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# Some Bonnesen-style Minkowski inequalities

Wenxue Xu[1,2], Chunna Zeng[3] and Jiazu Zhou[1,4*]

*Correspondence: zhoujz@swu.edu.cn
[1]School of Mathematics and Statistics, Southwest University, Chongqing, 400715, People's Republic of China
[4]School of Mathematical Science, Kaili University, Kaili, Guizhou 556000, People's Republic of China
Full list of author information is available at the end of the article

**Abstract**
In this paper, we obtain some Bonnesen-style Minkowski inequalities of mixed volumes of convex bodies $K$ and $L$ in the Euclidean space $\mathbb{R}^n$. Let $L$ be the unit ball; we get some better Bonnesen-style isoperimetric inequalities than Dinghas's result for $n \geq 3$.
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## 1 Introduction

It is well known that the ball has the maximum volume among bodies of fixed surface area in the Euclidean space $\mathbb{R}^n$. That is, of all domains $K$ with surface area $S(K)$ and volume $V(K)$ (*cf.* [1, 2]),

$$S(K)^n - n^n\omega_n V(K)^{n-1} \geq 0, \tag{1}$$

with equality if and only if $K$ is a ball. Here $\omega_n$ denotes the volume of the unit ball,

$$\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)},$$

where $\Gamma(\cdot)$ is the Gamma function.

The isoperimetric deficit

$$\Delta_n(K) = S(K)^n - n^n\omega_n V(K)^{n-1} \tag{2}$$

measures the deficit between the domain $K$ and a ball of radius $(S(K)/n\omega_n)^{1/(n-1)}$. A Bonnesen-style isoperimetric inequality is of the form (*cf.* [2–4])

$$\Delta_n(K) = S(K)^n - n^n\omega_n V(K)^{n-1} \geq B_K, \tag{3}$$

where the quantity $B_K$ is a non-negative invariant of geometric significance of $K$ and vanishes only when $K$ is a ball.

Bonnesen himself proved several inequalities of the form (3) in the Euclidean plane (*cf.* [5, 6]), but he was not able to obtain direct generalizations of his two-dimensional results. This was done much later, first by Hadwiger [7] for $n = 3$, and then by Dinghas [8] for arbitrary dimension. From then on, some Bonnesen-style inequalities in the higher dimensions and generalizations have been obtained by Osserman (*cf.* [1, 2]), Santaló (*cf.* [9]), Groemer and Schneider (*cf.* [10]), Zhang (*cf.* [11]), Zhou (*cf.* [4, 12]) and others. See references [13–36] for more details. The following well-known Bonnesen-style inequality for a convex body $K$ in the Euclidean space $\mathbb{R}^n$ is due to Dinghas (*cf.* [8]):

$$S(K)^n - n^n\omega_n V(K)^{n-1} \geq \left(S(K)^{1/(n-1)} - (n\omega_n)^{1/(n-1)} r\right)^{n(n-1)}, \tag{4}$$

where $r$ is the in-radius of $K$, and equality holds if and only if $K$ is a ball.

In [11], some different forms of Bonnesen-style isoperimetric inequalities have been established associated with the mean width of $K$. Zhang obtained (*cf.* [11])

$$\left(\frac{M(K)}{2}\right)^{n/(n-1)} - \left(\frac{V(K)}{\omega_n}\right)^{1/(n-1)} \geq \left(\frac{V(K)}{\omega_n}\right)^{n/(n-1)} \left(\left(\frac{V(K)}{\omega_n}\right)^{-1/n} - R^{-1}\right),$$

where $M(K)$ and $R$ are the mean width and out-radius of $K$, respectively.

The Minkowski inequality of mixed volume is a natural generalization of the isoperimetric inequality (1) in the Euclidean space $\mathbb{R}^n$ (*cf.* [27, 37–39]). Let $K$, $L$ be convex bodies in $\mathbb{R}^n$, then

$$V_1(K,L)^n \geq V(K)^{n-1} V(L), \tag{5}$$

where $V_1(K,L)$ is the mixed volume of $K$ and $L$ and the equality holds if and only if $K$ and $L$ are homothetic.

Motivated by (2), we define the Minkowski homothetic deficit as

$$\Delta_n(K,L) = V_1(K,L)^n - V(K)^{n-1} V(L). \tag{6}$$

The Minkowski homothetic deficit $\Delta_n(K,L)$ measures the homothety between $K$ and $L$. Then a Bonnesen-style Minkowski inequality would be of the form

$$\Delta_n(K,L) = V_1(K,L)^n - V(K)^{n-1} V(L) \geq B_{K,L}, \tag{7}$$

where the quantity $B_{K,L}$ is an invariant of geometric significance about $K$ and $L$ with the following basic properties:

1. $B_{K,L}$ is non-negative;
2. $B_{K,L}$ vanishes only when $K$ and $L$ are homothetic.

Note that let $L$ be the unit ball $B$ and by $S(K) = nV_1(K,B)$, the surface area of $K$, then the Minkowski homothetic deficit is just the isoperimetric deficit. Therefore, the Bonnesen-style Minkowski inequality (7) is more general than the Bonnesen-style isoperimetric inequality (3).

In this paper, we focus on Bonnesen-style Minkowski inequalities of type (7). Some $B_{K,L}$ are obtained. Let $L$ be the unit ball; then we obtain stronger Bonnesen-style isoperimetric inequalities $K$ than (4).

## 2 Preliminaries

A set of points $K$ in the Euclidean space $\mathbb{R}^n$ is *convex* if for all $x, y \in K$ and $0 \le \lambda \le 1$, $\lambda x + (1-\lambda)y \in K$. A *domain* is a set with nonempty interiors. A *convex body* is a compact convex domain. The set of convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$. Let $\mathcal{K}_o^n$ be the class of members of $\mathcal{K}^n$ containing the origin in their interiors. Write $V$ for an $n$-dimensional Lebesgue measure and $\mathcal{H}^{n-1}$ for an $(n-1)$-dimensional Hausdorff measure. $S^{n-1}$ denotes the surface of the unit ball in $\mathbb{R}^n$.

A convex body $K \subset \mathbb{R}^n$ is uniquely determined by its support function $h_K : \mathbb{R}^n \to \mathbb{R}$, where $h_K(x) = \max\{x \cdot y : y \in K\}$, for $x \in \mathbb{R}^n$. For the support function of the dilate $cK = \{cx : x \in K\}$ of a convex body $K$ we have

$$h_{cK} = ch_K, \quad c > 0. \tag{8}$$

Note that support functions are positively homogeneous of degree one and subadditive. It follows immediately from the definition of support functions that for convex bodies $K$ and $L$

$$K \subseteq L \quad \Longleftrightarrow \quad h_K \le h_L. \tag{9}$$

For a convex body $K$ and each Borel set $\omega \subset S^{n-1}$, the reverse spherical image $\tau(K, \omega)$, of $K$ at $\omega$ is the set of all boundary points of $K$ which have an outer unit normal belonging to the set $\omega$. Associated with each convex body $K \in \mathcal{K}_o^n$ there is a Borel measure $S_K$ on $S^{n-1}$ called the Aleksandrov-Fenchel surface area measure of $K$, defined by

$$S_K(\omega) = \mathcal{H}^{n-1}\big(\tau(K, \omega)\big),$$

for each Borel set $\omega \subseteq \mathbb{S}^{n-1}$. Observe that for the surface area measure of the dilate $cK$ of $K$ we have

$$S_{cK} = c^{n-1} S_K, \quad c > 0.$$

The Minkowski sum of convex sets $K_1, \ldots, K_m$ in $\mathbb{R}^n$ is defined by

$$K_1 + \cdots + K_m = \{x_1 + \cdots + x_m : x_1 \in K_1, \ldots, x_m \in K_m\}.$$

The mixed volume $V(K_1, \ldots, K_n)$ of compact convex sets $K_1, \ldots, K_n$ in $\mathbb{R}^n$ is defined by

$$V(K_1, \ldots, K_n) = \frac{1}{n!} \sum_{j=1}^{n} (-1)^{n+j} \sum_{i_1 < \cdots < i_k} V(K_{i_1} + \cdots + K_{i_k}).$$

The Aleksandrov-Fenchel inequality about the $i$th mixed volume is

$$V_i(K_1, K_2)^2 \ge V_{i+1}(K_1, K_2) V_{i-1}(K_1, K_2), \tag{10}$$

where

$$V_i(K_1, K_2) = V(\underbrace{K_1, \ldots, K_1}_{n-i}, \underbrace{K_2, \ldots, K_2}_{i})$$

with $K_1$ appears $n-i$ times and $K_2$ appears $i$ times and (10) holds as an equality if and only if $K$ and $L$ are homothetic.

Note that

$$V_n(K_1,K_2) = V(K_2), \qquad V_0(K_1,K_2) = V(K_1). \tag{11}$$

The following inequality for mixed volumes is the general Aleksandrov-Fenchel inequality: Let $K_1,\dots,K_n \in \mathcal{K}$ and $1 \le m \le n$. Then

$$V(K_1,\dots,K_n)^m \ge \prod_{i=1}^{m} V(K_i,\dots,K_i,K_{m+1},\dots,K_n).$$

Hence

$$V_1(K_1,K_2)^{n-1} \ge V(K_1)^{n-2} V_{n-1}(K_1,K_2). \tag{12}$$

Let $K_2 = B$, then $V_i(K_1,B) = W_i(K_1)$, the $i$th quermassintegral of the convex body $K_1$.

The mixed volume has monotonicity: If $K_1 \subset K_1'$, then

$$V(K_1,K_2,\dots,K_n) \le V\big(K_1',K_2,\dots,K_n\big).$$

The mixed volume $V_1(K,L)$ of the convex bodies $K,L \in \mathcal{K}_o^n$ has the integral form

$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L \, dS_K. \tag{13}$$

Since

$$V(K) = V_1(K,K),$$

we have

$$V(K) = \frac{1}{n}\int_{S^{n-1}} h_K \, dS_K.$$

If $B$ is the unit ball, then

$$nV_1(K,B) = S(K),$$

the surface area of $K$. The mean width $M(K)$ of $K$ is

$$M(K) = \frac{2}{\omega_n} V_1(B,K),$$

that is,

$$M(K) = \frac{2}{n\omega_n}\int_{S^{n-1}} h_K \, dS_K.$$

The in-radius $r(K, L)$, out-radius $R(K, L)$ of $K$ with respect to $L$ are, respectively, defined by

$$r(K, L) = \sup\{\lambda : x \in \mathbb{R}^n \text{ and } x + \lambda L \subset K\},$$
$$R(K, L) = \inf\{\lambda : x \in \mathbb{R}^n \text{ and } K \subset x + \lambda L\}.$$

Notice that always

$$r(K, L)R(L, K) = 1.$$

When $L$ is the unit ball, $r(K, L)$ and $R(K, L)$ are the radius of maximal inscribed and minimal circumscribed balls of $K$, respectively.

## 3 Bonnesen-style Minkowski inequalities associated with $r(K, L)$

In this section, we derive some Bonnesen-style Minkowski inequalities associated with in-radius $r(K, L)$ of $K$ with respect to $L$. In [26], Diskant improved the Minkowski inequality of mixed volumes as follows.

**Lemma 1** *Let $K$, $L$ be convex bodies in the Euclidean space $\mathbb{R}^n$, then*

$$V_1(K, L)^{n/(n-1)} - V(K)V(L)^{1/(n-1)} \geq \bigl(V_1(K, L)^{1/(n-1)} - r(K, L)V(L)^{1/(n-1)}\bigr)^n, \tag{14}$$

*with equality if and only if $K$ is homothetic to $L$.*

Note that the right-hand side of (14) is non-negative for $x + r(K, L)L \subseteq K$ ($x \in \mathbb{R}^n$). By (13) we have

$$V_1(K, L) = \frac{1}{n}\int_{S^{n-1}} h_L \, dS_K \geq \frac{1}{n}\int_{S^{n-1}} h_L \, dS_{r(K,L)L} \geq r(K, L)^{n-1}V(L).$$

From Lemma 1 and using the inequality $x^{n-1} - y^{n-1} \geq (x - y)^{n-1}$ (for $x \geq y \geq 0$), a lower bound of the Minkowski deficit follows (*cf.* [26, 27]).

**Proposition 1** *Let $K$, $L$ be convex bodies in the Euclidean space $\mathbb{R}^n$, then*

$$\Delta_n(K, L) \geq \bigl(V_1(K, L)^{1/(n-1)} - r(K, L)V(L)^{1/(n-1)}\bigr)^{n(n-1)}, \tag{15}$$

*where the inequality holds as an equality if and only if $K$ and $L$ are homothetic.*

The following Bonnesen-style Minkowski inequality is stronger than (15) for $n = 3$.

**Theorem 1** *Let $K$, $L$ be convex bodies in the Euclidean space $\mathbb{R}^3$, then*

$$\begin{aligned}\Delta_3(K, L) \geq{}& \bigl(V_1(K, L)^{1/2} - r(K, L)V(L)^{1/2}\bigr)^6 \\ &+ 2r(K, L)^3 V(L)^{3/2}\bigl(V_1(K, L)^{1/2} - r(K, L)V(L)^{1/2}\bigr)^3,\end{aligned} \tag{16}$$

*with equality if and only if $K$ is homothetic to $L$.*

*Proof* Since $V(K) \geq r(K,L)^3 V(L)$ and by $x^3 - y^3 \geq (x-y)^3$ (for $x \geq y \geq 0$), we have

$$\begin{aligned} V_1(K,L)^{3/2} + V(K)V(L)^{1/2} &\geq V_1(K,L)^{3/2} + r(K,L)^3 V(L)^{3/2} \\ &= V_1(K,L)^{3/2} - \big(r(K,L)V(L)^{1/2}\big)^3 + 2r(K,L)^3 V(L)^{3/2} \\ &\geq \big(V_1(K,L)^{1/2} - r(K,L)V(L)^{1/2}\big)^3 + 2r(K,L)^3 V(L)^{3/2}. \end{aligned}$$

Note that (14) can be rewritten as

$$V_1(K,L)^{3/2} - V(K)V(L)^{1/2} \geq \big(V_1(K,L)^{1/2} - r(K,L)V(L)^{1/2}\big)^3.$$

Multiplying by $V_1(K,L)^{3/2} + V(K)V(L)^{1/2}$ on both sides, we have

$$V_1(K,L)^3 - V(K)^2 V(L) \geq \big(V_1(K,L)^{1/2} - r(K,L)V(L)^{1/2}\big)^3 \big(V_1(K,L)^{3/2} + V(K)V(L)^{1/2}\big).$$

By these inequalities, we complete the proof of the theorem. □

Let $L$ be the unit ball and notice $S(K) = 3V_1(K,B)$ in (16), we obtain the following Bonnesen-style isoperimetric inequality that strengthens Dinghas's inequality (4) for $n = 3$.

**Corollary 1** *Let $K$ be a convex body in $\mathbb{R}^3$ and $r$ be the in-radius of $K$, then*

$$\Delta_3(K) \geq \big(S(K)^{1/2} - (4\pi)^{1/2} r\big)^6 + 16\pi^{3/2} r^3 \big(S(K)^{1/2} - (4\pi)^{1/2} r\big)^3, \tag{17}$$

*with equality if and only if $K$ is a ball.*

For $n \geq 4$, we obtain a stronger Bonnesen-style Minkowski inequality as follows.

**Theorem 2** *Let $K$, $L$ be convex bodies in the Euclidean space $\mathbb{R}^n$ ($n \geq 4$), then*

$$\begin{aligned} \Delta_n(K,L) \geq{} & \big(V_1(K,L)^{1/(n-1)} - r(K,L)V(L)^{1/(n-1)}\big)^{n(n-1)} \\ & + 2\big(r(K,L)V(L)^{1/(n-1)}\big)^{n(n-2)} \big(V_1(K,L)^{1/(n-1)} - r(K,L)V(L)^{1/(n-1)}\big)^n \\ & + \big(V_1(K,L)V(L)\big)^{n/(n-1)} r(K,L)^n \big(V_1(K,L)^{1/(n-1)} - r(K,L)V(L)^{1/(n-1)}\big)^{n(n-3)}, \end{aligned}$$

*with equality if and only if $K$ is homothetic to $L$.*

*Proof* Let $p = V_1(K,L)^{n/(n-1)}$ and $q = V(L)^{n/(n-1)} r(K,L)^n$, then $p \geq q$.

$$\begin{aligned} & \sum_{i=2}^{n} \big(V_1(K,L)^{n(n-i)/(n-1)} \big(V(K)^{n-1} V(L)\big)^{(i-2)/(n-1)}\big) \\ & \quad \geq \sum_{i=2}^{n} \big(V_1(K,L)^{n(n-i)/(n-1)} \big(V(L)^{1/(n-1)} r(K,L)\big)^{n(i-2)}\big) \\ & \quad = \big(p^{n-2} - q^{n-2}\big) + pq\big(p^{n-4} + p^{n-5}q + \cdots + p^{n-i}q^{i-4} + \cdots + q^{n-4}\big) + 2q^{n-2} \\ & \quad = \big(p^{n-2} - q^{n-2}\big) + pq \cdot \frac{p^{n-3} - q^{n-3}}{p-q} + 2q^{n-2} \end{aligned}$$

$$\begin{aligned}&\geq (p-q)^{n-2} + pq(p-q)^{n-4} + 2q^{n-2}\\&\geq \left(p^{1/n} - q^{1/n}\right)^{n(n-2)} + pq\left(p^{1/n} - q^{1/n}\right)^{n(n-4)} + 2q^{n-2}.\end{aligned}$$

That is,

$$\begin{aligned}&\sum_{i=2}^{n}\left(V_1(K,L)^{n(n-i)/(n-1)}\left(V(K)^{n-1}V(L)\right)^{(i-2)/(n-1)}\right)\\&\quad\geq \left(V_1(K,L)^{1/(n-1)} - V(L)^{1/(n-1)}r(K,L)\right)^{n(n-2)} + 2\left(V(L)^{1/(n-1)}r(K,L)\right)^{n(n-2)}\\&\qquad+ \left(V_1(K,L)V(L)\right)^{n/(n-1)}r(K,L)^n\left(V_1(K,L)^{1/(n-1)} - V(L)^{1/(n-1)}r(K,L)\right)^{n(n-4)}.\end{aligned}$$

Multiplying by $\sum_{i=2}^{n}(V_1(K,L)^{n(n-i)/n-1}(V(K)^{n-1}V(L))^{(i-2)/(n-1)})$ both sides of (14) and via the formula

$$a^{n-1} - b^{n-1} = (a-b)\left(a^{n-2} + a^{n-3}b + \cdots + a^{n-i}b^{i-2} + \cdots + ab^{n-3} + b^{n-2}\right),$$

we obtain

$$\begin{aligned}V_1(K,L)^n - V(K)^{n-1}V(L) &\geq \left(V_1(K,L)^{1/(n-1)} - r(K,L)V(L)^{1/(n-1)}\right)^n\\&\quad\times\left(\sum_{i=2}^{n}\left(V_1(K,L)^{n(n-i)/(n-1)}\left(V(K)^{n-1}V(L)\right)^{(i-2)/(n-1)}\right)\right).\end{aligned}$$

We complete the proof of Theorem 2. □

Let $L$ be the unit ball and by $S(K) = nV_1(K,B)$ in Theorem 2; we obtain the following stronger Bonnesen-style isoperimetric inequality than Dinghas's inequality (4) for $n \geq 4$.

**Corollary 2** *Let $K$ be a convex body in $\mathbb{R}^n$ ($n \geq 4$) and $r$ be the in-radius of $K$, then*

$$\begin{aligned}\Delta_n(K) &\geq \left(S(K)^{1/(n-1)} - (n\omega_n)^{1/(n-1)}r\right)^{n(n-1)}\\&\quad+ 2\left((n\omega_n)^{1/(n-1)}r\right)^{n(n-2)}\left(S(K)^{1/(n-1)} - (n\omega_n)^{1/(n-1)}r\right)^n\\&\quad+ \left(n\omega_nS(K)\right)^{n/(n-1)}r^n\left(S(K)^{1/(n-1)} - (n\omega_n)^{1/(n-1)}r\right)^{n(n-3)},\end{aligned}$$

*with equality if and only if $K$ is a ball.*

## 4 Bonnesen-style Minkowski inequalities associated with the mean width

In this section, we derive some Bonnesen-style Minkowski inequalities associated with the mean width.

**Lemma 2** *Let $K$, $L$ be convex bodies in $\mathbb{R}^n$, then*

$$\frac{V_0(K,L)}{V_1(K,L)} \leq \left(\frac{V_0(K,L)}{V_n(K,L)}\right)^{1/n} \leq \frac{V_{n-1}(K,L)}{V_n(K,L)} \leq \frac{V_1(K,L)^{n-1}}{V(K)^{n-2}V(L)}, \tag{18}$$

*with equality if and only if $K$ and $L$ are homothetic.*

*Proof* By inequality (12), we have

$$\frac{V_{n-1}(K,L)}{V(L)} \leq \frac{V_1(K,L)^{n-1}}{V(K)^{n-2}V(L)}.$$

By the Aleksandrov-Fenchel inequality (10) we have

$$\frac{V_0(K,L)}{V_1(K,L)} \leq \frac{V_1(K,L)}{V_2(K,L)} \leq \cdots \leq \frac{V_i(K,L)}{V_{i+1}(K,L)} \leq \cdots \leq \frac{V_{n-1}(K,L)}{V_n(K,L)}.$$

Therefore

$$\frac{V_0(K,L)}{V_1(K,L)} \leq \left(\frac{V_0(K,L)}{V_n(K,L)}\right)^{1/n} \leq \frac{V_{n-1}(K,L)}{V_n(K,L)}.$$ □

**Theorem 3** *Let $K$, $L$ be convex bodies in $\mathbb{R}^n$, then*

$$\Delta_n(K,L) \geq V(K)^{n-2}V(L)V_1(K,L)\left(\frac{V_{n-1}(K,L)}{V_n(K,L)} - \left(\frac{V_0(K,L)}{V_n(K,L)}\right)^{1/n}\right), \tag{19}$$

*with equality if and only if $K$ and $L$ are homothetic.*

*Proof* Via (18), we have

$$\frac{V_1(K,L)^{n-1}}{V(K)^{n-2}V(L)} - \frac{V_0(K,L)}{V_1(K,L)} \geq \frac{V_{n-1}(K,L)}{V_n(K,L)} - \left(\frac{V_0(K,L)}{V_n(K,L)}\right)^{1/n}.$$

That is

$$V_1(K,L)^n - V(K)^{n-1}V(L) \geq V(K)^{n-2}V(L)V_1(K,L)\left(\frac{V_{n-1}(K,L)}{V_n(K,L)} - \left(\frac{V_0(K,L)}{V_n(K,L)}\right)^{1/n}\right).$$ □

The following Bonnesen-style inequality is a direct consequence of Theorem 3.

**Theorem 4** *Let $K$ be a convex body in $\mathbb{R}^n$, then*

$$\Delta_n(K) \geq n^{n-1}\omega_n S(K)V(K)^{n-2}\left(\frac{M(K)}{2} - \left(\frac{V(K)}{\omega_n}\right)^{1/n}\right),$$

*with equality if and only if $K$ is a ball.*

**Lemma 3** *Let $K$, $L$ be convex bodies in $\mathbb{R}^n$, then*

$$\begin{aligned} &\frac{V_1(K,L)^{n-1} - \sqrt{V_1(K,L)^{n-2}(V_1(K,L)^n - V(K)^{n-1}V(L))}}{V(K)^{n-2}V(L)} \\ &\quad \leq \frac{V_0(K,L)}{V_1(K,L)} \leq \frac{V_{n-1}(K,L)}{V_n(K,L)} \leq \frac{V_1(K,L)^{n-1}}{V(K)^{n-2}V(L)}. \end{aligned} \tag{20}$$

*Proof* The Minkowski inequality (5) gives

$$\frac{V_1(K,L)^{n-1} - \sqrt{V_1(K,L)^{n-2}(V_1(K,L)^n - V(K)^{n-1}V(L))}}{V(K)^{n-2}V(L)} \leq \frac{V_0(K,L)}{V_1(K,L)}.$$

The above inequality together with (18) leads to Lemma 3. □

We are now in a position to prove the following Bonnesen-style Minkowski inequality.

**Theorem 5** *Let $K$, $L$ be convex bodies in $\mathbb{R}^n$, then*

$$\Delta_n(K,L) \geq \frac{V(K)^{2n-4}V(L)^2}{V_1(K,L)^{n-2}}\left(\frac{V_{n-1}(K,L)}{V_n(K,L)} - \frac{V_0(K,L)}{V_1(K,L)}\right)^2, \tag{21}$$

*with equality if and only if $K$ and $L$ are homothetic.*

*Proof* From (20) we have

$$\frac{\sqrt{V_1(K,L)^{n-2}(V_1(K,L)^n - V(K)^{n-1}V(L))}}{V(K)^{n-2}V(L)} \geq \frac{V_{n-1}(K,L)}{V_n(K,L)} - \frac{V_0(K,L)}{V_1(K,L)}. \qquad \square$$

The following Bonnesen-style inequality is a direct consequence of Theorem 5 when $L$ is the unit ball.

**Theorem 6** *Let $K$ be a convex body in $\mathbb{R}^n$, then*

$$S(K)^n - n^n\omega_n V(K)^{n-1} \geq \frac{n^{2n-2}\omega_n^2 V^{2n-4}}{S^{n-2}}\left(\frac{M(K)}{2} - \frac{nV(K)}{S(K)}\right)^2,$$

*with equality if and only if $K$ is a ball.*

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The authors read and approved the final manuscript.

**Author details**
[1]School of Mathematics and Statistics, Southwest University, Chongqing, 400715, People's Republic of China. [2]Faculty of Education, Southwest University, Chongqing, 400715, People's Republic of China. [3]College of Mathematics Science, Chongqing Normal University, Chongqing, 401331, People's Republic of China. [4]School of Mathematical Science, Kaili University, Kaili, Guizhou 556000, People's Republic of China.

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