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On the generalised sum of squared logarithms inequality

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Abstract

Assume $n \geq 2$. Consider the elementary symmetric polynomials $e_k(y_1, y_2, \ldots, y_n)$ and denote by $E_0, E_1, \ldots, E_{n-1}$ the elementary symmetric polynomials in reverse order $E_k(y_1, y_2, \ldots, y_n) := e_{n-k}(y_1, y_2, \ldots, y_n) = \sum_{i_1 < \cdots < i_{n-k}} y_{i_1} y_{i_2} \cdots y_{i_{n-k}}$, $k \in \{0, 1, \ldots, n-1\}$. Let, moreover, S be a nonempty subset of $\{0, 1, \ldots, n-1\}$. We investigate necessary and sufficient conditions on the function $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, such that the inequality $f(a_1) + f(a_2) + \cdots + f(a_n) \leq f(b_1) + f(b_2) + \cdots + f(b_n)$ $(*)$ holds for all $a = (a_1, a_2, \ldots, a_n) \in I^n$ and $b = (b_1, b_2, \ldots, b_n) \in I^n$ satisfying $E_k(a) < E_k(b)$ for $k \in S$ and $E_k(a) = E_k(b)$ for $k \in \{0, 1, \ldots, n-1\} \setminus S$. As a corollary, we obtain our inequality $(*)$ if $2 \leq n \leq 4$, $f(x) = \log^2 x$ and $S = \{1, \ldots, n-1\}$, which is the sum of squared logarithms inequality previously known for $2 \leq n \leq 3$.

MSC: 26D05; 26D07

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1 Introduction - the sum of squared logarithms inequality

In a previous contribution [1] the sum of squared logarithms inequality has been introduced and proved for the particular cases $n = 2, 3$. For $n = 3$ it reads: let $a_1, a_2, a_3, b_1, b_2, b_3 > 0$ be given positive numbers such that

$$\begin{aligned} &a_1 + a_2 + a_3 \leq b_1 + b_2 + b_3, \\ &a_1a_2 + a_1a_3 + a_2a_3 \leq b_1b_2 + b_1b_3 + b_2b_3, \\ &a_1a_2a_3 = b_1b_2b_3. \end{aligned}$$

Then

$$\log^2 a_1 + \log^2 a_2 + \log^2 a_3 \leq \log^2 b_1 + \log^2 b_2 + \log^2 b_3.$$

The general form of this inequality can be conjectured as follows.

Definition 1.1 The *standard elementary symmetric polynomials* $e_1, \ldots, e_{n-1}, e_n$ are

$$e_k(y_1, \ldots, y_n) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} y_{j_1} \cdot y_{j_2} \cdot \cdots \cdot y_{j_k}, \quad k \in \{1, 2, \ldots, n\}; \tag{1.1}$$

note that $e_n = y_1 \cdot y_2 \cdot \cdots \cdot y_n$.

Conjecture 1.2 (Sum of squared logarithms inequality) *Let $a_1, a_2, \dots, a_n$, $b_1, b_2, \dots, b_n$ be given positive numbers. Then the condition*

$$e_k(a_1, \dots, a_n) \le e_k(b_1, \dots, b_n), \quad k \in \{1, 2, \dots, n-1\}, \qquad e_n(a_1, \dots, a_n) = e_n(b_1, \dots, b_n)$$

implies that

$$\sum_{i=1}^{n} \log^2 a_i \le \sum_{i=1}^{n} \log^2 b_i.$$

Remark 1.3 Note that the conclusions of Conjecture 1.2 are trivial provided we have equality everywhere, *i.e.*

$$e_k(a_1, \dots, a_n) = e_k(b_1, \dots, b_n), \quad k \in \{1, 2, \dots, n\}. \tag{1.2}$$

In this case, the coefficients $a_1, \dots, a_n$, $b_1, \dots, b_n$ are equal up to permutations, which can be seen by looking at the characteristic polynomials of two matrices with eigenvalues $a_1, \dots, a_n$ and $b_1, \dots, b_n$. From this perspective, having equality just in the last product e_n and strict inequality else seems to be the most difficult case.

Based on extensive random sampling on $\mathbb{R}_+^n$ for small numbers n it has been conjectured that Conjecture 1.2 might be true for arbitrary $n \in \mathbb{N}$. The sum of squared logarithms inequality has immediate important applications in matrix analysis ([2]; see also [3]) as well as in nonlinear elasticity theory [4–7]. In matrix analysis it implies that the global minimiser over all rotations to

$$\inf_{Q \in \mathrm{SO}(n)} \left\| \operatorname{sym}_* \operatorname{Log} Q^T F \right\|^2 = \left\| \sqrt{F^T F} \right\|^2 \tag{1.3}$$

at given $F \in \mathrm{GL}^+(n)$ is realised by the orthogonal factor $R = \operatorname{polar}(F)$ (such that $R^T F = \sqrt{F^T F}$). Here, $\|X\|^2 := \sum_{i,j=1}^n X_{ij}^2$ denotes the Frobenius matrix norm and $\operatorname{Log} : \mathrm{GL}(n) \to \mathfrak{gl}(n) = \mathbb{R}^{n \times n}$ is the multivalued matrix logarithm, *i.e.* any solution $Z = \operatorname{Log} X \in \mathbb{C}^{n \times n}$ of $\exp(Z) = X$ and $\operatorname{sym}_*(Z) = \frac{1}{2}(Z^* + Z)$.

Recently, the case $n = 2$ was used to verify the polyconvexity condition in nonlinear elasticity [4, 5] for a certain class of isotropic energy functions. For more background information on the sum of squared logarithms inequality we refer the reader to [1].

In this paper we extend the investigation as to the validity of Conjecture 1.2 by considering arbitrary functions f instead of $f(x) = \log^2 x$. We formulate this more general problem and we are able to extend Conjecture 1.2 to the case $n = 4$. The same methods should also be useful for proving the statement for $n = 5, 6$. However, the necessary technicalities prevent us from discussing these cases in this paper.

In addition, we present ideas which might be helpful in attacking the fully general case, namely arbitrary f and arbitrary n.

2 The generalised inequality

In order to generalise Conjecture 1.2 in the directions hinted at in the introduction, we consider from now on a non-standard definition of the elementary symmetric polynomials. In fact, for $n \ge 2$ it will be more convenient for us to reverse their numbering and define $E_0, E_1, \dots, E_{n-1}$ by

$$E_k(y_1, \dots, y_n) := e_{n-k}(y_1, \dots, y_n) = \sum_{i_1 < \cdots < i_{n-k}} y_{i_1} \cdot y_{i_2} \cdot \cdots \cdot y_{i_{n-k}}, \quad k \in \{0, 1, \dots, n-1\}. \tag{2.1}$$

In particular, now

$$\begin{aligned} E_0(y_1, \dots, y_n) &:= e_n(y_1, \dots, y_n) = y_1 \cdot y_2 \cdot \cdots \cdot y_n, \\ E_{n-1}(y_1, \dots, y_n) &:= e_1(y_1, \dots, y_n) = y_1 + y_2 + \cdots + y_n. \end{aligned} \tag{2.2}$$

Let $I \subset \mathbb{R}$ be an open interval and let

$$\Delta_n := \{ y = (y_1, y_2, \dots, y_n) \in I^n \mid y_1 \le y_2 \le \cdots \le y_n \}. \tag{2.3}$$

Let S be a nonempty subset of $\{0, 1, \dots, n-1\}$ and assume that $a, b \in \Delta_n$ are such that

$$E_k(a) < E_k(b) \quad \text{for } k \in S \quad \text{and} \quad E_k(a) = E_k(b) \quad \text{for } k \in \{0, 1, \dots, n-1\} \setminus S. \tag{2.4}$$

In this section we investigate necessary and sufficient conditions for a (smooth) function $f : I \to \mathbb{R}$, such that the inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \le f(b_1) + f(b_2) + \cdots + f(b_n)$$

holds for all $a, b \in \Delta_n$ satisfying assumption (2.4).

Remark 2.1 The formulation of the above problem has a certain monotonicity structure: we assume that '$E(a) < E(b)$' and want to prove that '$F(a) < F(b)$'. Therefore our idea is to consider a curve y connecting the points a and b, such that $E(y(t))$ 'increases'. Then the function $g(t) = F(y(t))$ should also increase and therefore $g'(t) > 0$ must hold. From this we are able to derive necessary and sufficient conditions on the function f.

This approach motivates the following definition.

Definition 2.2 (*b dominates a*, $a \preceq b$) Let $a, b \in \Delta_n$. We will say that b *dominates* a and denote $a \preceq b$ if there exists a piecewise differentiable mapping $y : [0, 1] \to \Delta_n$ (*i.e.* y is continuous on $[0, 1]$ and differentiable in all but at most countably many points) such that $y(0) = a$, $y(1) = b$, $y_i(t) \ne y_j(t)$ for $i \ne j$ and all but at most countably many $t \in [0, 1]$ and the functions

$$A_k(t) := E_k(y(t)), \quad k \in \{0, 1, \dots, n-1\}$$

are nondecreasing on the interval $[0, 1]$.

If $a \preceq b$, then $E_k(a) = A_k(0) \le A_k(1) = E_k(b)$, so it follows from Definition 2.2 that a, b satisfy assumption (2.4) with S being the set of all k for which $A_k(t)$ is not a constant function on $[0, 1]$.

We are ready to formulate the main results of this section.

Theorem 2.3 *Assume that $a,b \in \Delta_n$ and let $a \preceq b$. Let $S \subseteq \{0,1,\dots,n-1\}$ denote the set of all integers k with $E_k(a) < E_k(b)$. Moreover, assume that $f \in C^n(I)$ be such that*

$$(-1)^{n+k}\left(x^k f'(x)\right)^{(n-1)} \le 0 \quad \text{for all } x \in I \text{ and all } k \in S. \tag{2.5}$$

Then the following inequality holds:

$$f(a_1)+f(a_2)+\cdots+f(a_n) \le f(b_1)+f(b_2)+\cdots+f(b_n). \tag{2.6}$$

A partially reverse statement is also true.

Theorem 2.4 *Let $f \in C^n(I)$ be such that the inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \le f(b_1)+f(b_2)+\cdots+f(b_n) \tag{2.7}$$

holds for all $a,b \in \Delta_n$ satisfying

$$E_k(a) \le E_k(b) \quad \text{for } k \in S \quad \text{and} \quad E_k(a) = E_k(b) \quad \text{for } k \in \{0,1,\dots,n-1\} \setminus S \tag{2.8}$$

for some subset $S \subseteq \{0,1,\dots,n-1\}$. Then f satisfies property (2.5), *i.e.*

$$(-1)^{n+k}\left(x^k f'(x)\right)^{(n-1)} \le 0 \quad \text{for all } x \in I \text{ and all } k \in S. \tag{2.9}$$

In this respect, we can formulate another conjecture.

Conjecture 2.5 *Let S be a nonempty subset of* $\{0,1,\dots,n-1\}$ *and assume that $a,b \in \Delta_n$ are such that* (2.4) *is satisfied, i.e.*

$$E_k(a) < E_k(b) \quad \text{for } k \in S \quad \text{and} \quad E_k(a) = E_k(b) \quad \text{for } k \in \{0,1,\dots,n-1\} \setminus S.$$

Then there exists a curve γ satisfying the conditions from Definition 2.2 *and thus $a \preceq b$.*

Remark 2.6 In concrete applications of Theorem 2.3 and Theorem 2.4 one would like to know whether condition (2.4) already implies $a \preceq b$. This is Conjecture 2.5. Unfortunately, we are able to prove Conjecture 2.5 only for $2 \le n \le 4$, $I = (0,\infty)$ and $S \subseteq \{1,2,\dots,n-1\}$ (see the next section).

Example 2.7 It is easy to see that if $I = (0,\infty)$ then the function $f(x) = \log^2 x$ satisfies property (2.5) for $S = \{1,2,\dots,n-1\}$. Indeed, we proceed by induction on n. For $n = 2$ and $k = 1$ the property is immediate. Moreover, for $k \ge 2$ and $n \ge 3$ we get

$$\begin{aligned}(-1)^{n+k}\left(x^k f'(x)\right)^{(n-1)} &= 2(-1)^{n+k}\left(x^{k-1}\log x\right)^{(n-1)} \\ &= 2(-1)^{n+k}\left((k-1)x^{k-2}\log x\right)^{(n-2)} + 2(-1)^{n+k}\left(x^{k-2}\right)^{(n-2)} \le 0 \end{aligned} \tag{2.10}$$

by the induction hypothesis, since the second summand vanishes. It remains to check property (2.5) for $k = 1$, which is also immediate.

Note also that property (2.5) is not true for $k = 0$. Therefore Theorem 2.3 and Theorem 2.4 for $f(x) = \log^2 x$ attain the following formulation.

Corollary 2.8 *Assume that $a, b \in \mathbb{R}_+^n$ be such that $a \preceq b$ and $a_1a_2\cdots a_n = b_1b_2\cdots b_n$. Then*

$$\log^2(a_1) + \log^2(a_2) + \cdots + \log^2(a_n) \le \log^2(b_1) + \log^2(b_2) + \cdots + \log^2(b_n)$$

and this inequality fails if the constraint $a_1a_2\cdots a_n = b_1b_2\cdots b_n$ is replaced by the weaker one $a_1a_2\cdots a_n \le b_1b_2\cdots b_n$.

In order to see that the weaker condition is not sufficient for the inequality to hold, consider the case

$$a = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right), \qquad b = (1, \ldots, 1).$$

Then $a \preceq b$ and $a_1a_2\cdots a_n \le b_1b_2\cdots b_n$, but

$$\log^2(a_1) + \log^2(a_2) + \cdots + \log^2(a_n) = n\log^2(n) > 0 = \log^2(b_1) + \log^2(b_2) + \cdots + \log^2(b_n).$$

Remark 2.9 Corollary 2.8 is a weaker statement than Conjecture 1.2 since we assume that $a \preceq b$. If Conjecture 2.5 is true, then Conjecture 1.2 follows.

Example 2.10 The function $f(x) = x^p$ $(x > 0)$ with $p \in (0,1)$ satisfies property (2.5) for the set $S = \{0, 1, \ldots, n-1\}$. Indeed, for each $n \ge 2$ and $0 \le k \le n-1$, we have

$$(-1)^{n+k}\bigl(x^k f'(x)\bigr)^{(n-1)} = (-1)^{n+k} p(k+p-1)(k+p-2)\cdots\bigl(k+p-(n-1)\bigr)x^{k+p-n}.$$

The above product is not greater than 0, because among the factors $k+p-1, k+p-2, \ldots, k+p-(n-1)$ there are exactly $n-1-k$ negative ones.

Similarly, the function $f(x) = x^p$ for $p \in (-1, 0)$ satisfies property (2.5) for the set $S = \{1, 2, \ldots, n-1\}$, because $p < 0$ and among the factors $k+p-1, k+p-2, \ldots, k+p-(n-1)$ there are exactly $n-k$ negative ones. On the other hand, property (2.5) is not true for $k = 0$.

Thus, like above, we have the following.

Corollary 2.11 *Assume that $a, b \in (0, \infty)^n$ be such that $a \preceq b$ and $a_1a_2\cdots a_n = b_1b_2\cdots b_n$. If $p \in (-1, 1)$, then*

$$a_1^p + a_2^p + \cdots + a_n^p \le b_1^p + b_2^p + \cdots + b_n^p.$$

This inequality fails for $-1 < p < 0$ (but remains true for $0 < p < 1$) if the constraint $a_1a_2\cdots a_n = b_1b_2\cdots b_n$ is replaced by the weaker one $a_1a_2\cdots a_n \le b_1b_2\cdots b_n$.

Proof of Theorem 2.3 If S is empty, then $E_k(a) = E_k(b)$ for all $k \in \{0, 1, \ldots, n-1\}$ and hence $a = b$, which immediately implies the inequality. We therefore assume that S is nonempty.

Let $y : [0,1] \to \Delta_n$ be the curve connecting points a and b as in Definition 2.2. Consider the function

$$\begin{aligned} p(t,x) &= \bigl(x + y_1(t)\bigr)\bigl(x + y_2(t)\bigr)\cdots\bigl(x + y_n(t)\bigr) = \sum_{k=0}^{n-1} x^k E_k\bigl(y(t)\bigr) + x^n \\ &= (x + a_1)(x + a_2)\cdots(x + a_n) + \sum_{k \in S} x^k A_k(t), \end{aligned} \tag{2.11}$$

where $A_k(t) = E_k(y(t)) - E_k(a)$ is a nondecreasing mapping. Our goal is to show that the function

$$\eta(t) = \sum_{i=1}^{n} f\bigl(y_i(t)\bigr) \tag{2.12}$$

is nondecreasing on $[0,1]$, *i.e.* we show that $\eta'(t) \geq 0$ a.e. on $(0,1)$.

To this end, fix $i \in \{1,2,\dots,n\}$. Since $p(t,-y_i(t)) = 0$ for all $t \in (0,1)$, we obtain

$$\partial_1 p\bigl(t,-y_i(t)\bigr) + \partial_2 p\bigl(t,-y_i(t)\bigr) \cdot \bigl(-y_i'(t)\bigr) = 0$$

for all $t \in (0,1)$ and therefore

$$\sum_{k\in S}\bigl(-y_i(t)\bigr)^k A_k'(t) + \prod_{j\neq i}\bigl(y_j(t) - y_i(t)\bigr) \cdot \bigl(-y_i'(t)\bigr) = 0, \tag{2.13}$$

which gives

$$y_i'(t) = \sum_{k\in S}\bigl(-y_i(t)\bigr)^k A_k'(t)\biggl(\prod_{j\neq i}\bigl(y_j(t) - y_i(t)\bigr)\biggr)^{-1}.$$

This equality holds, if $y_i(t) \neq y_j(t)$ for $i \neq j$, which is true for all but countably many values of $t \in (0,1)$. For those values of t we get

$$\begin{aligned}
\eta'(t) &= \sum_{i=1}^{n} f'\bigl(y_i(t)\bigr) \cdot y_i'(t) \\
&= \sum_{i=1}^{n} f'\bigl(y_i(t)\bigr) \cdot \sum_{k\in S}\bigl(-y_i(t)\bigr)^k A_k'(t)\biggl(\prod_{j\neq i}\bigl(y_j(t) - y_i(t)\bigr)\biggr)^{-1} \\
&= \sum_{k\in S} A_k'(t) \sum_{i=1}^{n} f'\bigl(y_i(t)\bigr) \cdot \bigl(-y_i(t)\bigr)^k \biggl(\prod_{j\neq i}\bigl(y_j(t) - y_i(t)\bigr)\biggr)^{-1}.
\end{aligned} \tag{2.14}$$

Fix $t \in (0,1)$ such that $y_i(t) \neq y_j(t)$ for $i \neq j$ and write $y_i = y_i(t)$ for simplicity. Since $A_k'(t) \geq 0$, we will be done if we show that

$$\widehat{D} := \sum_{i=1}^{n} f'(y_i) \cdot (-y_i)^k \biggl(\prod_{j\neq i}(y_j - y_i)\biggr)^{-1} \geq 0 \quad \text{for all } k \in S.$$

To this end, consider the polynomial

$$g(x) = \sum_{i=1}^{n} f'(y_i) \cdot (-y_i)^k \biggl(\prod_{j\neq i}(y_j - y_i)\biggr)^{-1} \cdot \prod_{j\neq i}(x - y_j).$$

The degree of g equals $n-1$ and the coefficient at x^{n-1} is equal to $\widehat{D}$. Moreover,

$$g(y_i) = f'(y_i) \cdot (-y_i)^k \cdot (-1)^{n-1} \quad (i = 1,2,\dots,n).$$

Therefore the function $h(x) = g(x) + (-1)^{n+k}x^k f'(x)$ has n different roots $y_1, y_2, \dots, y_n$ in the interval I. It follows that the function

$$h^{(n-1)}(x) = (n-1)!\widehat{D} + (-1)^{n+k}\bigl(x^k f'(x)\bigr)^{(n-1)} \tag{2.15}$$

has a root in the interval I, and since $(-1)^{n+k}(x^k f'(x))^{(n-1)} \leq 0$ for all $x \in I$, it follows that $\widehat{D} \geq 0$, which completes the proof of Theorem 2.3. □

Proof of Theorem 2.4 Suppose, to the contrary, that $(-1)^{k+n}(x^k f'(x))^{(n-1)} > 0$ for some $x \in I$ and some $k \in S$. Then $(-1)^{k+n}(x^k f'(x))^{(n-1)} > 0$ holds for all x belonging to some interval J contained in I. Choose the numbers $a_1 < a_2 < \cdots < a_n$ from J and consider

$$p(t,x) = (x+a_1)\cdot(x+a_2)\cdot\cdots\cdot(x+a_n) + tx^k.$$

Then for all sufficiently small t ($0 < t < \varepsilon$), there exist different numbers $y_i(t)$ belonging to J, such that

$$p(t,x) = \bigl(x+y_1(t)\bigr)\bigl(x+y_2(t)\bigr)\cdots\bigl(x+y_n(t)\bigr).$$

Then

$$x^n + \sum_{i=0}^{n-1} E_i(a)\cdot x^i + tx^k = p(t,x) = x^n + \sum_{i=0}^{n-1} E_i\bigl(y(t)\bigr)\cdot x^i,$$

and since $t > 0$, we see that a and $b = y(t)$ satisfy (2.8). We will be done if we show that

$$f(a_1) + f(a_2) + \cdots + f(a_n) > f\bigl(y_1(t)\bigr) + f\bigl(y_2(t)\bigr) + \cdots + f\bigl(y_n(t)\bigr).$$

We proceed in the same way as in the proof of Theorem 2.3. We define

$$\eta(t) = \sum_{i=1}^{n} f\bigl(y_i(t)\bigr) \quad \text{for } 0 < t < \varepsilon$$

and this time we want to show that $\eta'(t) < 0$ for $0 < t < \varepsilon$.

By the inverse mapping theorem (see the proof of Proposition 3.4 below for a more detailed explanation), $y \in C^1(0,\varepsilon)$ and therefore

$$\eta'(t) = \sum_{i=1}^{n} f'\bigl(y_i(t)\bigr)\cdot y_i'(t) = \sum_{i=1}^{n} f'\bigl(y_i(t)\bigr)\cdot\bigl(-y_i(t)\bigr)^k\Bigl(\prod_{j\neq i}\bigl(y_j(t)-y_i(t)\bigr)\Bigr)^{-1}. \tag{2.16}$$

Now, like previously, write $y_i = y_i(t)$ for simplicity. Our goal is therefore to prove that

$$\widehat{D} := \sum_{i=1}^{n} f'(y_i)\cdot(-y_i)^k\Bigl(\prod_{j\neq i}(y_j-y_i)\Bigr)^{-1} < 0.$$

Consider the polynomial

$$g(x) = \sum_{i=1}^{n} f'(y_i)\cdot(-y_i)^k\Bigl(\prod_{j\neq i}(y_j-y_i)\Bigr)^{-1}\cdot\prod_{j\neq i}(x-y_j).$$

The degree of g equals $n-1$ and the coefficient at x^{n-1} is equal to $\widehat{D}$. Moreover, the function $h(x) = g(x) + (-1)^{n+k}x^k f'(x)$ has n different roots $y_1, y_2, \dots, y_n$ in the interval J. It follows that the function

$$h^{(n-1)}(x) = (n-1)!\widehat{D} + (-1)^{n+k}\bigl(x^k f'(x)\bigr)^{(n-1)}$$

has a root in the interval J. Since $(-1)^{n+k}(x^k f'(x))^{(n-1)} > 0$ for all $x \in J$, it follows that $\widehat{D} < 0$, which completes the proof of Theorem 2.4. □

3 Construction of the connecting curve

In this section we prove that condition (2.4) implies $a \preceq b$, if $2 \leq n \leq 4$, $I = (0, \infty)$ and $S \subseteq \{1, 2, \dots, n-1\}$. However, we start with a construction of the desired curve for a general interval I, integer $n \geq 2$ and set $S \subseteq \{0, 1, \dots, n-1\}$.

For $a, b \in \Delta_n$, we say that $a < b$, if $a \neq b$ and $E_k(a) \leq E_k(b)$ for all $k = 0, 1, \dots, n-1$. We say that $a \leq b$, if $a < b$ or $a = b$.

Definition 3.1 For $a < b$ denote by $\mathcal{C}(a, b)$ the set of all piecewise differentiable (*i.e.* continuous and differentiable in all but at most countably many points) curves y in Δ_n satisfying:

(a) the curve $y(t)$ starts at a (*i.e.* $y(0) = a$, if the curve $y(t)$ is parametrised by the interval $[0, \varepsilon]$);

(b) $y(t) \in \operatorname{int}(\Delta_n)$ for all but at most countable many values t;

(c) the mappings $E_k(y(t))$ are nondecreasing in t and $E_k(y(t)) \leq E_k(b)$ for all t and each $k = 0, 1, \dots, n-1$.

Note that a curve in $\mathcal{C}(a, b)$ does not necessarily end at the point b.

Proposition 3.2 *Let $n \geq 2$ be a positive integer and let S be a nonempty subset of $\{0, 1, \dots, n-1\}$. Let, moreover, $a, b \in \Delta_n$ be such that* (2.4) *holds. Furthermore, suppose that for all $c \in \Delta_n$ with $a \leq c < b$ the set $\mathcal{C}(c, b)$ is nonempty. Then $a \preceq b$.*

Proof Each element (curve) of $\mathcal{C}(a, b)$ is a (closed) subset of Δ_n. We equip the set $\mathcal{C}(a, b)$ with the inclusion relation $\subseteq$, obtaining a nonempty partially ordered set $(\mathcal{C}(a, b), \subseteq)$. We are going to show that each chain $\{y_i\}_{i \in \mathcal{I}}$ has an upper bound in $\mathcal{C}(a, b)$.

To achieve this, consider the curve

$$y_0 = \overline{\bigcup_{i \in \mathcal{I}} y_i},$$

i.e. the concatenation of the curves y_i. Then obviously y_0 satisfies conditions (a) and (c) of Definition 3.1. To prove (b) assume that y_0 is parametrised on $[0, 1]$. Then for each positive integer k the curve y_k, defined as the restriction of y_0 to the interval $[0, 1 - \frac{1}{k}]$, is contained in some curve $y_i \in \mathcal{C}(a, b)$ of the given chain $\{y_i\}$. Therefore $y_k(t)$ is piecewise differentiable and satisfies condition (b) for each positive integer k. Moreover,

$$y_0 = \overline{\bigcup_{k=1}^{\infty} y_k}.$$

Hence y_0 is piecewise differentiable and satisfies (b) as well.

Now, by the Kuratowski-Zorn lemma, there exists a maximal element y in $(\mathcal{C}(a,b),\subseteq)$. We show that y is a desired curve connecting the points a and b, which will imply that $a \preceq b$.

To this end, it is enough to show that, if the curve y is parametrised on $[0,1]$, then $y(1) = b$. Suppose, to the contrary, that $y(1) = c \neq b$. Then $a \preceq c < b$, and hence the set $\mathcal{C}(c,b)$ is nonempty. Thus the curve y can be extended beyond the point c, which contradicts the fact that y is a maximal element in $\mathcal{C}(a,b)$. This completes the proof of Proposition 3.2. □

From now on assume that $I = (0,\infty)$ and S is a nonempty subset of $\{1,2,\dots,n-1\}$.

In order to prove that (2.4) implies $a \preceq b$, it suffices to show that the sets $\mathcal{C}(a,b)$ for $a,b \in \Delta_n$ with $a < b$ are nonempty. This is implied by the following conjecture, which we will prove later for $n \le 4$.

Conjecture 3.3 *Let $n \ge 2$ be an integer and $a \in \Delta_n$. Let S be a nonempty subset of $\{1,2,\dots,n-1\}$ with the property that there exist $A_k > 0$ for $k \in S$ such that all the roots of the polynomial*

$$q(x) = (x+a_1)(x+a_2)\cdots(x+a_n) + \sum_{k\in S} A_k x^k$$

are real (and hence negative). Then there exist mappings $B_k : [0,\varepsilon] \to \mathbb{R}$ $(k \in S)$ continuous on $[0,\varepsilon]$, differentiable on $(0,\varepsilon)$, and nondecreasing with $B_k(0) = 0$ such that $\sum_{k\in S} B_k(t)$ is increasing on $[0,\varepsilon]$ and for all sufficiently small values of $t > 0$ the polynomial

$$(x+a_1)(x+a_2)\cdots(x+a_n) + \sum_{k\in S} B_k(t) x^k$$

has n distinct real (and hence negative) roots.

Now we show how Conjecture 3.3 implies that the sets $\mathcal{C}(a,b)$ are nonempty.

Proposition 3.4 *Let n and S be such that the conjecture holds. Let, moreover, $a,b \in \Delta_n$ be such that (2.4) holds. Then the set $\mathcal{C}(a,b)$ is nonempty.*

Proof Consider the polynomials

$$p(x) = (x+a_1)(x+a_2)\cdots(x+a_n) \quad \text{and} \quad q(x) = (x+b_1)(x+b_2)\cdots(x+b_n).$$

Then

$$q(x) - p(x) = \sum_{k=0}^{n-1} \bigl(E_k(b) - E_k(a)\bigr) x^k = \sum_{k\in S} A_k x^k,$$

where $A_k > 0$ for all $k \in S$. According to the conjecture, there exist nondecreasing mappings $B_k : [0,\varepsilon] \to \mathbb{R}$, continuous on $[0,\varepsilon]$ and differentiable on $(0,\varepsilon)$, with $B_k(0) = 0$, such that $\sum_{k\in S} B_k(t)$ is increasing on $[0,\varepsilon]$ and for all $t \in (0,\varepsilon)$ the polynomial

$$p(x) + \sum_{k\in S} B_k(t) x^k$$

has n distinct real (and hence negative) roots $-y_n(t) < -y_{n-1}(t) < \cdots < -y_1(t) < 0$. We show that $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))$ defines a differentiable curve (parametrised on $[0,\varepsilon]$) that belongs to $\mathcal{C}(a,b)$, provided ε is chosen in such a way that $B_k(\varepsilon) \le A_k$ for $k \in S$.

Consider the mapping $\Psi : \overline{\Delta_n} \to \Psi(\overline{\Delta_n})$ given by

$$\Psi(y) = \bigl(E_{n-1}(y), E_{n-2}(y), \ldots, E_0(y)\bigr).$$

Then it follows from Remark 1.3 that the mapping Ψ is injective, hence Ψ is a continuous bijection defined on a closed subset of $\mathbb{R}^n$. Therefore the restriction $\Psi|_U$ of Ψ to a neighbourhood U of a is continuously invertible and thus

$$y(t) = \Psi^{-1}\bigl(\Psi(a) + \bigl(B_0(t), B_1(t), \ldots, B_{n-1}(t)\bigr)\bigr) \quad \bigl(t \in [0,\varepsilon]\bigr)$$

(here we put $B_k(t) = 0$ for $k \notin S$) is a curve starting at a; note that $\Psi(a) + (B_0(t), B_1(t), \ldots, B_{n-1}(t))$ is contained in $\Psi(U)$ for sufficiently small ε. Moreover $y(t) \in \Delta_n$. Hence condition (a) is satisfied. Since $y(t) \in \operatorname{int}(\Delta_n)$ for all $t \in (0,\varepsilon)$, condition (b) holds. It is also clear that (c) is satisfied, since $E_k(y(t)) = E_k(a) + B_k(t) \le E_k(a) + A_k = E_k(b)$ for all $k \in \{0, 1, \ldots, n-1\}$.

It remains to prove that $y(t)$ is differentiable on $(0,\varepsilon)$. This, however, is a consequence of the inverse mapping theorem, if we show that

$$\det\bigl[D\Psi(y)\bigr] \neq 0 \quad \text{for all } y \in \operatorname{int}(\Delta_n).$$

To this end, let $V(y)$ be the $n \times n$ Vandermonde-type matrix given by $V_{ij}(y) = (-y_i)^{n-j}$ $(1 \le i,j \le n)$. This matrix is obtained from the standard Vandermonde matrix

$$W(-y_1, -y_2, \ldots, -y_n) = \begin{pmatrix} 1 & -y_1 & (-y_1)^2 & \cdots & (-y_1)^{n-1} \\ 1 & -y_2 & (-y_2)^2 & \cdots & (-y_2)^{n-1} \\ 1 & -y_3 & (-y_3)^2 & \cdots & (-y_3)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -y_n & (-y_n)^2 & \cdots & (-y_n)^{n-1} \end{pmatrix} \tag{3.1}$$

by reversing the order of columns of W.

Since [8]

$$\bigl(D\Psi(y)\bigr)_{jk} = \frac{\partial}{\partial y_k} E_{n-j}(y) = \begin{cases} 1, & j = 1, \\ E_{n-j}(y^{(k)}), & j > 1, \end{cases}$$

where $y^{(k)} = (y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)$ is y with its kth component removed, it follows from the general formula

$$t^{n-1} + \sum_{j=0}^{n-2} t^j E_j(z_1, z_2, \ldots, z_{n-1}) = (t + z_1)(t + z_2) \cdots (t + z_{n-1}) \tag{3.2}$$

that

$$\begin{aligned}
\big(V(y)\cdot D\Psi(y)\big)_{ik} &= \sum_{j=1}^{n}\big(V(y)\big)_{ij}\cdot\big(D\Psi(y)\big)_{jk}\\
&= (-y_i)^{n-1} + \sum_{j=2}^{n}(-y_i)^{n-j}\cdot E_{n-j}\big(y^{(k)}\big)\\
&= (-y_i)^{n-1} + \sum_{j=0}^{n-2}(-y_i)^{j}\cdot E_{j}\big(y^{(k)}\big) = \prod_{j\neq k}(y_j - y_i)
\end{aligned}$$

and thus

$$V(y)\cdot D\Psi(y) = \operatorname{diag}\Bigl(\prod_{j\neq 1}(y_j - y_1), \prod_{j\neq 2}(y_j - y_2), \ldots, \prod_{j\neq n}(y_j - y_n)\Bigr). \tag{3.3}$$

It is well known that

$$\det\bigl[V(y)\bigr] = \prod_{i<j}(y_j - y_i) \neq 0 \quad (y \in \operatorname{int}\Delta_n).$$

Therefore we obtain

$$\det\bigl[D\Psi(y)\bigr] = \prod_{i<j}(y_i - y_j) \neq 0 \quad (y \in \operatorname{int}\Delta_n),$$

which completes the proof of Proposition 3.4. □

Lemma 3.5 *Assume that* $n \geq 3$ *is odd and let* $0 < a_1 \leq a_2 \leq \cdots \leq a_n$. *Let, moreover,* $A_k \geq 0$ *for* $k = 1, 2, \ldots, (n-1)/2$ *with at least one* A_k *not equal to* 0. *Consider the polynomials*

$$\begin{aligned}
P(x) &= (x + a_1)(x + a_2)\cdots(x + a_n) + \sum_{k=1}^{(n-1)/2} A_k x^{2k-1},\\
Q(x) &= (x + a_1)(x + a_2)\cdots(x + a_n) + \sum_{k=1}^{(n-1)/2} A_k x^{2k}.
\end{aligned} \tag{3.4}$$

Then the polynomial P *has exactly one root in the interval* $(-a_1, 0)$ *and at most two roots in the interval* $(-a_n, -a_{n-1})$. *Moreover, the polynomial* Q *has exactly one root in the interval* $(-\infty, -a_n)$ *and at most two roots in the interval* $(-a_2, -a_1)$.

Proof That P has exactly one root in $(-a_1, 0)$ follows immediately from the observation that $P(-a_1) < 0$, $P(0) > 0$ and $P'(x) > 0$ on $(-a_1, 0)$.

Now we show that Q has exactly one root in $(-\infty, -a_n)$.

Dividing the equation $Q(x) = 0$ by $x^n a_1 a_2 \cdots a_n$ and substituting $z = 1/x$ and $b_i = 1/a_i$ yield the equation $P_0(z) = 0$, where

$$P_0(z) = (z + b_1)(z + b_2)\cdots(z + b_n) + \sum_{k=1}^{(n-1)/2} B_k z^{2k-1}$$

for some nonnegative numbers B_k, not all equal to 0. We already know that P_0 has exactly one root in the interval $(-b_n, 0)$, so it follows that Q has exactly one root in the interval $(-\infty, -a_n)$.

Now we prove that Q has at most two roots in the interval $(-a_2, -a_1)$. To the contrary, suppose that Q has at least three roots in $(-a_2, -a_1)$. Since $Q(-a_2) > 0$ and $Q(-a_1) > 0$, it follows that Q has an even number, and hence at least four, roots in the interval $(-a_2, -a_1)$.

Let $0 > -c_1 \geq -c_2 \geq \cdots \geq -c_{n-1}$ be the roots of $p'(x) = 0$, where

$$p(x) = (x + a_1)(x + a_2)\cdots(x + a_n). \tag{3.5}$$

Then $a_1 < c_1 < a_2$. The polynomial $Q(x)$ is decreasing on the interval $[-a_2, -c_1]$, so it has at most one root in this interval. Therefore the polynomial Q has at least three roots in the interval $(-c_1, -a_1)$, and consequently the equation $Q''(x) = 0$ has a root in $(-c_1, -a_1)$. But $Q''(x) > 0$ for all $x > -c_1$, a contradiction. Hence Q must have at most two roots in $(-a_2, -a_1)$.

Finally, to prove that P has at most two roots in the interval $(-a_n, -a_{n-1})$, divide the equation $P(x) = 0$ by $x^n a_1 a_2 \cdots a_n$ and substitute $z = 1/x$ and $b_i = 1/a_i$. This reduces to the equation $Q_0(z) = 0$, where

$$Q_0(z) = (z + b_1)(z + b_2)\cdots(z + b_n) + \sum_{k=1}^{(n-1)/2} B_k z^{2k}$$

for some nonnegative numbers B_k, not all equal to 0. We already know that Q_0 has at most two roots in the interval $(-b_{n-1}, -b_n)$, so it follows that P has at most two roots in the interval $(-a_n, -a_{n-1})$. This completes the proof of Lemma 3.5. □

The same proof yields an analogous result for even values of n.

Lemma 3.6 *Assume that* $n \geq 2$ *is even and let* $0 < a_1 \leq a_2 \leq \cdots \leq a_n$. *Let, moreover,* $A_k \geq 0$ *for* $k = 1, 2, \ldots, n/2$ *and not all of the* A_k*'s are equal to* 0. *Consider the polynomials*

$$\begin{aligned} P(x) &= (x + a_1)(x + a_2)\cdots(x + a_n) + \sum_{k=1}^{n/2} A_k x^{2k-1}, \\ Q(x) &= (x + a_1)(x + a_2)\cdots(x + a_n) + \sum_{k=1}^{n/2-1} A_k x^{2k}. \end{aligned} \tag{3.6}$$

Then the polynomial P *has exactly one root in each of the intervals* $(-\infty, -a_n)$ *and* $(-a_1, 0)$ *and* Q *has at most two roots in each of the intervals* $(-a_n, -a_{n-1})$ *and* $(-a_2, -a_1)$.

Proof The same proof as that for Lemma 3.5 can be used. □

Now we turn to the proof of Conjecture 3.3 for $2 \leq n \leq 4$ and an arbitrary nonempty set $S \subseteq \{1, 2, \ldots, n-1\}$.

We first make some useful general remarks.

Let $I(a) = \{i \in \{1, 2, \dots, n-1\} : a_i = a_{i+1}\}$. If $I(a)$ is empty, then the conjecture holds. Indeed, if $k \in S$, then all the roots of the polynomial

$$(x + a_1)(x + a_2) \cdots (x + a_k) + tx^k$$

are, for all sufficiently small $t > 0$, real and distinct.

On the other hand, if $I(a) = \{1, 2, \dots, n-1\}$, then only the set $S = \{1, 2, \dots, n-1\}$ possibly satisfies the assumptions of the conjecture. Indeed, suppose that $l \notin S$ and let $-b_1 \geq -b_2 \geq \cdots \geq -b_n$ be the roots of

$$q(x) = (x + a_1)^n + \sum_{k \in S} A_k x^k.$$

Then by the inequality of arithmetic and geometric means, we obtain

$$\frac{E_l(a)}{\binom{n}{l}} = \frac{E_l(b)}{\binom{n}{l}} \geq \bigl(E_0(b)\bigr)^{(n-l)/n} = \bigl(E_0(a)\bigr)^{(n-l)/n} = \frac{E_l(a)}{\binom{n}{l}}, \tag{3.7}$$

and hence $b_1 = b_2 = \cdots = b_n$. Since $E_0(a) = E_0(b)$, it follows that $a = b$, *i.e.* $A_k = 0$ for all $k \in S$. A contradiction.

Let I be a nonempty subset of $\{1, 2, \dots, n-1\}$. We observe that the conjecture is true for a set S and all $a \in \Delta_n$ with $I(a) = I$, if it is true for a set $T = \{n - k : k \in S\}$ and all $b \in \Delta_n$ with $I(b) = \{n - i : i \in I\}$. Indeed, if all the roots of the polynomial

$$q(x) = (x + a_1)(x + a_2) \cdots (x + a_n) + \sum_{k \in S} A_k x^k$$

are real, then substituting $x = 1/z$ and $a_i = 1/b_i$, we infer that all the roots of the polynomial

$$r(z) = (z + b_1)(z + b_2) \cdots (z + b_n) + \sum_{l \in T} B_l z^l$$

are real. Hence there exist mappings $C_l(t)$ with $C_l(0) = 0$, continuous on $[0, \varepsilon]$, differentiable on $(0, \varepsilon)$ and nondecreasing such that the polynomial

$$(z + b_1)(z + b_2) \cdots (z + b_n) + \sum_{l \in T} C_l(t) z^l$$

has n distinct real roots. Substituting $z = 1/x$ and $b_i = 1/a_i$, we infer that the polynomial

$$(x + a_1)(x + a_2) \cdots (x + a_n) + \sum_{k \in S} C_{n-k}(t) x^k$$

has n distinct real roots.

For $n = 2$ the only possibility for the set S is $\{1\}$ and it is enough to notice that the polynomial $(x + a_1)(x + a_2) + tx$ has two distinct real roots for any $t > 0$.

Assume now $n = 3$. Then, in view of the above remarks, we have to consider two cases: (1) $a_1 < a_2 = a_3$; (2) $a_1 = a_2 = a_3$.

(1) If $2 \notin S$, then the condition of Conjecture 3.3 cannot be satisfied since for $A_1 > 0$, according Lemma 3.5, the polynomial

$$P(x) = (x + a_1)(x + a_2)^2 + A_1 x$$

has only one real root in the interval $(-a_1, 0)$ and obviously no roots on $\mathbb{R} \setminus (-a_1, 0)$. Thus P has only one real root for all $A_1 > 0$. We can therefore assume $2 \in S$, and for all sufficiently small $t > 0$, the polynomial

$$(x + a_1)(x + a_2)^2 + tx^2$$

has three distinct real roots.

(2) According to the above remarks, $S = \{1, 2\}$. Then the polynomial $(x + a_1)^3 + ta_1 x + tx^2$ has three distinct real roots for all sufficiently small $t > 0$.

Assume $n = 4$. In this case we have five possibilities: (1) $a_1 = a_2 < a_3 < a_4$; (2) $a_1 < a_2 = a_3 < a_4$; (3) $a_1 < a_2 = a_3 = a_4$; (4) $a_1 = a_2 < a_3 = a_4$; (5) $a_1 = a_2 = a_3 = a_4$.

(1) We note that $S \neq \{2\}$, since, by Lemma 3.6, the polynomial

$$Q(x) = (x + a_1)^2(x + a_3)(x + a_4) + A_2 x^2 \quad \text{for } A_2 > 0$$

has at most two real roots in the interval $(-a_4, -a_3)$ and obviously no roots on $\mathbb{R} \setminus (-a_4, -a_3)$. Thus Q has at most two real roots. Therefore S contains an odd integer k. Then for all sufficiently small $t > 0$, the polynomial $(x + a_1)^2(x + a_3)(x + a_4) + tx^k$ has four distinct real roots.

(2) Note that $2 \in S$, since by Lemma 3.6, the polynomial

$$(x + a_1)(x + a_2)^2(x + a_4) + A_1 x + A_3 x^3 \quad \text{for } A_1, A_3 > 0$$

has at most two real roots. Then for all sufficiently small $t > 0$, the polynomial

$$(x + a_1)(x + a_2)^2(x + a_4) + tx^2$$

has four distinct real roots.

(3) We observe that $\{1, 2\} \subset S$ or $\{2, 3\} \subset S$, since by Lemma 3.6, each of the polynomials

$$(x + a_1)(x + a_2)^3 + A_1 x + A_3 x^3 \quad \text{and} \quad (x + a_1)(x + a_2)^3 + A_2 x^2 \quad \text{for } A_1, A_2, A_3 > 0$$

as well as

$$(x + a_1)(x + a_2)^3 + A_1 x \quad \text{and} \quad (x + a_1)(x + a_2)^3 + A_3 x^3 \quad \text{for } A_1, A_3 > 0$$

has at most two real roots. Moreover, we prove that $S \neq \{1, 2\}$.

Suppose that the polynomial $Q(x) = (x + a_1)(x + a_2)^3 + A_1 x + A_2 x^2$ has four real roots. Let $Q_1(x) = (x + a_1)(x + a_2)^3$ and $Q_2(x) = A_1 x + A_2 x^2$. Let $-c \neq a_2$ be the root of the polynomial $Q_1'(x)$ and let $-d$ be the root of $Q_2'(x)$.

If $d < c$, then Q is decreasing on $(-\infty, -c]$, so Q has at most one root in this interval. Therefore Q has at least three roots in the interval $(-c, 0)$. Thus $Q''(x)$ has a root in the interval $(-c, 0)$, which is impossible, since $Q''(x) > 0$ on $(-c, 0)$.

If $a_2 \geq d \geq c$, then Q is increasing on the interval $[-c, 0)$ and decreasing on the interval $(-\infty, -d]$, so Q must have at least two roots in the interval $(-d, -c)$. But $Q(x) < 0$ on this interval.

Finally, if $d > a_2$, then Q may only have roots in the union $(-\infty, a_2) \cup (-a_1, 0)$. But Q is increasing on $(-a_1, 0)$, so Q has three roots in $(-\infty, a_2)$. This, however, is impossible, since $Q''(x) > 0$ for $x \in (-\infty, a_2)$. Thus $\{2, 3\} \subseteq S$ and the polynomial

$$(x + a_1)(x + a_2)^3 + tx^2(x + a_2)$$

has, for all sufficiently small $t > 0$, four distinct roots.

(4) Since the polynomial $(x + a_1)^2(x + a_3)^2 + A_2x^2$ has no real roots, $1 \in S$ or $3 \in S$. Then the polynomial $(x + a_1)^2(x + a_3)^2 + tx^k$ for $k = 1, 3$ has, for all sufficiently small $t > 0$, four distinct real roots.

(5) In view of the above remarks, $S = \{1, 2, 3\}$. Consider

$$r(x) = (x + a_1)^4 + tx^3 + 2ta_1x^2 + t\left(a_1^2 - t^2\right)x = (x + a_1)^4 + tx\left((x + a_1)^2 - t^2\right).$$

Then for all sufficiently small $t > 0$, $a_1^2 - t^2 > 0$, and the polynomial r has four distinct real roots, because

$$r(-a_1 - 2t) = t^3(10t - 3a_1) < 0, \qquad r(-a_1) = a_1t^3 > 0 \quad \text{and}$$
$$r(-a_1 + 2t) = t^3(22t - 3a_1) < 0.$$

Thus we have proved the following.

Corollary 3.7 *Conjecture* 3.3 *is true if* $2 \leq n \leq 4$ *and* S *is an arbitrary nonempty subset of* $\{1, 2, \ldots, n - 1\}$.

This implies that the sum of squared logarithms inequality (Conjecture 1.2) holds also for $n = 4$.

Corollary 3.8 (Sum of squared logarithms inequality for $n = 4$) *Let* $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 > 0$ *be given positive numbers such that*

$$a_1 + a_2 + a_3 + a_4 \leq b_1 + b_2 + b_3 + b_4,$$
$$a_1a_2 + a_1a_3 + a_2a_3 + a_1a_4 + a_2a_4 + a_3a_4 \leq b_1b_2 + b_1b_3 + b_2b_3 + b_1b_4 + b_2b_4 + b_3b_4,$$
$$a_1a_2a_3 + a_1a_2a_4 + a_2a_3a_4 + a_1a_3a_4 \leq b_1b_2b_3 + b_1b_2b_4 + b_2b_3b_4 + b_1b_3b_4,$$
$$a_1a_2a_3a_4 = b_1b_2b_3b_4.$$

Then

$$\log^2 a_1 + \log^2 a_2 + \log^2 a_3 + \log^2 a_4 \leq \log^2 b_1 + \log^2 b_2 + \log^2 b_3 + \log^2 b_4.$$

Proof Use Corollary 3.7 and observe that S may be an arbitrary subset of $\{1, 2, 3\}$. □

Corollary 3.9 *Let $n \geq 2$ be an integer and let T be an arbitrary subset of $\{1, 2, \dots, n-1\}$. Assume that the Conjecture 3.3 holds for n and for any nonempty subset S of T. Let, moreover, $f \in C^n(0, \infty)$. Then the inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \leq f(b_1) + f(b_2) + \cdots + f(b_n)$$

holds for all $a, b \in \Delta_n$ satisfying

$$E_k(a) \leq E_k(b) \quad \text{for } k \in T \quad \text{and} \quad E_k(a) = E_k(b) \quad \text{for } k = 0 \text{ or } k \notin T \tag{3.8}$$

if and only if

$$(-1)^{n+k}\left(x^k f'(x)\right)^{(n-1)} \leq 0 \quad \text{for all } x > 0 \text{ and all } k \in T. \tag{3.9}$$

Proof Assume first (3.9) holds and let $a, b \in \Delta_n$ satisfy (3.8). Consider any $c \in \Delta_n$ with $a \leq c < b$. Then the pair c, b satisfies condition (2.4) for some nonempty subset S of T. Therefore by Proposition 3.4, the set $\mathcal{C}(c, b)$ is nonempty and hence by Proposition 3.2, $a \preceq b$. Now Theorem 2.3 implies that inequality (2.6) holds.

Conversely, if (2.6) holds for all $a, b \in \Delta_n$ satisfying (3.8), then (2.6) also holds for all $a, b \in \Delta_n$ satisfying condition (2.4) with $S = T$. Thus Theorem 2.4 implies (3.9). This completes the proof. □

4 Outlook

Our result generalises and extends previous results on the sum of squared logarithms inequality. Indeed, compared to the proof in [1] our development here views the problem from a different angle in that it is not the logarithm function that defines the problem, but a certain monotonicity property in the geometry of polynomials, explicitly stated in Conjecture 3.3.

If one tries to adopt the above proof of Conjecture 3.3 for $n \leq 4$ to the case $n \geq 5$, one has to deal with approximately 2^n cases considered separately. Therefore it is clear that the extension to natural numbers n beyond $n = 6$, say, is out of reach with such a method. Instead, a general argument should be found to prove or disprove Conjecture 3.3 for general n. Furthermore, it might be worthwhile to develop a better understanding of the differential inequality condition $(-1)^{n+k}(x^k f'(x))^{(n-1)} \leq 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed fully to all parts of this paper. Both authors read and approved the final manuscript.

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