta a} : M(P,Q)(\mathbb{R}^d) \to M(P,Q)(\mathbb{R}^d)$$

is bounded for every $\zeta \in \mathbb{R}^{2d}$, *with*

$$\|A^{\varphi,\varphi}_{M_\zeta a}\| \le C\|a\|_{W(L^r,L^s)}$$

for some $C > 0$.

Proof Since $a \in W(L^r,L^s)$, then $M_\zeta a \in W(L^r,L^s)$ for every $\zeta \in \mathbb{R}^{2d}$. Also since $s \le r$, there exists $1 \le t_0 \le \infty$ such that $s \le t_0 \le r$. Then $W(L^r,L^s)(\mathbb{R}^{2d}) \subset L^{t_0}(\mathbb{R}^{2d})$ and

$$\|M_\zeta a\|_{t_0} = \|a\|_{t_0} \le \|a\|_{W(L^r,L^s)} = \|M_\zeta a\|_{W(L^r,L^s)} \qquad (2.10)$$

for all $a \in W(L^r,L^s)(\mathbb{R}^{2d})$. Let $B(M(P,Q)(\mathbb{R}^d), M(P,Q)(\mathbb{R}^d))$ be the space of the bounded linear operators from $M(P,Q)(\mathbb{R}^d)$ into $M(P,Q)(\mathbb{R}^d)$. Also let T be an operator from $L^1(\mathbb{R}^{2d})$ into $B(M(P,Q)(\mathbb{R}^d), M(P,Q)(\mathbb{R}^d))$ by $T(a) = A^{\varphi,\varphi}_{M_\zeta a}$. Take any $f \in M(P,Q)(\mathbb{R}^d)$ and $h \in M(P',Q')(\mathbb{R}^d)$. Assume that $a \in W(L^1,L^1)(\mathbb{R}^{2d}) = L^1(\mathbb{R}^{2d})$. By the Hölder inequality we get

$$\begin{aligned}
|\langle T(a)f,h\rangle| &= |\langle A^{\varphi,\varphi}_{M_\zeta a} f,h\rangle| = |\langle M_\zeta a V_\varphi f, V_\varphi h\rangle| \\
&\le \iint_{\mathbb{R}^{2d}} |M_\zeta a(x,w)| |V_\varphi f(x,w)| |V_\varphi h(x,w)|\,dx\,dw \\
&= \iint_{\mathbb{R}^{2d}} |a(x,w)| |\langle f, M_w T_x\varphi\rangle| |\langle h, M_w T_x\varphi\rangle|\,dx\,dw \\
&\le \iint_{\mathbb{R}^{2d}} |a(x,w)| \|f\|_{M(P,Q)} \|M_w T_x\varphi\|_{M(P',Q')} \|h\|_{M(P',Q')} \\
&\qquad \times \|M_w T_x\varphi\|_{M(P,Q)}\,dx\,dw \\
&= \|f\|_{M(P,Q)} \|\varphi\|_{M(P',Q')} \|h\|_{M(P',Q')} \|\varphi\|_{M(P,Q)} \|a\|_1. \qquad (2.11)
\end{aligned}$$

Hence by (2.11)

$$\begin{aligned}\|T(a)f\|_{M(P,Q)} &= \|A^{\varphi,\varphi}_{M_\zeta a}f\|_{M(P,Q)} = \sup_{0\neq h\in M(P',Q')}\frac{|\langle A^{\varphi,\varphi}_{M_\zeta a}f,h\rangle|}{\|h\|_{M(P',Q')}}\\ &\leq \|\varphi\|_{M(P',Q')}\|\varphi\|_{M(P,Q)}\|f\|_{M(P,Q)}\|a\|_1.\end{aligned}$$

Then

$$\|T(a)\| = \|A^{\varphi,\varphi}_{M_\zeta a}\| = \sup_{0\neq f\in M(P,Q)}\frac{\|A^{\varphi,\varphi}_{M_\zeta a}f\|_{M(P,Q)}}{\|f\|_{M(P,Q)}} \leq \|\varphi\|_{M(P',Q')}\|\varphi\|_{M(P,Q)}\|a\|_1. \tag{2.12}$$

Thus the operator

$$T : L^1(\mathbb{R}^{2d}) \to B\big(M(P,Q)(\mathbb{R}^d), M(P,Q)(\mathbb{R}^d)\big) \tag{2.13}$$

is bounded. Now let $a \in W(L^\infty, L^\infty)(\mathbb{R}^{2d}) = L^\infty(\mathbb{R}^{2d})$. Take any $f \in M(P,Q)(\mathbb{R}^d)$ and $h \in M(P',Q')(\mathbb{R}^d)$. Then $V_\varphi f \in L(P,Q)(\mathbb{R}^{2d})$, $V_\varphi h \in L(P',Q')(\mathbb{R}^{2d})$. Applying the Hölder inequality

$$\begin{aligned}\big|\langle T(a)f,h\rangle\big| &= \big|\langle A^{\varphi,\varphi}_{M_\zeta a}f,h\rangle\big| = \big|\langle M_\zeta a V_\varphi f, V_\varphi h\rangle\big|\\ &\leq \iint_{\mathbb{R}^{2d}} \big|M_\zeta a(x,w)\big|\big|V_\varphi f(x,w)\big|\big|V_\varphi h(x,w)\big|\,dx\,dw\\ &\leq \|a\|_\infty \iint_{\mathbb{R}^{2d}} \big|V_\varphi f(x,w)\big|\big|V_\varphi h(x,w)\big|\,dx\,dw\\ &\leq \|a\|_\infty\|V_\varphi f\|_{PQ}\|V_\varphi h\|_{P'Q'}.\end{aligned} \tag{2.14}$$

By using (2.14) we write

$$\|T(a)f\|_{M(P,Q)} = \|A^{\varphi,\varphi}_{M_\zeta a}f\|_{M(P,Q)} = \sup_{0\neq h\in M(P',Q')}\frac{|\langle A^{\varphi,\varphi}_{M_\zeta a}f,h\rangle|}{\|h\|_{M(P',Q')}} \leq \|a\|_\infty\|f\|_{M(P,Q)}. \tag{2.15}$$

Hence by (2.15)

$$\|T(a)\| = \|A^{\varphi,\varphi}_{M_\zeta a}\| = \sup_{0\neq f\in M(P,Q)}\frac{\|A^{\varphi,\varphi}_{M_\zeta a}f\|_{M(P,Q)}}{\|f\|_{M(P,Q)}} \leq \|a\|_\infty.$$

That means the operator

$$T : L^\infty(\mathbb{R}^{2d}) \to B\big(M(P,Q)(\mathbb{R}^d), M(P,Q)(\mathbb{R}^d)\big) \tag{2.16}$$

is bounded. Combining (2.13) and (2.16) we obtain

$$T : L^t(\mathbb{R}^{2d}) \to B\big(M(P,Q)(\mathbb{R}^d), M(P,Q)(\mathbb{R}^d)\big)$$

is bounded by interpolation theorem for $1 \leq t \leq \infty$. That means the localization operator

$$A^{\varphi,\varphi}_{M_\zeta a} : M(P,Q)(\mathbb{R}^d) \to M(P,Q)(\mathbb{R}^d)$$

is bounded for $1 \leq t \leq \infty$. Hence there exists $C > 0$ such that

$$\| T(a) \| = \| A^{\varphi,\varphi}_{M_\zeta a} \| \leq C \|a\|_t. \tag{2.17}$$

This implies that it is also true for $1 \leq t_0 \leq \infty$. From (2.10) and (2.17) we write

$$\| T(a) \| = \| A^{\varphi,\varphi}_{M_\zeta a} \| \leq C \|a\|_{t_0} \leq C \|a\|_{W(L^r, L^s)}. \qquad \square$$

Proposition 2.1 *Let* $\varphi \in \bigcap_{1 \leq R, S < \infty} M(R,S)(\mathbb{R}^d)$, *where* $R = (r_1, r_2)$, $S = (s_1, s_2)$. *If* $0 < s \leq 1$ *and* $a \in W(L^1, L^s)(\mathbb{R}^{2d})$ *then*

$$A^{\varphi,\varphi}_{M_\zeta a} : M(P,Q)\big(\mathbb{R}^d\big) \to M(P,Q)\big(\mathbb{R}^d\big)$$

is bounded.

Proof Let $0 < s \leq 1$ and let $a \in W(L^1, L^s)(\mathbb{R}^{2d})$. Then $M_\zeta a \in W(L^1, L^s)$ for every $\zeta \in \mathbb{R}^{2d}$. Since $W(L^1, L^s)(\mathbb{R}^{2d}) \subset L^1(\mathbb{R}^{2d})$, there exists a number $C > 0$ such that $\|M_\zeta a\|_1 \leq C \|M_\zeta a\|_{W(L^1, L^s)}$. Hence by (2.12),

$$\begin{aligned} \| A^{\varphi,\varphi}_{M_\zeta a} \| &\leq \|\varphi\|_{M(P',Q')} \|\varphi\|_{M(P,Q)} \|M_\zeta a\|_1 \\ &\leq C \|\varphi\|_{M(P',Q')} \|\varphi\|_{M(P,Q)} \|M_\zeta a\|_{W(L^1,L^s)} \\ &= C \|\varphi\|_{M(P',Q')} \|\varphi\|_{M(P,Q)} \|a\|_{W(L^1,L^s)}. \end{aligned}$$

Then the localization operator from $M(P,Q)(\mathbb{R}^d)$ into $M(P,Q)(\mathbb{R}^d)$ is bounded for $0 < s \leq 1$. $\square$

Proposition 2.2 *Let* $\varphi \in \bigcap_{1 \leq R, S < \infty} M(R,S)(\mathbb{R}^d)$, *where* $R = (r_1, r_2)$, $S = (s_1, s_2)$. *If* $1 \leq P, Q < \infty$ *and* $a \in L(P', Q')(\mathbb{R}^{2d})$ *then the localization operator*

$$A^{\varphi,\varphi}_{M_\zeta a} : M(P,Q)\big(\mathbb{R}^d\big) \to M(P,Q)\big(\mathbb{R}^d\big)$$

is bounded, where $\frac{1}{P} + \frac{1}{P'} = 1$, $\frac{1}{Q} + \frac{1}{Q'} = 1$.

Proof Let $a \in L(P', Q')(\mathbb{R}^{2d})$. Then $M_\zeta a \in L(P', Q')(\mathbb{R}^{2d})$ for every $\zeta \in \mathbb{R}^{2d}$ with $\|M_\zeta a\|_{P'Q'} = \|a\|_{P'Q'}$. Take any $f \in M(P,Q)(\mathbb{R}^d)$ and $h \in M(P', Q')(\mathbb{R}^d)$. Applying the Hölder inequality we have by (2.11)

$$\begin{aligned} \big| \langle A^{\varphi,\varphi}_{M_\zeta a} f, h \rangle \big| &\leq \iint_{\mathbb{R}^{2d}} \big| M_\zeta a(x,w) \big| \big| V_\varphi f(x,w) \big| \big| \langle h, M_w T_x \varphi \rangle \big| \, dx \, dw \\ &\leq \iint_{\mathbb{R}^{2d}} \big| a(x,w) \big| \big| V_\varphi f(x,w) \big| \|h\|_{M(P',Q')} \|M_w T_x \varphi\|_{M(P,Q)} \, dx \, dw \\ &= \|h\|_{M(P',Q')} \|\varphi\|_{M(P,Q)} \iint_{\mathbb{R}^{2d}} \big| a(x,w) \big| \big| V_\varphi f(x,w) \big| \, dx \, dw \\ &\leq \|h\|_{M(P',Q')} \|\varphi\|_{M(P,Q)} \|f\|_{M(P,Q)} \|a\|_{P'Q'}. \end{aligned}$$

Similarly to (2.12), we get

$$\left\| A^{\varphi,\varphi}_{M_\zeta a} \right\| \le \|\varphi\|_{M(P,Q)} \|a\|_{P'Q'}.$$

Then the localization operator $A^{\varphi,\varphi}_{M_\zeta a}$ from $M(P,Q)(\mathbb{R}^d)$ into $M(P,Q)(\mathbb{R}^d)$ is bounded. □

Corollary 2.1 *It is known by Proposition* 2 *in* [8] *that* $\mathcal{S}(\mathbb{R}^d) \subset M(R,S)(\mathbb{R}^d)$ *for* $1 \le R,S < \infty$. *Then* $\mathcal{S}(\mathbb{R}^d) \subset \bigcap_{1\le R,S<\infty} M(R,S)(\mathbb{R}^d)$. *So, Theorem* 2.4, *Propositions* 2.1 *and* 2.2 *are still true under the same hypotheses for them if* $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Corollary 2.2 *It is known* [8] *that if* $P = p$ *and* $Q = q$, *then Lorentz mixed-normed modulation space* $M(P,Q)(\mathbb{R}^d)$ *is the Lorentz type modulation space* $M(p,q)(\mathbb{R}^d)$. *Therefore our theorems hold for a Lorentz type modulation space rather than for a Lorentz mixed-normed modulation space.*

Competing interests
The author declares that she has no competing interests.

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