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A viscosity splitting algorithm for solving inclusion and equilibrium problems

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Abstract

In this paper, we present a viscosity splitting algorithm with computational errors for solving common solutions of inclusion and equilibrium problems. Strong convergence theorems are established in the framework of real Hilbert spaces. Applications are also provided to support the main results.

Keywords: equilibrium problem; variational inequality; splitting algorithm; nonexpansive mapping; fixed point

1 Introduction

Splitting algorithms have recently received much attention due to the fact that many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two nonlinear operators; see, for example, [1–4] and the references therein. The central problem is to iteratively find a zero point of the sum of two monotone operators.

One of the classical methods of studying the problem $0 \in Tx$, where T is a maximal monotone operator, is the proximal point algorithm (PPA) which was initiated by Martinet [5] and further developed by Rockafellar [6]. The PPA and its dual version in the context of convex programming, the method of multipliers of Hesteness and Powell, have been extensively studied and are known to yield as special cases decomposition methods such as the method of partial inverses [7], the Douglas-Rachford splitting method, and the alternating direction method of multipliers [8]. In the case of $T = A + B$, where A and B are monotone mappings, the splitting method $x_{n+1} = (I + r_n B)^{-1}(I - r_n A)x_n$, $n = 0, 1, \ldots$, where $r_n > 0$, was proposed by Lions and Mercier [9] and by Passty [10]. There are many nonlinear problems arising in engineering areas needing more than one constraint. Solving such problems, we have to obtain some solution which is simultaneously the solution of two or more subproblems or the solution of one subproblem on the solution set of another subproblem; see [11–18] and the references therein. The viscosity approximation method, which was introduced by Moudafi [19], has been extensively investigated by many authors for solving the problems; see, for example, [20–23] and the references therein. The highlight of the viscosity approximation method is that the desired limit point is not only a solution of a nonlinear problem but a unique solution of a classical monotone variational inequality. The aim of this paper is to introduce and investigate a viscosity splitting algorithm with computational errors for solving common solutions of inclusion and equi-

librium problems. The common solution is a unique solution of a monotone variational inequality. Strong convergence is guaranteed without the aid of compactness assumptions or metric projections. The main results mainly improve the corresponding results in [11–15].

The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a viscosity splitting algorithm with computational errors is investigated. A strong convergence theorem is established. In Section 4, applications of the main results are discussed.

2 Preliminaries

From now on, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C is a nonempty, closed, and convex subset of H.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. We consider the following equilibrium problem in the terminology of Blum and Oettli [24], which is also known as the Ky Fan inequality [25]:

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{2.1}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F)$, *i.e.*, $EP(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}$.

To study equilibrium problem (2.1), we may assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F\big(tz + (1 - t)x, y\big) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Let S be a mapping on C. $F(S)$ stands for the fixed point set of S. Recall that S is said to be contractive iff there exists a constant $\kappa \in (0, 1)$ such that

$$\|Sx - Sy\| \leq \kappa \|x - y\|, \quad \forall x, y \in C.$$

S is said to be nonexpansive iff

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

If C is closed, convex, and bounded, then $F(S)$ is not empty; see [26] and the references therein.

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone iff

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, we say that A is an α-strongly monotone mapping. A is said to be inverse-strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, we say that A is an α-inverse-strongly monotone mapping. It is clear that A is inverse-strongly monotone if and only if A^{-1} is strongly monotone.

A set-valued mapping $T : H \to 2^H$ is said to be monotone iff for all $x, y \in H, f \in Tx$, and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal iff the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping T is maximal iff, for any $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone mapping of C into H and $N_C v$ the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{ w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C \},$$

and define a mapping T on C by

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ iff $\langle Av, u - v \rangle \geq 0$ for all $u \in C$; see [27] and the references therein. Let I denote the identity operator on H and $B : H \to 2^H$ be a maximal monotone operator. Then we can define, for each $\lambda > 0$, a nonexpansive single valued mapping $J_r : H \to H$ by $J_r = (I + rB)^{-1}$. It is called the resolvent of B. We know that $B^{-1}0 = F(J_r)$ for all $r > 0$ and J_r is firmly nonexpansive, that is,

$$\|J_r x - J_r y\|^2 \leq \langle J_r x - J_r y, x - y \rangle, \quad x, y \in H.$$

In order to prove our main results, we also need the following lemmas.

Lemma 2.1 [24] *Let C be a nonempty, closed, and convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}$$

for all $r > 0$ and $x \in H$. Then, the following hold:

(a) *$F(T_r) = EP(F)$;*
(b) *T_r is single-valued;*

(c) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y \rangle;$$

(d) *The solution set is closed and convex.*

Lemma 2.2 *Let C be a nonempty, closed, and convex subset of H. Let* $A : C \to H$ *be a mapping and let* $B : H \rightrightarrows H$ *be a maximal monotone operator. Then* $F(J_r(I - rA)) = (A + B)^{-1}(0)$.

Proof

$$
\begin{aligned}
p \in F\big(J_r(I - rA)\big) \quad &\Longleftrightarrow \quad p = J_r(I - rA)p \\
&\Longleftrightarrow \quad p + rBp = p - rAp \\
&\Longleftrightarrow \quad p \in (A + B)^{-1}(0). \qquad \square
\end{aligned}
$$

The following lemma appears implicitly in [16]. For the sake of completeness, we still give the proof.

Lemma 2.3 *Let C be a nonempty, closed, and convex subset of H and let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A1)-(A4). T_t *and* T_s *are defined as in Lemma* 2.1. *Then*

$$\|T_s x - T_t x\| \le \frac{|s - t|}{t} \|x - T_t x\|.$$

Proof Putting $\varsigma = T_s x$ and $\tau = T_t x$, we find that $F(\tau, \varsigma) + \frac{1}{t}\langle \varsigma - \tau, \tau - x \rangle \ge 0$ and $F(\varsigma, \tau) + \frac{1}{s}\langle \tau - \varsigma, \varsigma - x \rangle \ge 0$. It follows that

$$\frac{1}{s}\langle \tau - \varsigma, \varsigma - x \rangle + \frac{1}{t}\langle \varsigma - \tau, \tau - x \rangle \ge 0.$$

Hence, we have

$$\left\langle \tau - \varsigma, \varsigma - x + \frac{s}{t}(x - \tau) \right\rangle \ge 0.$$

That is,

$$\|\tau - \varsigma\| \le \left|\frac{t - s}{t}\right| \|\tau - x\|.$$

This proves the lemma. $\qquad \square$

Lemma 2.4 [28] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in H and let* $\{\beta_n\}$ *be a sequence in* $(0, 1)$ *with* $0 < \liminf_{n \to \infty} \beta_n \le \limsup_{n \to \infty} \beta_n < 1$. *Suppose that* $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all integers* $n \ge 0$ *and*

$$\limsup_{n \to \infty} \big(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\big) \le 0.$$

Then $\lim_{n \to \infty} \|y_n - x_n\| = 0$.

Lemma 2.5 [29] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1 - \gamma_n)\alpha_n + \delta_n + e_n,$$

where $\{\gamma_n\}$ is a real number sequence in $(0, 1)$, and $\{\delta_n\}$ and $\{e_n\}$ are nonnegative real number sequences such that

 (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
 (ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ or $\sum_{n=1}^{\infty} \delta_n < \infty$;
 (iii) $\sum_{n=1}^{\infty} e_n < \infty$.
Then $\lim_{n\to\infty} \alpha_n = 0$.

3 Main results

Theorem 3.1 *Let C be a nonempty, closed, and convex subset of H and let F be a bifunction from $C \times C$ to R which satisfies (A1)-(A4). Let $f : C \to C$ be a contraction with the contractive constant $\kappa \in (0, 1)$. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $B : H \rightrightarrows H$ be a maximal monotone mapping. Assume $(A + B)^{-1}(0) \cap EP(F) \ne \emptyset$. Let $\{r_n\}$ and $\{s_n\}$ be positive real number sequences. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{e_n\}$ be a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n$, where $\{y_n\}$ is a sequence in C such that $F(y_n, y) + \frac{1}{s_n}\langle y - y_n, y_n - J_{r_n}(x_n - r_n A x_n + e_n)\rangle \ge 0$, $\forall y \in C$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$, and $\{s_n\}$ satisfy the conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$, $0 < \liminf_{n\to\infty} s_n$, $0 < r \le r_n \le r' < 2\alpha$, $\lim_{n\to\infty} |r_n - r_{n+1}| = \lim_{n\to\infty} |s_n - s_{n+1}| = 0$, where r and r' are real constants. Then $\{x_n\}$ converges strongly to $q = P_{(A+B)^{-1}(0)\cap EP(F)} f(q)$.*

Proof From Lemmas 2.1 and 2.2, we see that $(A + B)^{-1}(0)$ and $EP(F)$ are closed and convex. It follows that the projection onto the intersection $(A + B)^{-1}(0) \cap EP(F)$ is well defined. For any $x, y \in C$, we see that

$$\begin{aligned}
&\left\|(I - r_n A)x - (I - r_n A)y\right\|^2 \\
&\quad = \|x - y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2 \\
&\quad \le \|x - y\|^2 - r_n(2\alpha - r_n)\|Ax - Ay\|^2.
\end{aligned}$$

By using the condition imposed on $\{r_n\}$, we see that $\|(I - r_n A)x - (I - r_n A)y\| \le \|x - y\|$. This proves that $I - r_n A$ is nonexpansive. Let $p \in (A + B)^{-1}(0) \cap EP(F)$ be fixed arbitrarily. By using Lemma 2.1, we find that $y_n = T_{s_n} J_{r_n}(x_n - r_n A x_n + e_n)$. It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &\le \alpha_n\|f(x_n) - p\| + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\| \\
&\le \alpha_n\|x_n - f(p)\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| \\
&\quad + \gamma_n\|J_{r_n}(x_n - r_n A x_n + e_n) - J_{r_n}(p - r_n A p)\| \\
&\le \alpha_n\kappa\|x_n - p\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| \\
&\quad + \gamma_n\|(x_n - r_n A x_n + e_n) - (p - r_n A p)\| \\
&\le (1 - \alpha_n(1 - \kappa))\|x_n - p\| + \alpha_n\|f(p) - p\| + \|e_n\|
\end{aligned}$$

$$\leq \max\left\{\|x_n - p\|, \frac{\|f(p) - p\|}{1 - \kappa}\right\} + \|e_n\|$$

$$\leq \cdots$$

$$\leq \max\left\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1 - \kappa}\right\} + \sum_{i=1}^{\infty} \|e_i\| < \infty.$$

This implies that $\{x_n\}$ is bounded, and so is $\{y_n\}$. Letting $\xi_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$, we have

$$\xi_{n+1} - \xi_n = \frac{\alpha_{n+1}(f(x_{n+1}) - y_n) + (1 - \beta_{n+1})y_{n+1}}{1 - \beta_{n+1}}$$

$$- \frac{\alpha_n(f(x_n) - y_n) + (1 - \beta_n)y_n}{1 - \beta_n}$$

$$= \frac{\alpha_{n+1}(f(x_{n+1}) - y_{n+1})}{1 - \beta_{n+1}} - \frac{\alpha_n(f(x_n) - y_n)}{1 - \beta_n} + y_{n+1} - y_n.$$

It follows that

$$\|\xi_{n+1} - \xi_n\| \leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1 - \beta_n}\|f(x_n) - y_n\| + \|y_{n+1} - y_n\|. \tag{3.1}$$

Put $u_n = x_n - r_n A x_n + e_n$. Since B is monotone, we see that

$$\left\langle J_{r_{n+1}} u_n - J_{r_n} u_n, \frac{u_n - J_{r_{n+1}} u_n}{r_{n+1}} - \frac{u_n - J_{r_n} u_n}{r_n}\right\rangle \geq 0.$$

It follows that

$$\left\langle J_{r_{n+1}} u_n - J_{r_n} u_n, \left(1 - \frac{r_{n+1}}{r_n}\right)(u_n - J_{r_n} u_n)\right\rangle \geq \|J_{r_{n+1}} u_n - J_{r_n} u_n\|^2.$$

This in turn implies that

$$\|J_{r_{n+1}} u_n - J_{r_n} u_n\| \leq \frac{|r_{n+1} - r_n|}{r_n}\|u_n - J_{r_n} u_n\|. \tag{3.2}$$

Putting $z_n = J_{r_n}(x_n - r_n A x_n + e_n)$, we have

$$\|z_{n+1} - z_n\| \leq \left\|J_{r_{n+1}}(x_{n+1} - r_{n+1}A x_{n+1} + e_{n+1}) - J_{r_{n+1}}(x_n - r_n A x_n + e_n)\right\|$$

$$+ \left\|J_{r_{n+1}}(x_n - r_n A x_n + e_n) - J_{r_n}(x_n - r_n A x_n + e_n)\right\|$$

$$\leq \left\|(x_{n+1} - r_{n+1}A x_{n+1} + e_{n+1}) - (x_n - r_n A x_n + e_n)\right\|$$

$$+ \left\|J_{r_{n+1}}(x_n - r_n A x_n + e_n) - J_{r_n}(x_n - r_n A x_n + e_n)\right\|. \tag{3.3}$$

From (3.2) and (3.3), we find that

$$
\begin{aligned}
\|z_{n+1} - z_n\| &\leq \left\| (x_{n+1} - r_{n+1}Ax_{n+1} + e_{n+1}) - (x_n - r_nAx_n + e_n) \right\| \\
&\quad + \frac{|r_{n+1} - r_n|}{r_n} \|u_n - J_{r_n}u_n\| \\
&\leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n| \left(\|Ax_n\| + \frac{\|u_n - J_{r_n}u_n\|}{r_n} \right) \\
&\quad + \|e_{n+1}\| + \|e_n\|.
\end{aligned}
\tag{3.4}
$$

By using Lemma 2.3, we find that

$$
\begin{aligned}
\|y_{n+1} - y_n\| &\leq \left\| T_{s_{n+1}}J_{r_{n+1}}(x_{n+1} - r_{n+1}Ax_{n+1} + e_{n+1}) - T_{s_{n+1}}J_{r_n}(x_n - r_nAx_n + e_n) \right\| \\
&\quad + \left\| T_{s_{n+1}}J_{r_n}(x_n - r_nAx_n + e_n) - T_{s_n}J_{r_n}(x_n - r_nAx_n + e_n) \right\| \\
&\leq \|z_{n+1} - z_n\| + \frac{\|T_{s_n}z_n - z_n\|}{s_n} |s_{n+1} - s_n|.
\end{aligned}
\tag{3.5}
$$

From (3.4) and (3.5), we find that

$$
\begin{aligned}
\|y_{n+1} - y_n\| &\leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n| \left(\|Ax_n\| + \frac{\|u_n - J_{r_n}u_n\|}{r_n} \right) \\
&\quad + \frac{\|T_{s_n}z_n - z_n\|}{s_n} |s_{n+1} - s_n| + \|e_{n+1}\| + \|e_n\|.
\end{aligned}
\tag{3.6}
$$

Substituting (3.6) into (3.1), we arrive at

$$
\begin{aligned}
\|\xi_{n+1} - \xi_n\| - \|x_{n+1} - x_n\| &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1 - \beta_n} \|f(x_n) - y_n\| \\
&\quad + |r_{n+1} - r_n| \left(\|Ax_n\| + \frac{\|u_n - J_{r_n}u_n\|}{r_n} \right) \\
&\quad + \frac{\|T_{s_n}z_n - z_n\|}{s_n} |s_{n+1} - s_n| + \|e_{n+1}\| + \|e_n\|.
\end{aligned}
$$

It follows from the conditions that

$$
\limsup_{n \to \infty} \left(\|\xi_{n+1} - \xi_n\| - \|x_{n+1} - x_n\| \right) = 0.
$$

By using Lemma 2.4, we see that $\lim_{n \to \infty} \|\xi_n - x_n\| = 0$, which in turn implies that

$$
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\tag{3.7}
$$

Since J_{r_n} is firmly nonexpansive, we find that

$$
\begin{aligned}
\|z_n - p\|^2 &\leq \left\langle (x_n - r_nAx_n + e_n) - (p - r_nAp), z_n - p \right\rangle \\
&= \frac{1}{2} \Big(\left\| (x_n - r_nAx_n + e_n) - (p - r_nAp) \right\|^2 + \|z_n - p\|^2 \\
&\quad - \left\| \big((x_n - r_nAx_n + e_n) - (p - r_nAp) \big) - (z_n - p) \right\|^2 \Big)
\end{aligned}
$$

$$\leq \frac{1}{2}\Big(\|x_n - p\|^2 + \|e_n\|\big(\|e_n\| + 2\|x_n - p\|\big) + \|z_n - p\|^2$$

$$- \big\|x_n - z_n - r_n(Ax_n - Ap) + e_n\big\|^2\Big)$$

$$\leq \frac{1}{2}\Big(\|x_n - p\|^2 + \|e_n\|\big(\|e_n\| + 2\|x_n - p\|\big) + \|z_n - p\|^2 - \|x_n - z_n\|^2$$

$$- \big\|r_n(Ax_n - Ap) - e_n\big\|^2 + 2\|x_n - z_n\|\big\|r_n(Ax_n - Ap) - e_n\big\|\Big).$$

It follows that

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 + \|e_n\|\big(\|e_n\| + 2\|x_n - p\|\big) - \|x_n - z_n\|^2$$

$$+ 2r_n\|x_n - z_n\|\|Ax_n - Ap\| + 2\|x_n - z_n\|\|e_n\|. \tag{3.8}$$

Since T_{s_n} is firmly nonexpansive, we find from (3.8) that

$$\|x_{n+1} - p\|^2 \leq \alpha_n\big\|f(x_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2$$

$$\leq \alpha_n\big\|f(x_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|z_n - p\|^2$$

$$\leq \alpha_n\big\|f(x_n) - p\big\|^2 + \|x_n - p\|^2 + \|e_n\|\big(\|e_n\| + 2\|x_n - p\|\big) - \gamma_n\|x_n - z_n\|^2$$

$$+ 2r_n\|x_n - z_n\|\|Ax_n - Ap\| + 2\|x_n - z_n\|\|e_n\|.$$

It follows that

$$\gamma_n\|x_n - z_n\|^2 \leq \alpha_n\big\|f(x_n) - p\big\|^2 + \|e_n\|\big(\|e_n\| + 2\|x_n - p\| + 2\|x_n - z_n\|\big)$$

$$+ 2r_n\|x_n - z_n\|\|Ax_n - Ap\| + \|x_n - x_{n+1}\|\big(\|x_n - p\| + \|x_{n+1} - p\|\big). \tag{3.9}$$

Since A is inverse-strongly monotone, we find that

$$\|z_n - p\|^2 \leq \big\|(x_n - r_n Ax_n) - (p - r_n Ap) + e_n\big\|^2$$

$$\leq \big\|(x_n - p) - r_n(Ax_n - Ap)\big\|^2 + \|e_n\|\big(\|e_n\| + 2\|x_n - p\|\big)$$

$$\leq \|x_n - p\|^2 - r_n(2\alpha_n - r_n)\|Ax_n - Ap\|^2 + \|e_n\|\big(\|e_n\| + 2\|x_n - p\|\big).$$

Hence, we have

$$\big\|x_{n+1} - p\big\|^2 \leq \alpha_n\big\|f(x_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2$$

$$\leq \alpha_n\big\|f(x_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\big\|(x_n - r_n Ax_n + e_n) - (p - r_n Ap)\big\|^2$$

$$\leq \alpha_n\big\|f(x_n) - p\big\|^2 + \|x_n - p\|^2 - r_n(2\alpha_n - r_n)\gamma_n\|Ax_n - Ap\|^2$$

$$+ \|e_n\|\big(\|e_n\| + 2\|e_n\|\|x_n - p\|\big).$$

This implies that

$$r_n(2\alpha_n - r_n)\gamma_n\|Ax_n - Ap\|^2 \leq \alpha_n\big\|f(x_n) - p\big\|^2 + \|x_n - x_{n+1}\|\big(\|x_n - p\| + \|x_{n+1} - p\|\big)$$

$$+ \|e_n\|\big(\|e_n\| + 2\|x_n - p\|\big).$$

By using the conditions imposed on the control sequences, we find from (3.7) that $\lim_{n\to\infty} \|Ax_n - Ap\| = 0$. This in turn implies from (3.9) and the conditions that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.10}$$

Since T_{s_n} is firmly nonexpansive, we find that

$$\|y_n - p\|^2 \leq \langle z_n - p, y_n - p \rangle$$
$$= \frac{1}{2}\left(\|z_n - p\|^2 + \|y_n - p\|^2 - \|y_n - z_n\|^2\right).$$

That is,

$$\|y_n - p\|^2 \leq \|z_n - p\|^2 - \|y_n - z_n\|^2$$
$$\leq \|x_n - p\|^2 + \|e_n\|\left(\|e_n\| + 2\|x_n - p\|\right) - \|y_n - z_n\|^2.$$

It follows that

$$\|x_{n+1} - p\|^2 \leq \alpha_n\left\|f(x_n) - p\right\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2$$
$$\leq \alpha_n\left\|f(x_n) - p\right\|^2 + \|x_n - p\|^2 + \|e_n\|\left(\|e_n\| + 2\|x_n - p\|\right) - \gamma_n\|y_n - z_n\|^2.$$

This implies that

$$\gamma_n\|y_n - z_n\|^2 \leq \alpha_n\left\|f(x_n) - p\right\|^2 + \left(\|x_n - p\| + \|x_{n+1} - p\|\right)\|x_n - x_{n+1}\|$$
$$+ \|e_n\|\left(\|e_n\| + 2\|x_n - p\|\right).$$

By using the conditions imposed on the control sequences, we find from (3.7) that

$$\lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.12}$$

Since the mapping $P_{(A+B)^{-1}(0)\cap EP(F)}f$ is contractive, we see that there exists a unique fixed point. Next, we use q to denote the unique fixed point of the mapping. That is, $q = P_{(A+B)^{-1}(0)\cap EP(F)}f(q)$. Now, we are in a position to show $\limsup_{n\to\infty}\langle f(q) - q, x_n - q\rangle \leq 0$. To show it, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(q) - q, x_n - q\rangle = \lim_{i\to\infty}\langle f(q) - q, x_{n_i} - q\rangle.$$

Since $\{x_{n_i}\}$ is bounded, we can choose a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly some point $\bar{x}$. We may assume, without loss of generality, that $\{x_{n_i}\}$ converges weakly to $\bar{x}$. Now, we are in a position to show $\bar{x} \in (A + B)^{-1}(0) \cap EP(F)$.

First, we prove $\bar{x} \in (A + B)^{-1}(0)$. Notice that

$$\frac{x_n - z_n + e_n}{r_n} - Ax_n \in Bz_n.$$

Let $\mu \in Bv$. Since B is monotone, we find that

$$\left\langle \frac{x_n - z_n + e_n}{r_n} - Ax_n - \mu, z_n - v \right\rangle \geq 0.$$

It follows from (3.10) that $\langle -A\bar{x} - \mu, \bar{x} - v \rangle \geq 0$. This implies that $-A\bar{x} \in B\bar{x}$, that is, $\bar{x} \in (A + B)^{-1}(0)$.

Next, we prove $\bar{x} \in EP(F)$. Notice that

$$F(y_n, y) + \frac{1}{s_n} \langle y - y_n, y_n - z_n \rangle \geq 0, \quad \forall y \in C.$$

By using condition (A2), we see that $\frac{1}{s_n} \langle y - y_n, y_n - z_n \rangle \geq F(y, y_n)$, $\forall y \in C$. Replacing n by n_i, we arrive at

$$\left\langle y - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{s_{n_i}} \right\rangle \geq F(y, y_{n_i}), \quad \forall y \in C.$$

By using the condition $\liminf_{n \to \infty} s_n > 0$, (3.11), and (3.12), we find that $\{y_{n_i}\}$ converges weakly to $\bar{x}$. It follows that $0 \geq F(y, \bar{x})$. This proves that $\bar{x} \in EP(F)$. This shows that $\limsup_{n \to \infty} \langle f(q) - q, x_n - q \rangle \leq 0$. Note that

$$\|x_{n+1} - q\|^2 \leq \alpha_n \langle f(x_n) - q, x_{n+1} - q \rangle + \beta_n \|x_n - q\| \|x_{n+1} - q\|$$
$$+ \gamma_n \|y_n - q\| \|x_{n+1} - q\|$$
$$\leq \alpha_n \langle f(x_n) - f(q), x_{n+1} - q \rangle + \alpha_n \langle f(q) - q, x_{n+1} - q \rangle$$
$$+ \beta_n \|x_n - q\| \|x_{n+1} - q\| + \gamma_n (\|x_n - q\| + e_n) \|x_{n+1} - q\|$$
$$\leq \frac{\alpha_n \kappa + \beta_n + \gamma_n}{2} (\|x_n - q\|^2 + \|x_{n+1} - q\|^2)$$
$$+ \alpha_n \langle f(q) - q, x_{n+1} - q \rangle + e_n \|x_{n+1} - q\|.$$

It follows that

$$\|x_{n+1} - q\|^2 \leq (1 - \alpha_n(1 - \kappa)) \|x_n - q\|^2 + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle + 2\|x_{n+1} - q\| e_n.$$

By using Lemma 2.5, we find that $\lim_{n \to \infty} \|x_n - q\| = 0$. This completes the proof. □

From Theorem 3.1, we have the following result on the equilibrium problem immediately.

Corollary 3.2 *Let C be a nonempty, closed, and convex subset of H and let F be a bifunction from $C \times C$ to R which satisfies (A1)-(A4). Assume $EP(F) \neq \emptyset$. Let $f : C \to C$ be a contraction with the contractive constant $\kappa \in (0, 1)$. Let $\{s_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$.*

Let $\{e_n\}$ be a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Let $\{x_n\}$ be a sequence gener-ated in the following process: $x_1 \in C$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n$, where $\{y_n\}$ is a se-quence such that $F(y_n, y) + \frac{1}{s_n}\langle y - y_n, y_n - x_n - e_n\rangle \geq 0$, $\forall y \in C$. Assume that the con-trol sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{s_n\}$ satisfy the conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $0 < \liminf_{n\to\infty} s_n$, $\lim_{n\to\infty} |s_n - s_{n+1}| = 0$. Then $\{x_n\}$ converges strongly to $q = P_{EP(F)}f(q)$.

From Theorem 3.1, we have the following result on the inclusion problem immediately.

Corollary 3.3 *Let C be a nonempty, closed, and convex subset of H and let $f : C \to C$ be a contraction with the contractive constant $\kappa \in (0, 1)$. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $B : H \rightrightarrows H$ be a maximal monotone mapping such that the domain of B is in C. Assume $(A + B)^{-1}(0) \neq \emptyset$. Let $\{r_n\}$ be a positive real number se-quence. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{e_n\}$ be a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{r_n}(x_n - r_n A x_n + e_n)$, $n \geq 1$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ satisfy the following conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $0 < r \leq r_n \leq r' < 2\alpha$, $\lim_{n\to\infty} |r_n - r_{n+1}| = 0$, where r and r' are real constants. Then $\{x_n\}$ converges strongly to $q = P_{(A+B)^{-1}(0)}f(q)$.*

4 Applications

In this section, we give some results on equilibrium problems, variational inequalities, and convex functions.

Lemma 4.1 [12] *Let G be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4), and let W be a multivalued mapping of H into itself defined by*

$$Wx = \begin{cases} \{z \in H : G(x, y) \geq \langle y - x, z\rangle, \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{4.1}$$

Then W is a maximal monotone operator with the domain $D(W) \subset C$ and $EP(G) = W^{-1}(0)$.

Theorem 4.2 *Let C be a nonempty, closed, and convex subset of H and let F and G be two bifunctions from $C \times C$ to R which satisfies (A1)-(A4). Assume that $EP(G) \cap EP(F) \neq \emptyset$. Let $f : C \to C$ be a contraction with the contractive constant $\kappa \in (0, 1)$. Let $\{r_n\}$ and $\{s_n\}$ be pos-itive real number sequences. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{e_n\}$ be a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Let $\{x_n\}$ be a se-quence generated in the following process: $x_1 \in C$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n$, $n \geq 1$, where $F(y_n, y) + \frac{1}{s_n}\langle y - y_n, y_n - (I + r_n W)^{-1}(x_n + e_n)\rangle \geq 0$, $\forall y \in C$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$, and $\{s_n\}$ satisfy the following conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $0 < \liminf_{n\to\infty} s_n$, $0 < r \leq r_n$, $\lim_{n\to\infty} |r_n - r_{n+1}| = \lim_{n\to\infty} |s_n - s_{n+1}| = 0$, where r is a real constant. Then $\{x_n\}$ converges strongly to $q = P_{EP(G)\cap EP(F)}f(q)$.*

Recall that the classical variational inequality is to find an $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C, \tag{4.2}$$

where $A : C \to H$ is a monotone mapping. The solution set of (4.2) is denoted by $VI(C, A)$. Projection methods have been recently investigated for solving variational inequality (4.2). It is well known that x is a solution to (4.2) iff x is a fixed point of the mapping $\text{Proj}_C(I - rA)$, where Proj_C is the metric projection from H onto C and I denotes the identity on H. If A is inverse-strongly monotone, then $\text{Proj}_C(I - rA)$ is nonexpansive. Moreover, if C is bounded, closed, and convex, then the existence of solutions of the variational inequality is guaranteed by the nonexpansivity of the mapping $\text{Proj}_C(I - rA)$. Next, we consider solutions of variational inequality (4.2). Let i_C be a function defined by

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

It is easy to see that i_C is a proper lower and semicontinuous convex function on H, and the subdifferential ∂i_C of i_C is maximal monotone. Define the resolvent $J_r := (I + r\partial i_C)^{-1}$ of the subdifferential operator ∂i_C. Letting $x = J_r y$, we find that

$$y \in x + r\partial i_C x \iff y \in x + rN_C x$$

$$\iff \langle y - x, v - x \rangle \leq 0, \quad \forall v \in C$$

$$\iff x = \text{Proj}_C y,$$

where $N_C x := \{e \in H : \langle e, v - x \rangle, \forall v \in C\}$. Putting $B = \partial i_C$ in Theorems 3.1, we find the following result immediately.

Theorem 4.3 *Let C be a nonempty, closed, and convex subset of H and let $f : C \to C$ be a contraction with the contractive constant $\kappa \in (0, 1)$. Let $A : C \to H$ be an α-inverse-strongly monotone mapping such that $VI(C, A) \neq \emptyset$. Let $\{r_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{e_n\}$ be a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n$, $n \geq 1$, $y_n = \text{Proj}_C(x_n - r_n A x_n + e_n)$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ satisfy the conditions: $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$, $0 < r \leq r_n \leq r' < 2\alpha$, $\lim_{n \to \infty} |r_n - r_{n+1}| = 0$, where r and r' are real constants. Then $\{x_n\}$ converges strongly to $q = P_{VI(C,A)} f(q)$.*

Proof Put $F(x, y) = 0$ for any $x, y \in C$ and $s_n = 1$. From Theorem 3.1, we draw the desired conclusion immediately. □

Now, we are in a position to consider the problem of finding minimizers of proper lower semicontinuous convex functions. For a proper lower semicontinuous convex function $g : H \to (-\infty, \infty]$, the subdifferential mapping ∂g of g is defined by $\partial g(x) = \{x^* \in H : g(x) + \langle y - x, x^* \rangle \leq g(y), \forall y \in H\}$, $\forall x \in H$. Rockafellar [30] proved that ∂g is a maximal monotone operator. It is easy to verify that $0 \in \partial g(v)$ iff $g(v) = \min_{x \in H} g(x)$.

Theorem 4.4 *Let $g : H \to (-\infty, +\infty]$ be a proper convex lower semicontinuous function such that $(\partial g)^{-1}(0)$ is not empty. Let f be a contraction on H with the contractive constant $\kappa \in (0, 1)$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{e_n\}$ be a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n$, $n \geq 1$, $y_n = \arg\min_{z \in H} \{g(z) + \frac{\|z - x_n + e_n\|^2}{2r_n}\}$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ satisfy the conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $0 < r \leq r_n < \infty$ $\lim_{n\to\infty} |r_n - r_{n+1}| = 0$, where r is a real constant. Then $\{x_n\}$ converges strongly to $q = P_{(\partial g)^{-1}(0)} f(q)$.*

Proof Since $g : H \to (-\infty, \infty]$ is a proper convex and lower semicontinuous function, we see that subdifferential ∂g of g is maximal monotone. Putting $F(x, y) = 0$ for any $x, y \in C$, $s_n = 1$, and $A = 0$, we find $y_n = J_{r_n}(x_n + e_n)$. It follows that

$$y_n = \arg\min_{z \in H} \left\{ g(z) + \frac{\|z - x_n - e_n\|^2}{2r_n} \right\}$$

is equivalent to

$$0 \in \partial g(y_n) + \frac{1}{r_n}(y_n - x_n - e_n).$$

Hence, we have

$$x_n + e_n \in y_n + r_n \partial g(y_n).$$

By using Theorem 3.1, we find the desired conclusion immediately. $\square$

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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