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On some differential inequalities in the unit disk with applications

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Abstract

In this paper we obtain a number of interesting relations associated with some differential inequalities in the open unit disk, $\mathbb{U} = \{z : |z| < 1\}$. Some applications of the main results are also obtained.

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1 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $\mathbb{U} = \{z : |z| < 1\}$. Also, we denote by K the class of functions $f(z) \in A$ that are convex in $\mathbb{U}$.

A function $f(z)$ in the class A is said to be in the class $S^*(\alpha)$ of starlike functions of order α $(0 \le \alpha < 1)$ if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}) \tag{1.2}$$

for some α $(0 \le \alpha < 1)$. Also, we write $S(0) = S^*$, the class of starlike functions in $\mathbb{U}$.

A function $f(z) \in A$ is in S^λ $(|\lambda| < \frac{\pi}{2})$, the class of λ-spiral-like functions, if it satisfies

$$\operatorname{Re}\left(e^{i\lambda}\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{U}). \tag{1.3}$$

Definition 1.1 Let $f(z)$ and $F(z)$ be analytic functions. The function $f(z)$ is said to be *subordinate* to $F(z)$, written $f(z) \prec F(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| \le 1$, and such that $f(z) = F(w(z))$. If $F(z)$ is univalent, then $f(z) \prec F(z)$ *if and only if* $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.

Let $\mathbb{D}$ be the set of analytic functions $q(z)$ injective on $\overline{\mathbb{U}}\backslash E(q)$, where

$$E(q) = \left\{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} q(z) = \infty\right\}$$

and $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(q)$. Further, let $\mathbb{D}_a = \{q(z) \in \mathbb{D} : q(0) = a\}$.

In this paper we obtain some interesting relations associated with some differential inequalities in $\mathbb{U}$. These relations extend and generalize the Carathéodory functions in $\mathbb{U}$ which have been studied by many authors *e.g.*, see [1–14].

2 Main results

To prove our results, we need the following lemma due to Miller and Mocanu [15, p.24].

Lemma 2.1 *Let $q(z) \in \mathbb{D}_a$ and let*

$$p(z) = b + b_n z^n + \cdots$$

be analytic in $\mathbb{U}$ with $p(z) \neq b$. If $p(z) \not\prec q(z)$, then there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}\backslash E(q)$ and on $m \geq n \geq 1$ for which

(i) $p(z_0) = q(\zeta_0)$,

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$.

Theorem 2.1 *Let*

$$P : \mathbb{U} \to \mathbb{C}$$

with

$$\operatorname{Re}\big(\overline{a}P(z)\big) > 0 \quad (a \in \mathbb{C}).$$

If p is a function analytic in $\mathbb{U}$ with $p(0) = 1$ and

$$\operatorname{Re}\big(p(z) + P(z)zp'(z)\big) > \frac{E}{2|a|^2 \operatorname{Re}(\overline{a}P(z))}, \tag{2.1}$$

then

$$\operatorname{Re}\big(ap(z)\big) > \alpha,$$

where

$$\begin{aligned} E = &-\big(\operatorname{Re}(a) - \alpha\big)\big(\operatorname{Re}\big(\overline{a}P(z)\big)\big)^2 \\ &+ 2\operatorname{Re}\big(\overline{a}P(z)\big)\big[\big(\operatorname{Im}(a)\big)^2 + 2\alpha\operatorname{Re}(a)\big] \\ &+ \big(\operatorname{Re}(a) - \alpha\big)\big(\operatorname{Im}(a)\big)^2, \end{aligned} \tag{2.2}$$

with $\operatorname{Re}(a) > \alpha$.

Proof Let us define $q(z)$ and $h(z)$ as follows:

$$q(z) = ap(z)$$

and

$$h(z) = \frac{a - (2\alpha - \overline{a})z}{1 - z} \quad \big(\operatorname{Re}(a) > \alpha\big).$$

The functions q and h are analytic in $\mathbb{U}$ with $q(0) = h(0) = a \in \mathbb{C}$ with

$$h(\mathbb{U}) = \{w : \operatorname{Re}(w) > \alpha\}.$$

Now, we suppose that $q(z) \not\prec h(z)$. Therefore, by using Lemma 2.1, there exist points

$$z_0 \in \mathbb{U} \quad \text{and} \quad \zeta_0 \in \partial\mathbb{U}\backslash\{1\}$$

such that $q(z_0) = h(\zeta_0)$ and $z_0 q'(z_0) = m\zeta_0 h'(\zeta_0)$, $m \geq n \geq 1$.

We note that

$$\zeta_0 = h^{-1}(q(z_0)) = \frac{q(z_0) - a}{q(z_0) - (2\alpha - \overline{a})} \tag{2.3}$$

and

$$\zeta_0 h'(\zeta_0) = \frac{-|q(z_0) - a|^2}{2\operatorname{Re}(a - q(z_0))}. \tag{2.4}$$

We have $h(\zeta_0) = \alpha + \rho i$ $(\alpha, \rho \in \mathbb{R})$, therefore

$$\begin{aligned}
&\operatorname{Re}(p(z_0) + P(z_0)z_0 p'(z_0)) \\
&\quad = \operatorname{Re}\left(\frac{1}{a}h(\zeta_0) + \frac{1}{a}P(z_0)m\zeta_0 h'(\zeta_0)\right) \\
&\quad = \operatorname{Re}\left(\frac{\alpha + \rho i}{a}\right) - m\frac{|\alpha + \rho i - a|^2}{2\operatorname{Re}(a - \alpha)}\operatorname{Re}\left(\frac{P(z_0)}{a}\right) \\
&\quad \leq \operatorname{Re}\left(\frac{\alpha + \rho i}{a}\right) - \frac{|\alpha + \rho i - a|^2}{2\operatorname{Re}(a - \alpha)}\operatorname{Re}\left(\frac{P(z_0)}{a}\right) \\
&\quad = A\rho^2 + B\rho + C \\
&\quad = g(\rho),
\end{aligned} \tag{2.5}$$

where

$$A = -\frac{\operatorname{Re}(\overline{a}P(z_0))}{2|a|^2\operatorname{Re}(a - \alpha)},$$
$$B = \frac{\operatorname{Im}(a)}{|a|^2}\left(1 + \frac{\operatorname{Re}(\overline{a}P(z_0))}{\operatorname{Re}(a) - \alpha}\right)$$

and

$$C = \frac{1}{|a|^2}\left(\alpha\operatorname{Re}(a) - \frac{\alpha^2 + |a|^2 - 2\alpha\operatorname{Re}(a)\operatorname{Re}(\overline{a}P(z_0))}{2(\operatorname{Re}(a) - \alpha)}\right).$$

We can see that the function $g(\rho)$ in (2.5) takes the maximum value at ρ_1 given by

$$\rho_1 = \operatorname{Im}(a)\left(1 + \frac{\operatorname{Re}(a) - \alpha}{\operatorname{Re}(\overline{a}P(z_0))}\right).$$

Hence, we have

$$\begin{aligned}
&\operatorname{Re}\bigl(p(z_0) + P(z_0)zp'(z_0)\bigr) \\
&\quad \le g(\rho_1) \\
&\quad = \frac{E}{2|a|^2 \operatorname{Re}(\overline{a}P(z))},
\end{aligned}$$

where E is defined by (2.2). This is in contradiction to (2.1). Then we obtain $\operatorname{Re}(ap(z)) > \alpha$. $\square$

Theorem 2.2 *Let $p(z)$ a nonzero analytic function in $\mathbb{U}$ with $p(0) = 1$. If*

$$\left|p(z) + \frac{zp'(z)}{p(z)} - 1\right| < \frac{3\operatorname{Re}(a-\alpha)}{2|a|}\bigl|p(z)\bigr|, \tag{2.6}$$

then

$$\operatorname{Re}\left(\frac{a}{p(z)}\right) > \alpha,$$

where $\operatorname{Re}(a) > \alpha$.

Proof Let us define both $q(z)$ and $h(z)$ as follows:

$$q(z) = a/p(z)$$

and

$$h(z) = \frac{a - (2\alpha - \overline{a})z}{1-z} \quad \bigl(\operatorname{Re}(a) > \alpha\bigr).$$

The functions q and h are analytic in $\mathbb{U}$ with $q(0) = h(0) = a \in \mathbb{C}$ with

$$h(\mathbb{U}) = \bigl\{w : \operatorname{Re}(w) > \alpha\bigr\}.$$

Now, we suppose that $q(z) \not\prec h(z)$. Therefore, by using Lemma 2.1, there exist points

$$z_0 \in \mathbb{U} \quad \text{and} \quad \zeta_0 \in \partial\mathbb{U}\backslash\{1\}$$

such that $q(z_0) = h(\zeta_0)$ and $z_0 q'(z_0) = m\zeta_0 h'(\zeta_0)$, $m \ge n \ge 1$.

We note that

$$\zeta_0 h'(\zeta_0) = \frac{-|q(z_0) - a|^2}{2\operatorname{Re}(a - q(z_0))}. \tag{2.7}$$

We have $h(\zeta_0) = \alpha + \rho i$ ($\rho \in \mathbb{R}$); therefore,

$$\begin{aligned}
\frac{|p(z_0) + \frac{zp'(z_0)}{p(z_0)} - 1|}{|p(z_0)|} &= \left|\frac{\alpha + \rho i}{a} - \frac{m}{a}\frac{|a - \alpha - i\rho|^2}{2\operatorname{Re}(a-\alpha)} - 1\right| \\
&\ge \frac{1}{|a|}\left|\frac{m|a - \alpha - i\rho|^2}{2\operatorname{Re}(a-\alpha)} + \operatorname{Re}(a-\alpha)\right|
\end{aligned}$$

$$\geq \frac{1}{|a|}\left(\frac{|a-\alpha-i\rho|^2}{2\operatorname{Re}(a-\alpha)}+\operatorname{Re}(a-\alpha)\right)$$
$$\geq \frac{1}{2|a|\operatorname{Re}(a-\alpha)}\left(3\left(\operatorname{Re}(a-\alpha)\right)^2+\left(\operatorname{Im}(a)-\rho\right)^2\right)$$
$$\geq \frac{3\operatorname{Re}(a-\alpha)}{2|a|}.$$

This is in contradiction to (2.6). Then we obtain $\operatorname{Re}(\frac{a}{p(z)}) > \alpha$. □

3 Applications and examples

Putting $P(z) = \beta$ ($\beta > 0$; real) in Theorem 2.1 we have the following corollary.

Corollary 3.1 *If p is a function analytic in $\mathbb{U}$ with $p(0) = 1$ and*

$$\operatorname{Re}\left(p(z)+\beta zp'(z)\right) > \frac{E}{2\beta|a|^2\operatorname{Re}(a)},$$

then

$$\operatorname{Re}\left(ap(z)\right) > \alpha,$$

where

$$E = -\left(\operatorname{Re}(a)-\alpha\right)\beta^2\left(\operatorname{Re}(a)\right)^2 + 2\beta\operatorname{Re}(a)\left[\left(\operatorname{Im}(a)\right)^2+2\alpha\operatorname{Re}(a)\right] + \left(\operatorname{Re}(a)-\alpha\right)\left(\operatorname{Im}(a)\right)^2,$$

with $\operatorname{Re}(a) > \alpha$ ($\alpha \geq 0$).

Putting $\beta = 1$ in Corollary 3.1, we obtain the following corollary.

Corollary 3.2 *If p is a function analytic in $\mathbb{U}$ with $p(0) = 1$ and*

$$\operatorname{Re}\left(p(z)+zp'(z)\right) > \frac{1}{2\operatorname{Re}(a)}\left(3\operatorname{Re}(a)-\alpha\right) - \frac{\operatorname{Re}(a)}{|a|^2}\left(2\operatorname{Re}(a)-3\alpha\right),$$

then

$$\operatorname{Re}\left(ap(z)\right) > \alpha,$$

with $\operatorname{Re}(a) > \alpha$ ($\alpha \geq 0$).

Corollary 3.3 *Let $f(z) \in A$, $(g(z))^a \in S^*$ and*

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > \frac{E}{2|a|^2\operatorname{Re}(\overline{a}\frac{g(z)}{zg'(z)})},$$

then

$$\operatorname{Re}\left(a\frac{f(z)}{g(z)}\right) > \alpha,$$

where $\operatorname{Re}(a) > \alpha$ ($\alpha \geq 0$) *and E is defined by* (2.2) *with* $P(z) = \frac{g(z)}{zg'(z)}$.

Proof Putting $p(z) = \frac{f(z)}{g(z)}$ and $P(z) = \frac{g(z)}{zg'(z)}$ in Theorem 2.1, we have

$$\operatorname{Re}\bigl(p(z) + P(z)zp'(z)\bigr) = \operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right).$$

Since $(g(z))^a \in S^*$, which gives $\operatorname{Re}(a\frac{zg'(z)}{g(z)}) > 0$, therefore, $\operatorname{Re}(\overline{a}P(z)) > 0$. This completes the proof of the corollary. □

Example 3.1 Let $f(z) \in A$ and

$$\operatorname{Re}\bigl(f'(z)\bigr) > \frac{1}{2\operatorname{Re}(a)}\bigl(3\operatorname{Re}(a) - \alpha\bigr) - \frac{\operatorname{Re}(a)}{|a|^2}\bigl(2\operatorname{Re}(a) - 3\alpha\bigr),$$

then

$$\operatorname{Re}\left(a\frac{f(z)}{z}\right) > \alpha,$$

where $\operatorname{Re}(a) > \alpha$.

Example 3.2 Let $f(z) \in A$ and

$$\operatorname{Re}\left(\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\frac{zf'(z)}{f(z)}\right) > \frac{1}{2\operatorname{Re}(a)}\bigl(3\operatorname{Re}(a) - \alpha\bigr) - \frac{\operatorname{Re}(a)}{|a|^2}\bigl(2\operatorname{Re}(a) - 3\alpha\bigr),$$

then

$$\operatorname{Re}\left(a\frac{zf'(z)}{f(z)}\right) > \alpha,$$

where $\operatorname{Re}(a) > \alpha$.

(1) Putting $a = e^{i\lambda}$ ($|\lambda| < \frac{\pi}{2}$) and $\alpha = 0$ in Theorem 2.1, we have Theorem 1 due to Kim and Cho [4].
(2) Putting $a = e^{i\lambda}$ ($|\lambda| < \frac{\pi}{2}$), $P(z) = \beta$ ($\beta > 0$; real) and $\alpha = 0$ in Theorem 2.1, we have Corollary 1 due to Kim and Cho [4].
(3) Putting $a = \alpha = 0$ and $P(z) = 1$ in Theorem 2.1, we have the result due to Nunokawa *et al.* [16].
(4) Putting $a = e^{i\lambda}$ ($|\lambda| < \frac{\pi}{2}$), $P(z) = 1$ and $\alpha = 0$ in Theorem 2.1, we have Corollary 2 due to Kim and Cho [4].

Putting $p(z) = \frac{zf'(z)}{f(z)}$ in Theorem 2.2, we have the following corollary.

Corollary 3.4 *Let $p(z)$ a nonzero analytic function in U with $p(0) = 1$. If*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{3\operatorname{Re}(a - \alpha)}{2|a|}\left|\frac{zf'(z)}{f(z)}\right|,$$

then

$$\operatorname{Re}\left(\frac{1}{a}\frac{zf'(z)}{f(z)}\right) > \alpha,$$

where $\operatorname{Re}(a) > \alpha$.

Remark

(1) Putting $a = 1$ and $\alpha = 0$ in Corollary 3.4, we have the result due to Attiya and Nasr [1].

(2) Putting $a = 1$ and $\alpha = 0$ in Corollary 3.4, we have the result due to Kim and Cho [4].

Competing interests
The author declares that he has no competing interests.

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