academicJournals

Vol. 9(7), pp. 140-145, 16 April, 2014
DOI: 10.5897/IJPS2013.4079
Article Number:63FC13844156
ISSN 1992 - 1950

http://www.academicjournals.org/IJPS

International Journal of Physical Sciences

Full Length Research Paper

On the Bishop curvatures of involute-evolute curve couple in E^3

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Received 24 November, 2013; Accepted 19 February, 2014

In this study, the relations between of the Bishop curvatures of involute-evolute curve couple were examined in E^3. Also, the relation between of the Bishop frames of these curves was given. Moreover, the relations between of curvature radii of these curves were investigated. Lastly, in the case of these curves were planar curves, some corollaries were obtained.

Key words. Bishop frame, Bishop curvatures, involute-evolute curve couple.

INTRODUCTION

The idea of a string involute is due to Christian Huygens (1658), who is also known for his work in optics. He discovered involutes while trying to build a more accurate clock (Boyer, 1968). The involute-evolute curve couple is a well-known concept in E^3 (Hacisalihoglu, 1995). The specific curve pairs are the most popular subjects in curve and surface theory and involute-evolute curve couple is one of them. We can see in most textbooks various applications not only in curve theory but also in surface theory and mechanics.

An evolute and its involute, are defined in mutual pairs. The evolute of any curve is defined as the locus of the centers of curvature of the curve. The original curves is then defined as the involute of the evolute. The simplest case is that of a circle, which has only one center of curvature (its center), which is a degenerate evolute. The circle itself is the involte of this point. There are extensive literature on this subject, for instance, Mustafa and Çalişkan (2002), Bükcü and Karacan (2007), Mustafa (2009), Boyer (1968), Hacisalihoglu (1995), and Turgut and Yilmaz (2008).

It is well-known that, if a curve differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. These vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves. However, the Frenet-Serret frame is not defined for all points along every curve as curvature may vanish at some points on the curve; that is, second derivative of the curve may be zero. In this situation, we need an alternative frame. Therefore, in 1975, R.L. Bishop defined a new frame for a curve and he called it ‘relatively parallel adapted frames’ which is well defined even if the curve has vanishing second derivative in Euclidean 3-space. After Bishop’s study, the ralatively parallel adapted frame have been called Bishop frame in other studies and also we will use the Bishop frame instead of the relatively parallel adapted frame. The advantages of Bishop frame

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Mathematics Subject Classification: 53A04, 53A05.

and comparable Bishop frame with the Frenet frame in Euclidean 3-space were given by Bishop and by Hanson. After the invention of the Bishop frame, many studies were performed in Bükcü and Karacan (2008a, b, 2009), Andrew and Hui (1995), Bishop (1975), Yilmaz and Turgut (2010), and Yilmaz (2010).

In this paper, Bishop curvatures and curvatures radii of involute-evolute curve couple were obtained. The relations between Bishop curvatures and curvatures radii of these curve were computed and some results were given.

PRELIMINARIES

Let $\alpha: I \subset IR \to E^3$ be a differentiable curve with arc length parameter s and $\{t, n, b\}$ be the Frenet frame of α at the point $\alpha(s)$. These vectors are expressed by

$$t(s) = \alpha'(s), \quad n(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \quad b(s) = t(s) \wedge n(s)$$

where t, n and b are called tangent vector, principal normal vector and binormal vector, respectively. The Frenet formulas of α are

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \tag{1}$$

Here κ and τ are called curvature and torsion of α (Hacisalihoglu, 1995).

Definition 1

Let α and β be two curves given by (I, α) and (I, β) coordinate neighbourhoods, respectively. Let Frenet frames of α and β be $\{t, n, b\}$ and $\{\bar{t}, \bar{n}, \bar{b}\}$, respectively. β is called the involute of α (β is called the evolute of α), if

$$\langle t, \bar{t} \rangle = 0.$$

Theorem 1

Let (β, α) be involute-evolute curve couple. Then, the relation between Frenet frames $\{t, n, b\}$ and $\{\bar{t}, \bar{n}, \bar{b}\}$ of α and β is given as follows, respectively:

$$\begin{bmatrix} \bar{t} \\ \bar{n} \\ \bar{b} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \cos\psi(s) & 0 & \sin\psi(s) \\ -\sin\psi(s) & 0 & \cos\psi(s) \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}. \tag{2}$$

Here, ψ is angle between vectors t and $\bar{n}$ (Mustafa and Çalişkan, 2002).

Theorem 2

Let (β, α) be involute-evolute curve couple given by (I, α) and (I, β) coordinate neighbourhoods, respectively. Then,

$$d(\alpha(s), \beta(s)) = |c - s|$$

where, $\forall s \in I$ and $c = constant$ (Hacisalihoglu, 1995).

Theorem 3

Let (β, α) be involute-evolute curve couple and Frenet frames of α and β be $\{t, n, b\}$ and $\{\bar{t}, \bar{n}, \bar{b}\}$, respectively. Then, we have

$$\bar{\kappa}^2 = \frac{\kappa^2 + \tau^2}{\kappa^2 (c - s)^2}, \quad \bar{\tau}^2 = \frac{\kappa\tau' - \kappa'\tau}{\kappa(c - s)(\kappa^2 + \tau^2)}$$

Here, κ and τ are curvature and torsion of α and $\bar{\kappa}$ and $\bar{\tau}$ are curvature and torsion of β (Mustafa, 2009).

The Bishop frame (relatively parallel adapted frame) is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. We can parallelly transport an orthonormal frame along a curve by simply transporting each component of the frame parallelly. The parallel transport frame is based on the observation that, while t for a given curve model is unique, we may choose any convenient arbitrary basis (N_1, N_2) for the remainder of the frame, so long as it is in the normal plane perpendicular to t at each point (Yilmaz, 2010). Therefore, the type-1 Bishop (frame) formulas is expressed as

$$\begin{bmatrix} t' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} t \\ N_1 \\ N_2 \end{bmatrix} \tag{3}$$

Here, we shall call the set $\{t, N_1, N_2\}$ as Bishop frame and k_1 and k_2 as Bishop curvatures (Bükcü and Karacan, 2009; Bishop, 1975). The relation between Frenet and Bishop frames can be written as

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(s) & \sin\theta(s) \\ 0 & -\sin\theta(s) & \cos\theta(s) \end{bmatrix} \begin{bmatrix} t \\ N_1 \\ N_2 \end{bmatrix}. \tag{4}$$

Where

$$\theta(s) = acrtan\left(\frac{k_2}{k_1}\right), \tag{5}$$

$$\tau(s) = \theta'(s), \tag{6}$$

$$\kappa(s) = \sqrt{k_1^2 + k_2^2}. \tag{7}$$

Here, Bishop curvatures are defined by

$$\begin{cases} k_1 = \kappa cos\theta \\ k_2 = \kappa sin\theta \end{cases} \tag{8}$$

Definition 2

A regular curve $\alpha: I \to E^3$ is called a slant helix provided the unit vector $N_1(s)$ of α has constant angle θ with some fixed unit vector u; that is, $\langle N_1(s), u\rangle = cos\theta$ for all $s \in I$.

Theorem 4

Let $\alpha: I \to E^3$ be a unit speed curve with nonzero natural curvatures. Then α is a slant helix if and only if

$$\frac{k_1}{k_2} = const.$$

(Bükcü and Karacan, 2009).

ON THE BISHOP CURVATURES OF INVOLUTE-EVOLUTE CURVE COUPLE IN E^3

Let (β, α) be involute-evolute curve couple and Frenet frames of α and β be $\{t, n, b\}$ and $\{\bar{t}, \bar{n}, \bar{b}\}$, respectively. Let $\{t, N_1, N_2\}$ and $\{\bar{t}, \bar{N}_1, \bar{N}_2\}$ be Bishop frames of α and β, respectively. Moreover, let k_1, k_2 and $\bar{k}_1$, $\bar{k}_2$ be Bishop curvatures of α and β, respectively and κ and τ be curvature and torsion of α.
The relation between Frenet frame and Bishop frames of β is

$$\begin{bmatrix} \bar{t} \\ \bar{n} \\ \bar{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\bar{\theta} & sin\bar{\theta} \\ 0 & -sin\bar{\theta} & cos\bar{\theta} \end{bmatrix} \begin{bmatrix} \bar{t} \\ \bar{N}_1 \\ \bar{N}_2 \end{bmatrix}.$$

Here, $\bar{\theta}$ is angle between vectors $\bar{n}$ and $\bar{N}_1$. By using the last equation and Equation (2), we obtain

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 0 & cos\psi cos\bar{\theta} + sin\psi sin\bar{\theta} & cos\psi sin\bar{\theta} - cos\bar{\theta} sin\psi \\ 1 & 0 & 0 \\ 0 & cos\bar{\theta} sin\psi - cos\psi sin\bar{\theta} & cos\psi cos\bar{\theta} + sin\psi sin\bar{\theta} \end{bmatrix} \begin{bmatrix} \bar{t} \\ \bar{N}_1 \\ \bar{N}_2 \end{bmatrix},$$

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 0 & cos(\psi - \bar{\theta}) & -sin(\psi - \bar{\theta}) \\ 1 & 0 & 0 \\ 0 & sin(\psi - \bar{\theta}) & cos(\psi - \bar{\theta}) \end{bmatrix} \begin{bmatrix} \bar{t} \\ \bar{N}_1 \\ \bar{N}_2 \end{bmatrix}$$

or

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 0 & \cos\sigma & -sin\sigma \\ 1 & 0 & 0 \\ 0 & sin\sigma & cos\sigma \end{bmatrix} \begin{bmatrix} \bar{t} \\ \bar{N}_1 \\ \bar{N}_2 \end{bmatrix}$$

Here, $\measuredangle(t, \bar{N}_1) = \sigma = \psi - \bar{\theta}$ and $\measuredangle(t, \bar{n}) = \psi$. Substituting Equation (4) into the last equation, we can write

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta(s) & sin\theta(s) \\ 0 & -sin\theta(s) & cos\theta(s) \end{bmatrix} \begin{bmatrix} t \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 0 & \cos\sigma & -sin\sigma \\ 1 & 0 & 0 \\ 0 & sin\sigma & cos\sigma \end{bmatrix} \begin{bmatrix} \bar{t} \\ \bar{N}_1 \\ \bar{N}_2 \end{bmatrix}.$$

Then, the relation between Bishop frames of α and β is

$$\begin{bmatrix} t \\ N_1 \\ N_2 \end{bmatrix}. = \begin{bmatrix} 0 & cos\sigma & -sin\sigma \\ cos\theta & -sin\sigma sin\theta & -cos\sigma sin\theta \\ sin\theta & sin\sigma cos\theta & cos\sigma cos\theta \end{bmatrix} \begin{bmatrix} \bar{t} \\ \bar{N}_1 \\ \bar{N}_2 \end{bmatrix}. \tag{9}$$

From Figure 1,

$$\beta(\bar{s}) = \alpha(s) + \lambda(s) t(s) \tag{10}$$

for some function $\lambda(s)$. By taking the derivative of (10) with respect to s and applying the Bishop formulas, we have

$$\langle \bar{t}, t\rangle \frac{d\bar{s}}{ds} = (1 + \lambda')\langle t, t\rangle + \lambda k_1 N_1 + \lambda k_2 N_2.$$

Multiplying the last equation with t we get

$$\langle \bar{t}, t\rangle \frac{d\bar{s}}{ds} = (1 + \lambda')\langle t, t\rangle + \lambda k_1 \langle N_1, t\rangle + \lambda k_2 \langle N_2, t\rangle.$$

From the definition of the involute-evolute curve couple $\langle \bar{t}, t\rangle$, we obtain

$$\lambda = c - s, \quad c = constant \tag{11}$$

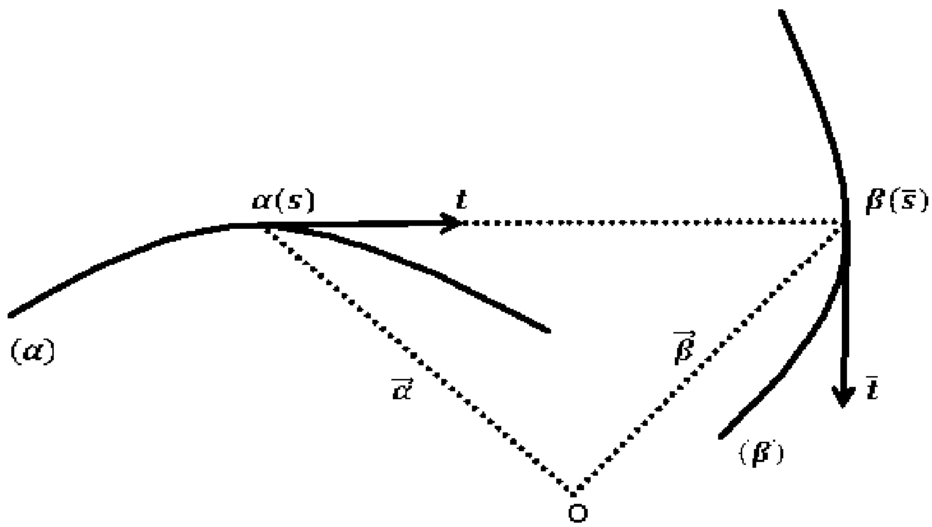

Figure 1. Involute-Evolute Curve Couple.

Then, from Equation (7) we can write

$$\bar{t}\frac{d\bar{s}}{ds} = \sqrt{k_1^2 + k_2^2}(c-s)n \tag{12}$$

Then, n and $\bar{t}$ are linearly dependent. Moreover, from Equation (2), we have

$$\frac{d\bar{s}}{ds} = \sqrt{k_1^2 + k_2^2}(c-s). \tag{13}$$

Theorem 5

Let (β, α) be involute-evolute curve couple. The relations between Bishop curvatures of α and β are given as follows:

$$\begin{cases} \bar{k}_1 = \dfrac{-\cos\sigma\sqrt{k_1^2+k_2^2} + \theta'\sin\sigma}{(c-s)\sqrt{k_1^2+k_2^2}} \\ \bar{k}_2 = \dfrac{-\sin\sigma\sqrt{k_1^2+k_2^2} + \theta'\cos\sigma}{(c-s)\sqrt{k_1^2+k_2^2}} \end{cases}$$

Proof

By taking the derivative of Equation (12) with respect to s, we have

$$\bar{t}\left(\frac{d\bar{s}}{ds}\right)^2 = \left[(c-s)\sqrt{k_1^2+k_2^2}\right]' \tag{14}$$

By using the last Equations (1), (3), (6) and (13), we get

$$\bar{k}_1\bar{N}_1 + \bar{k}_2\bar{N}_2 = \frac{-1}{(c-s)^2(k_1^2+k_2^2)}\begin{bmatrix} (c-s)^2(k_1^2+k_2^2)t \\ +\dfrac{(k_1^2+k_2^2)-(k_1k_1'+k_2k_2')(c-s)}{\sqrt{k_1^2+k_2^2}}n \\ -(c-s)\theta'\sqrt{k_1^2+k_2^2}b \end{bmatrix}. \tag{15}$$

Multiplying the last equation with $\bar{N}_1$, we have

$$\bar{k}_1 = \frac{-\cos\sigma\sqrt{k_1^2+k_2^2} + \theta'\sin\sigma}{(c-s)\sqrt{k_1^2+k_2^2}} \tag{16}$$

Similarly, multiplying Equation (15) with $\bar{N}_2$, we have

$$\bar{k}_2 = \frac{-\sin\sigma\sqrt{k_1^2+k_2^2} + \theta'\cos\sigma}{(c-s)\sqrt{k_1^2+k_2^2}} \tag{17}$$

Thus, the following corollaries can be written.

Corollary 1

From Equations (16) and (17) it can be easily seen that

$$\bar{k}_1^2 + \bar{k}_2^2 = \frac{k_1^2+k_2^2+\theta'}{(c-s)^2(k_1^2+k_2^2)} \tag{18}$$

Corollary 2

From Equations (16) and (17), we can write

$$\tau = \theta' = (c-s)\sqrt{k_1^2 + k_2^2}(\bar{k}_1 sin\sigma + \bar{k}_2 sin\sigma),$$

$$\lambda = (c-s) = \frac{1}{-\bar{k}_1 cos\sigma + \bar{k}_2 cos\sigma}.$$

Corollary 3

If (β, α) is a planar curve, then we can write

$$\lambda = (c-s) = \pm\frac{1}{\sqrt{k_1^2 + k_2^2}}. \quad (19)$$

Proof

From theorem 3, Equations (6) and (18) can be easily seen.

Theorem 6

Let (β, α) be involute-evolute curve couple. If α is a slant helix, then β is a slant helix⇔ σ = constant.

Proof

From theorem 4, Equations (8), (16) and (17), the proof is clear.

Theorem 7

Let (β, α) be involute-evolute curve couple. The relation between Bishop curvatures of α is

$$k_1^2 + k_2^2 = \frac{1}{2}(c-s)(k_1^2 + k_2^2)'. \quad (20)$$

Proof

Multiplying Equation (15) with $\bar{t}$, we have

$$k_1^2 + k_2^2 = (c-s)(k_1 k_1' + k_2 k_2')$$

and

$$k_1^2 + k_2^2 = \frac{1}{2}(c-s)(k_1^2 + k_2^2)'. \quad (21)$$

Corollary 4

Let (β, α) be involute-evolute curve couple. If α is a planar curve, then the relations between Bishop curvatures of α and β are

$$(\bar{k}_1^2 + \bar{k}_2^2)^{\frac{1}{2}} = \pm\frac{1}{2}\frac{(k_1^2 + k_2^2)'}{k_1^2 + k_2^2}.$$

Proof

From Equation (19) and (21), the proof is clear.

Theorem 8

Let (β, α) be involute-evolute curve couple and curvature radiuses of α and β be r and $\bar{r}$ at the points $\alpha(s)$ and $\beta(\bar{s})$, respectively. Then, the relation between r and $\bar{r}$ is given as follows:

$$\bar{r} = \frac{1}{r^2}\frac{|c-s|}{\sqrt{k_1^2 + k_2^2 + \theta'}}$$

Proof

Let M and $\bar{M}$ be curvature centers of α and β at the points $\alpha(s)$ and $\beta(\bar{s})$,, respectively. Then, we can write

$$r = \|\alpha M\| = \frac{1}{\kappa} = \frac{1}{\sqrt{k_1^2 + k_2^2}} \quad (22)$$

$$\bar{r} = \|\beta \bar{M}\| = \frac{1}{\bar{\kappa}} = \frac{1}{\sqrt{\bar{k}_1^2 + \bar{k}_2^2}}. \quad (23)$$

Substituting Equation (18) into Equation (23), we get

$$\bar{r} = \frac{|c-s|(k_1^2 + k_2^2)}{\sqrt{\bar{k}_1^2 + \bar{k}_2^2 + \theta'}}.$$

From Equation (22), we obtain

$$\bar{r} = \frac{1}{r^2}\frac{|c-s|}{\sqrt{k_1^2+k_2^2+\theta\prime}} \qquad (24)$$

From Equations (6) and (24), the following corollary can be given.

Corollary 5

Let (β, α) be involute-evolute curve couple. If α is a planar curve, then the relation between curvature radii of α and β are

$$\bar{r} = \frac{1}{r}|c-s|.$$

Conflict of Interests

The author(s) have not declared any conflict of interests.

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