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On the complete convergence for arrays of rowwise ψ-mixing random variables

Yong-Feng Wu[1] and Hui Ding[2*]

*Correspondence:
dinghuicz@126.com
[2]School of Mathematical Sciences,
Chuzhou University, Chuzhou,
239000, P.R. China
Full list of author information is
available at the end of the article

Abstract

Some sufficient conditions for complete convergence for maximal weighted sums $\max_{1\le j\le n}|\sum_{k=1}^{j} a_{nk}X_{nk}|$ and weighted sums $\sum_{k=1}^{n} a_{nk}X_{nk}$ are presented, where $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ is an array of rowwise ψ-mixing random variables, and $\{a_{nk}, 1 \le k \le n, n \ge 1\}$ is an array of constants. The obtained results extend and improve the corresponding result in the previous literature.

MSC: 60F15

Keywords: ψ-mixing random variable; weighted sums; complete convergence

1 Introduction

The following notion was given firstly by Hsu and Robbins [1].

Definition 1.1 A sequence of random variables $\{U_n, n \ge 1\}$ is said to converge completely to a constant θ if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\big(|U_n - \theta| > \varepsilon\big) < \infty.$$

In this case, we write $U_n \to \theta$ completely. In view of the Borel-Cantelli lemma, the result above implies that $U_n \to \theta$ almost surely. Therefore, the complete convergence is a very important tool in establishing almost sure convergence of summation of random variables as well as weighted sums of random variables.

Let $\{X_n, n \ge 1\}$ be a sequence of random variables, defined on a probability space $(\Omega, \mathcal{F}, P)$, and denote σ-algebras

$$\mathcal{F}_n^m = \sigma(X_k, n \le k \le m), \quad 1 \le n \le m \le \infty.$$

As usual, for a σ-algebra $\mathcal{F}$, we denote by $\mathcal{L}^2(\mathcal{F})$ the class of all $\mathcal{F}$-measurable random variables with the finite second moment. Given σ-algebras $\mathcal{A}$, $\mathcal{B}$ in $\mathcal{F}$, let

$$\psi(\mathcal{A},\mathcal{B}) = \sup_{A\in\mathcal{A},B\in\mathcal{B},P(A)P(B)>0}\left|\frac{P(AB)}{P(A)P(B)} - 1\right|,$$

$$\varphi(\mathcal{A},\mathcal{B}) = \sup_{A\in\mathcal{A},B\in\mathcal{B},P(A)>0}\big|P(B|A) - P(B)\big|.$$

Define the mixing coefficients by

$$\psi(n) = \sup_{k\geq 1} \psi\left(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty\right), \qquad \varphi(n) = \sup_{k\geq 1} \varphi\left(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty\right), \quad n \geq 0.$$

The concepts of ψ-mixing and φ-mixing random variables were introduced by Blum *et al.* [2] and Dobrushin [3], respectively.

Definition 1.2 A sequence of random variables $\{X_n, n \geq 1\}$ is said to be a ψ-mixing (φ-mixing) sequence of random variables if $\psi(n) \downarrow 0$ ($\varphi(n) \downarrow 0$) as $n \to \infty$.

Clearly, from the definition above, we know that the independence implies ψ-mixture and φ-mixture. It is easily seen that the ψ-mixing condition is stronger than the φ-mixing. Therefore, the family of ψ-mixing is a special case of φ-mixing. Years after the appearance of Dobrushin [3], many works of investigation concerning the convergence properties of φ-mixing random variables have emerged. We refer the reader to Ibragimov [4], Cogburn [5], Sen [6], Choi and Sung [7], Utev [8], Chen [9], Shao [10], Rüdiger [11], Chen *et al.* [12], Zhou [13], Wang *et al.* [14, 15], Guo [16].

However, according to our knowledge, few papers discuss the subjects for sequences or arrays of ψ-mixing random variables except Blum *et al.* [2], Bradley [17], Yang [18], Wu and Zhu [19], Wang *et al.* [14, 15], and Yang and Liu [20]. The goal of this paper is to study a complete convergence for arrays of rowwise ψ-mixing random variables.

Then we recall that the following concept of stochastic domination is a slight generalization of identical distribution.

Definition 1.3 An array of rowwise random variables $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ is said to be stochastically dominated by a nonnegative random variable X (write $\{X_{nk}\} \prec X$) if there exists a constant $C > 0$ such that

$$\sup_{n,k} P\left(|X_{nk}| > x\right) \leq CP(X > x), \quad \forall x > 0.$$

Stochastic dominance of $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ by the random variable X implies that $E|X_{nk}|^p \leq CEX^p$ if the p-moment of X exists, *i.e.*, if $EX^p < \infty$.

Hu *et al.* [21] obtained the following result in the complete convergence.

Theorem A *Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise independent random variables with $EX_{nk} = 0$. Suppose that $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ are uniformly bounded by some random variable X. If $E|X|^{2p} < \infty$ for some $1 \leq p < 2$, then*

$$n^{-1/p} \sum_{k=1}^n X_{nk} \to 0 \quad \textit{completely.}$$

Taylor *et al.* [22], Baek *et al.* [23] extended and generalized Theorem A to rowwise negatively dependent (ND) random variables.

The main purpose of this article is to discuss the complete convergence for weighted sums of ψ-mixing random variables. We shall extend Theorem A by considering ψ-mixing instead of independent. It is worthy to point out that our main methods differ from those used by Hu *et al.* [21].

Below, C will be used to denote various positive constants, whose value may vary from one application to another. For a finite set A, the symbol $\sharp(A)$ denotes the number of elements in the set A. $I_{(A)}$ will indicate the indicator function of A.

2 Main results and some lemmas

Now, we state our main results. The proofs will be given in Section 3.

Theorem 2.1 *Let $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of rowwise ψ-mixing random variables with $\sum_{m=1}^{\infty} \psi(m) < \infty$ and $EX_{nk} = 0$. Suppose that $\{X_{nk}\} \prec X$ and $EX^{2p} < \infty$ for some $p > 0$. Let $\{a_{nk}, 1 \le k \le n, n \ge 1\}$ be a real numbers array satisfying $\max_{1 \le k \le n} |a_{nk}| = O(n^{-\alpha})$ for some $\alpha > 1/(2p)$. Furthermore, when $p \ge 1$, we suppose that there exists a constant $\theta > 0$ such that $\sum_{k=1}^{n} a_{nk}^2 \le Cn^{-\theta}$. Then*

$$\sum_{n=1}^{\infty} n^{2\alpha p-2} P\left(\max_{1 \le j \le n}\left|\sum_{k=1}^{j} a_{nk}X_{nk}\right| > \varepsilon\right) < \infty, \quad \forall \varepsilon > 0.$$

Take $a_{nk} = n^{-1/p}$ and $1 \le p < 2$ in Theorem 2.1, we can have the following corollary.

Corollary 2.1 *Let $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of rowwise ψ-mixing random variables with $\sum_{m=1}^{\infty} \psi(m) < \infty$ and $EX_{nk} = 0$. Suppose that $\{X_{nk}\} \prec X$ and $EX^{2p} < \infty$ for some $1 \le p < 2$, then*

$$\sum_{n=1}^{\infty} P\left(\max_{1 \le j \le n}\left|\sum_{k=1}^{j} X_{nk}\right| > n^{1/p}\varepsilon\right) < \infty, \quad \forall \varepsilon > 0.$$

Remark 2.1 Since the independence implies ψ-mixture, Theorem 2.1 and Corollary 2.1 hold for arrays of rowwise independent random variables. Therefore, Theorem 2.1 and Corollary 2.1 extend and improve Theorem A.

Theorem 2.2 *Let $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of rowwise ψ-mixing random variables with $EX_{nk} = 0$. Suppose that $\{X_{nk}\} \prec X$ and $EX^{2p} < \infty$ for some $p \ge 1$. Let $\{a_{nk}, 1 \le k \le n, n \ge 1\}$ be a real numbers array satisfying $\max_{1 \le k \le n} |a_{nk}| = O(n^{-\alpha})$ for some $\alpha > 1/(2p)$. Suppose that the following statements hold.*

(i) *There exists a positive constant $\lambda < \min\{\frac{1}{2p}, \frac{2\alpha p-1}{4p}\}$ such that $\sum_{n=1}^{\infty} \psi^{\frac{\lambda}{1-\lambda}}(n) < \infty$;*
(ii) *$\log n \sum_{k=1}^{n} a_{nk}^2 = o(1)$ if $\frac{1}{2p} < \alpha \le \frac{1}{2}$. Then*

$$\sum_{n=1}^{\infty} n^{2\alpha p-2} P\left(\left|\sum_{k=1}^{n} a_{nk}X_{nk}\right| > \varepsilon\right) < \infty, \quad \forall \varepsilon > 0.$$

Remark 2.2 Compared with Theorem 2.1, Theorem 2.2 requires a stronger mixing rate, but weakens the requirement of $\sum_{k=1}^{n} a_{nk}^2$. In fact, $\log n \sum_{k=1}^{n} a_{nk}^2 = o(1)$ holds if $\sum_{k=1}^{n} a_{nk}^2 \le Cn^{-\theta}$, $\theta > 0$.

Now, we state some lemmas which will be used in the proofs of our main results.

Lemma 2.1 (Wang *et al.* [14, 15]) *Let $\{X_n, n \ge 1\}$ be a sequence of ψ-mixing random variables satisfying $\sum_{m=1}^{\infty} \psi(m) < \infty$, $q \ge 2$. Assume that $EX_n = 0$ and $E|X_n|^q < \infty$ for each*

$n \geq 1$. *Then there exists a constant* C *depending only on* q *and* $\psi(\cdot)$ *such that*

$$E \max_{1\leq j\leq n} \left| \sum_{k=1}^{j} X_k \right|^q \leq C \left\{ \sum_{k=1}^{n} E|X_k|^q + \left(\sum_{k=1}^{n} EX_k^2 \right)^{q/2} \right\}.$$

Lemma 2.2 (Yang [18]) *Let* $\{X_n, n \geq 1\}$ *be a sequence of* ψ*-mixing random variables with* $EX_k = 0$, $|X_k| \leq d < \infty$ *a.s.*, $k = 1, 2, \dots$, $0 < \lambda < 1$, $m = [n^\lambda]$. *Then* $\forall \varepsilon > 0$,

$$P\left(\left| \sum_{k=1}^{n} X_k \right| > \varepsilon \right) \leq 2eC_1 \exp\{-t\varepsilon + C_2 t^2 B_n\},$$

where $B_n = \sum_{k=1}^n EX_k^2$, $tmd \leq 1/4$, $C_1 = \exp\{2en^{1-\lambda}\psi(m)\}$, $C_2 = 4(1 + 4\sum_{k=1}^{2m} \psi(k))$.

Lemma 2.3 *Let* $\{X_n, n \geq 1\}$ *be a sequence of* ψ*-mixing random variables, and let* $A_j = \{|X_j| \geq x_j\}$, $x_j \in \mathbb{R}^+$, $j = 1, 2, \dots, N$, *then*

$$P(A_1, A_2, \dots, A_N) \leq \big(1 + \psi(1)\big)^N \prod_{j=1}^{N} P(A_j).$$

Proof By the definition of ψ-mixing, we have

$$\begin{aligned} P(A_1, A_2, \dots, A_N) &\leq \big(1 + \psi(1)\big) P(A_1) P(A_2, \dots, A_N) \\ &\leq \cdots \\ &\leq \big(1 + \psi(1)\big)^N P(A_1) P(A_2) \cdots P(A_N). \end{aligned}$$

The proof is complete. □

3 Proofs

In this section, we state the proofs of our main results.

Proof of Theorem 2.1 Let $S_{nj} = \sum_{k=1}^{j} a_{nk} X_{nk}$, $1 \leq j \leq n$. Since $a_{nk} = a_{nk}^+ - a_{nk}^-$, without loss of generality, we may assume that $0 < a_{nk} \leq Cn^{-\alpha}$. Let $0 < \rho < \frac{(2\alpha p-1)(N-1)}{2pN}$, where N is a positive integer with $N > 1$. Let

$$\begin{aligned} &X'_{nk} = X_{nk} I_{(a_{nk}|X_{nk}| \leq n^{-\rho})}, \qquad X''_{nk} = X_{nk} I_{(a_{nk}|X_{nk}| > \varepsilon/N)}, \\ &X'''_{nk} = X_k - X'_{nk} - X''_{nk} = X_{nk} I_{(n^{-\rho} < a_{nk}|X_{nk}| \leq \varepsilon/N)}, \\ &S'_{nj} = \sum_{k=1}^{j} a_{nk} X'_{nk}, \qquad S''_{nj} = \sum_{k=1}^{j} a_{nk} X''_{nk}, \qquad S'''_{nj} = \sum_{k=1}^{j} a_{nk} X'''_{nk}. \end{aligned}$$

Firstly, we prove $\sum_{n=1}^{\infty} n^{2\alpha p-2} P(\max_{1\leq j\leq n} |S'_{nj}| > \varepsilon) < \infty$. By $\{X_n\} \prec X$, we know that $E|X_{nk}|^{2p} \leq EX^{2p} < \infty$. If $0 < p \leq 1/2$, we have

$$\begin{aligned} \max_{1\leq j\leq n} \left| \sum_{k=1}^{j} a_{nk} EX'_{nk} \right| &\leq \sum_{k=1}^{n} a_{nk} \left| EX'_{nk} \right| \leq \sum_{k=1}^{n} a_{nk} E|X_{nk}| I_{(a_{nk}|X_{nk}| \leq n^{-\rho})} \\ &\leq \sum_{k=1}^{n} \frac{a_{nk}^{2p} E|X_{nk}|^{2p}}{n^{-2\rho p}} n^{-\rho} \leq Cn^{\rho(2p-1)-(2\alpha p-1)} \to 0, \quad n \to \infty. \qquad (1) \end{aligned}$$

If $p > 1/2$, by $EX_{nk} = 0$ and $\rho < \frac{2\alpha p-1}{2p} < \frac{2\alpha p-1}{2p-1}$, we also have

$$\begin{aligned}\max_{1\le j\le n}\left|\sum_{k=1}^{j} a_{nk}EX'_{nk}\right| &\le \sum_{k=1}^{n} a_{nk}\left|EX'_{nk}\right| \le \sum_{k=1}^{n} a_{nk}E|X_{nk}|I_{(a_{nk}|X_{nk}|>n^{-\rho})} \\ &\le \sum_{k=1}^{n}\frac{a_{nk}^{2p}E|X_{nk}|^{2p}}{n^{-2\rho p}}n^{-\rho} \le Cn^{\rho(2p-1)-(2\alpha p-1)} \to 0, \quad n\to\infty.\end{aligned}$$

Therefore, we know that (1) holds for $p > 0$. Let $S^*_{nj} = \sum_{k=1}^{j} a_{nk}(X'_{nk} - EX'_{nk})$. To prove $\sum_{n=1}^{\infty} n^{2\alpha p-2}P(\max_{1\le j\le n}|S'_{nj}| > \varepsilon) < \infty$, it suffices to show that $\sum_{n=1}^{\infty} n^{2\alpha p-2}P(\max_{1\le j\le n}|S^*_{nj}| > \varepsilon) < \infty$.

If $0 < p < 1$, by Markov's inequality and Lemma 2.1, we have

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{2\alpha p-2}P\Big(\max_{1\le j\le n}\left|S^*_{nj}\right| > \varepsilon\Big) \\ &\quad\le C\sum_{n=1}^{\infty} n^{2\alpha p-2}E\Big(\max_{1\le j\le n}\left|S^*_{nj}\right|\Big)^2 \le C\sum_{n=1}^{\infty} n^{2\alpha p-2}\sum_{k=1}^{n} a_{nk}^2 E\big(X'_{nk}\big)^2 \\ &\quad\le \sum_{n=1}^{\infty} n^{2\alpha p-2}\sum_{k=1}^{n} a_{nk}^{2p}E\big(\left|a_{nk}X'_{nk}\right|^{2-2p}\left|X'_{nk}\right|^{2p}\big) \\ &\quad\le \sum_{n=1}^{\infty} n^{2\alpha p-2}n^{-\rho(2-2p)}\sum_{k=1}^{n} a_{nk}^{2p}E\left|X'_{nk}\right|^{2p} \le C\sum_{n=1}^{\infty} n^{-1-2\rho(1-p)} < \infty.\end{aligned} \tag{2}$$

If $p \ge 1$, take $q > \max\{2p, 2(2\alpha p - 1)/\theta\}$. By $q > 2$ and Lemma 2.1, we have

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{2\alpha p-2}P\Big(\max_{1\le j\le n}\left|S^*_{nj}\right| > \varepsilon\Big) \\ &\quad\le \sum_{n=1}^{\infty} n^{2\alpha p-2}\left[\sum_{k=1}^{n} a_{nk}^q E\left|X'_{nk}\right|^q + \left(\sum_{k=1}^{n} a_{nk}^2 E\big(X'_{nk}\big)^2\right)^{q/2}\right].\end{aligned} \tag{3}$$

By a similar argument as in the proof of (2) (replacing exponent 2 into q), we can get

$$\sum_{n=1}^{\infty} n^{2\alpha p-2}\sum_{k=1}^{n} a_{nk}^q E\left|X'_{nk}\right|^q < \infty. \tag{4}$$

Note that $E|X'_{nk}|^2 \le EX^2 < \infty$ and the definition of q, we have

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{2\alpha p-2}\left(\sum_{k=1}^{n} a_{nk}^2 E\big(X'_{nk}\big)^2\right)^{q/2} \\ &\quad\le \sum_{n=1}^{\infty} n^{2\alpha p-2}\left(\sum_{k=1}^{n} a_{nk}^2\right)^{q/2}\big(EX^2\big)^{q/2} \le C\sum_{n=1}^{\infty} n^{-1+(2\alpha p-1)-\theta q/2}\big(EX^2\big)^{q/2} < \infty.\end{aligned} \tag{5}$$

From (3)-(5), we know that (2) still holds for $p \geq 1$. By (1) and (2), we have

$$\sum_{n=1}^{\infty} n^{2\alpha p-2} P\Big(\max_{1\leq j\leq n} \big|S'_{nj}\big| > \varepsilon\Big) < \infty.$$

Secondly, we prove $\sum_{n=1}^{\infty} n^{2\alpha p-2} P(\max_{1\leq j\leq n} |S''_{nj}| > \varepsilon) < \infty$. Let $\varphi_n(j) = \sharp\{1 \leq k \leq n : a_{nk} > \varepsilon/(jN)\}$ and $\phi_j = [(jCN/\varepsilon)^{1/\alpha}]$, then

$$\begin{aligned} P\Big(\max_{1\leq j\leq n} \big|S''_{nj}\big| > \varepsilon\Big) &\leq P\left(\bigcup_{k=1}^{n} \big\{a_{nk}|X_{nk}| > \varepsilon/N\big\}\right) \\ &\leq \sum_{k=1}^{n} P\big(a_{nk}|X_{nk}| > \varepsilon/N\big) \leq C\sum_{k=1}^{n} P(a_{nk}X > \varepsilon/N) \\ &= C\sum_{j=1}^{\infty}\sum_{k=1}^{n} P(a_{nk}X > \varepsilon/N, j-1 \leq X < j) \\ &\leq C\sum_{j=1}^{\infty} \varphi_n(j) P(j-1 \leq X < j). \end{aligned} \tag{6}$$

By $a_{nk}X > \varepsilon/N$, we know $a_{nk} > \varepsilon/(jN)$. Note $a_{nk} \leq Cn^{-\alpha}$, we have $n < (jCN/\varepsilon)^{1/\alpha}$. Hence, we have $n \leq \phi_j$, then

$$\begin{aligned} \sum_{n=1}^{\infty} n^{2\alpha p-2} P\Big(\max_{1\leq j\leq n} \big|S''_{nj}\big| > \varepsilon\Big) &\leq C\sum_{n=1}^{\infty} n^{2\alpha p-2} \sum_{j=1}^{\infty} \varphi_n(j) P(j-1 \leq X < j) \\ &\leq C\sum_{j=1}^{\infty}\sum_{n=1}^{\phi_j} n^{2\alpha p-2} \varphi_n(j) P(j-1 \leq X < j). \end{aligned} \tag{7}$$

Take $\nu \in (0, \frac{2\alpha p-1}{\alpha})$, then $\sum_{k=1}^{n} a_{nk}^{\nu} \geq \varphi_n(j)\varepsilon^{\nu}/(jN)^{\nu}$. From $\sum_{k=1}^{n} a_{nk}^{\nu} \leq Cn^{1-\alpha\nu}$, we have

$$\varphi_n(j) \leq N^{\nu} j^{\nu}/\varepsilon^{\nu} \sum_{k=1}^{n} a_{nk}^{\nu} \leq Cn^{1-\alpha\nu} j^{\nu}. \tag{8}$$

Therefore, by (7) and (8), we have

$$\sum_{n=1}^{\infty} n^{2\alpha p-2} P\Big(\max_{1\leq j\leq n} \big|S''_{nj}\big| > \varepsilon\Big) \leq C\sum_{j=1}^{\infty}\sum_{n=1}^{\phi_j} n^{2\alpha p-\alpha\nu-1} j^{\nu} P(j-1 \leq X < j). \tag{9}$$

By the definition of ν, we have $2\alpha p - \alpha\nu - 1 > 0$, then

$$\sum_{n=1}^{\phi_j} n^{2\alpha p-\alpha\nu-1} \leq \phi_j^{\alpha(2p-\nu)} \leq Cj^{2p-\nu}. \tag{10}$$

From (9), (10) and $EX^{2p} < \infty$, then

$$\sum_{n=1}^{\infty} n^{2\alpha p-2} P\Big(\max_{1\leq j\leq n} \big|S''_{nj}\big| > \varepsilon\Big) \leq C\sum_{j=1}^{\infty} j^{2p} P(j-1 \leq X < j) < \infty.$$

Finally, we prove $\sum_{n=1}^{\infty} n^{2\alpha p-2}P(\max_{1\le j\le n}|S_{nj}'''| > \varepsilon) < \infty$. Obviously, we know that $P(\max_{1\le j\le n}|S_{nj}'''| > \varepsilon) \le P(\sum_{k=1}^{n} a_{nk}|X_{nk}'''| > \varepsilon)$. Let $M = \sharp\{1 \le k \le n : n^{-\rho} < a_{nk}|X_{nk}| \le \varepsilon/N\}$. We must let M be at least N such that $\sum_{k=1}^{n} a_{nk}|X_{nk}'''| > \varepsilon$. Take $W_k = \{a_{nk}|X_{nk}| > n^{-\rho}\}$, we have

$$P\Bigl(\max_{1\le j\le n}\bigl|S_{nj}'''\bigr| > \varepsilon\Bigr) \le P\bigl(M \text{ is at least } N,\ \text{such that } a_{nk}|X_{nk}| > n^{-\rho}\bigr) \le \sum_{1\le i_1<i_2<\cdots<i_N\le n} P(W_{i_1}W_{i_2}\cdots W_{i_N}). \tag{11}$$

By Lemma 2.3, we have

$$P(W_{i_1}W_{i_2}\cdots W_{i_N}) \le \bigl(1+\psi(1)\bigr)^N P(W_{i_1})P(W_{i_2})\cdots P(W_{i_N}). \tag{12}$$

From (11) and (12), we have

$$\begin{aligned}
&n^{2\alpha p-2}P\Bigl(\max_{1\le j\le n}\bigl|S_{nj}'''\bigr| > \varepsilon\Bigr)\\
&\quad\le \bigl(1+\psi(1)\bigr)^N n^{2\alpha p-2}\sum_{1\le i_1<i_2<\cdots<i_N\le n} P(W_{i_1})P(W_{i_2})\cdots P(W_{i_N})\\
&\quad\le Cn^{2\alpha p-2}\binom{n}{N}P^N\bigl(X > a_{nk}^{-1}n^{-\rho}\bigr)\\
&\quad\le Cn^{2\alpha p-2}n^N P^N\bigl(X > C^{-1}n^{-\rho+\alpha}\bigr)\\
&\quad\le Cn^{-1-(2\alpha p-1)(N-1)+2\rho pN}\bigl(EX^{2p}\bigr)^N.
\end{aligned} \tag{13}$$

Noting that $0 < \rho < \frac{(2\alpha p-1)(N-1)}{2pN}$. We have

$$-(2\alpha p-1)(N-1) + 2\rho pN < 0,$$

then

$$\sum_{n=1}^{\infty} n^{2\alpha p-2}P\Bigl(\max_{1\le j\le n}\bigl|S_{nj}'''\bigr| > \varepsilon\Bigr) \le C\sum_{n=1}^{\infty} n^{-1-(2\alpha p-1)(N-1)+2\rho pN} < \infty.$$

The proof is completed. □

Proof of Theorem 2.2 Following the notations of X_{nk}', X_{nk}'' and X_{nk}''', but let

$$T_n' = \sum_{k=1}^{n} a_{nk}X_{nk}', \qquad T_n'' = \sum_{k=1}^{n} a_{nk}X_{nk}'',$$
$$T_n''' = \sum_{k=1}^{n} a_{nk}X_{nk}''', \qquad T_n^* = \sum_{k=1}^{n} a_{nk}\bigl(X_{nk}' - EX_{nk}'\bigr).$$

Obviously, by following the methods used in the proof of (1), we have

$$\left|\sum_{k=1}^{n} a_{nk}EX_{nk}'\right| \to 0, \quad n \to \infty. \tag{14}$$

By similar arguments as in the proofs of

$$\sum_{n=1}^{\infty} n^{2\alpha p-2} P\Big(\max_{1\le j\le n} \big|S''_{nj}\big| > \varepsilon\Big) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} n^{2\alpha p-2} P\Big(\max_{1\le j\le n} \big|S'''_{nj}\big| > \varepsilon\Big) < \infty,$$

we can prove

$$\sum_{n=1}^{\infty} n^{2\alpha p-2} P\big(\big|T''_n\big| > \varepsilon\big) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} n^{2\alpha p-2} P\big(\big|T'''_n\big| > \varepsilon\big) < \infty.$$

Here, we omit the details. Therefore, we need only to show

$$\sum_{n=1}^{\infty} n^{2\alpha p-2} P\big(\big|T^*_n\big| > \varepsilon\big) < \infty.$$

Take $\lambda < \rho < \frac{(2\alpha p-1)(N-1)}{2pN}$, by $0 < \lambda < \frac{1}{2p}$ and $p \ge 1$, we know $0 < \frac{\lambda}{1-\lambda} < 1$. Hence, from condition (i), we have

$$C_2 = 4\left(1 + 4\sum_{k=1}^{2m} \psi(k)\right) \le 4\left(1 + 4\sum_{k=1}^{\infty} \psi(k)\right) < \infty, \qquad \psi(m) = o\big(m^{\frac{\lambda-1}{\lambda}}\big),$$

where $m = [n^\lambda]$. Therefore, $C_1 = \exp\{2en^{1-\lambda}\psi(m)\} \ll \exp\{2e\} < \infty$.

Take $t = \frac{2\alpha p \log n}{\varepsilon}$. Clearly, $t \le n^{\rho-\lambda}/8$ when n is sufficiently large. Note that $|a_{nk}(X'_{nk} - EX'_{nk})| \le 2n^{-\rho} = d$, then $tmd \le 1/4$ when n is sufficiently large. By Lemma 2.2, we have

$$\begin{aligned} P\big(\big|T^*_n\big| > \varepsilon\big) &\le C\exp\left\{-t\varepsilon + C_0 C_2 t^2 \sum_{k=1}^{n} a_{nk}^2\right\} \\ &= C\exp\left\{-2\alpha p \log n + C_0 C_2 \frac{(2\alpha p)^2 \log^2 n}{\varepsilon^2} \sum_{k=1}^{n} a_{nk}^2\right\}, \end{aligned} \tag{15}$$

where $C_0 = E(X'_{nk})^2 \le EX^{2p} < \infty$. Note that $\log n \sum_{k=1}^{n} a_{nk}^2 = o(1)$ holds if $\alpha > \frac{1}{2}$. Therefore, by condition (ii), we have

$$C_0 C_2 \frac{(2\alpha p)^2 \log n}{\varepsilon^2} \sum_{k=1}^{n} a_{nk}^2 \to 0, \quad n \to \infty. \tag{16}$$

Hence, when n is sufficiently large, by (14) and (15), we have

$$P\big(\big|T^*_n\big| > \varepsilon\big) \le C\exp\big(-2\alpha p \log n + 2^{-1} \log n\big) = Cn^{-2\alpha p+1/2}.$$

Then

$$\sum_{n=1}^{\infty} n^{2\alpha p-2} P\big(\big|T^*_n\big| > \varepsilon\big) \le C\sum_{n=1}^{\infty} n^{-\frac{3}{2}} < \infty.$$

The proof is completed. $\square$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

YFW carried out the proofs of the main results in the manuscript. HD participated in the design of the study and drafted the manuscript. All authors read and approved the final manuscript.

Author details

[1]College of Mathematics and Computer Science, Tongling University, Tongling, 244000, P.R. China. [2]School of Mathematical Sciences, Chuzhou University, Chuzhou, 239000, P.R. China.

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