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New characterizations for the products of differentiation and composition operators between Bloch-type spaces

Yu-Xia Liang[1] and Xing-Tang Dong[2*]

*Correspondence: dongxingtang@163.com
[2]Department of Mathematics, Tianjin University, Tianjin, 300072, P.R. China
Full list of author information is available at the end of the article

Abstract

We use a brief way to give various equivalent characterizations for the boundedness and the essential norm of the operator $C_\varphi D^m$ acting on Bloch-type spaces. At the same time, we use this method to easily get a known characterization for the operator DC_φ on Bloch-type spaces.

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1 Introduction and preliminaries

Recently there has been a considerable interest on various product-type operators (see, *e.g.* [1–19]), and among them on products of composition and differentiation operators (see, *e.g.* [1, 2, 4, 5, 7–12, 15–19]). One of the problems of interest is to characterize the boundedness and compactness of the composition operator C_φ acting on Bloch-type spaces in terms of the nth power of the analytic self-mapping φ of the unit disk $\mathbb{D}$. Very recently, the first author and Zhou have given the characterizations for the boundedness and the essential norm of the products of differentiation and composition operator $C_\varphi D^m$ and DC_φ acting on Bloch-type spaces in [9, 10], respectively. Inspired by [20], we present here an easier way to research the corresponding problem. Moreover, by this brief method, we first give new equivalent characterizations for the boundedness and the essential norm of the operator $C_\varphi D^m$, and then we obtain the same results for the operator DC_φ as in the paper [10].

Let $\mathbb{D}$ denote the unit disk in the complex plane $\mathbb{C}$. Denote $H(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the collection of all holomorphic self-mappings on $\mathbb{D}$. The composition operator C_φ is defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$.

The Bloch space of ν-type

$$\mathcal{B}_\nu = \left\{ f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}_\nu} = \sup_{z\in\mathbb{D}} \nu(z) \left| f'(z) \right| < \infty \right\}$$

is a Banach space endowed with the norm $|f(0)| + \|f\|_{\mathcal{B}_\nu}$, where the weight $\nu : \mathbb{D} \to \mathbb{R}_+$ is a continuous, strictly positive and bounded function.

For the standard weights $\nu_\alpha(z) = (1-|z|^2)^\alpha$ for $\alpha > 0$, we denote $\mathcal{B}_\nu = \mathcal{B}^\alpha$ and

$$\|f\|_\alpha = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f'(z)|.$$

Similarly, $\mathcal{B}^\alpha$ is a Banach space under the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha$. When $\alpha = 1$, we get the classical Bloch space $\mathcal{B}$. We refer the readers to the book [21] for more information as regards the above spaces.

The weighted Banach space of analytic functions

$$H_\nu^\infty = \Big\{f \in H(\mathbb{D}) : \|f\|_\nu = \sup_{z\in\mathbb{D}} \nu(z)|f(z)| < \infty\Big\}$$

is a Banach space endowed with the norm $\|\cdot\|_\nu$. The weight ν is called *radial*, if $\nu(z) = \nu(|z|)$ for all $z \in \mathbb{D}$. For a weight ν, the *associated* weight $\tilde{\nu}(z)$ is defined by

$$\tilde{\nu}(z) = \big(\sup\big\{|f(z)| : f \in H_\nu^\infty, \|f\|_\nu \leq 1\big\}\big)^{-1}, \quad z \in \mathbb{D}.$$

For the standard weights $\nu_\alpha(z) = (1-|z|^2)^\alpha$ $(0 < \alpha < \infty)$, we have $\tilde{\nu}_\alpha(z) = \nu_\alpha(z)$. We refer the interested readers to [22, p.39]. In this case, we denote $H_\nu = H_{\nu_\alpha}^\infty$ and

$$\|f\|_{\nu_\alpha} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f(z)|.$$

Then $H_{\nu_\alpha}^\infty$ is a Banach space under the norm $\|f\|_{\nu_\alpha}$.

For $\varphi \in S(\mathbb{D})$, $u \in H(\mathbb{D})$, the weighted composition operator uC_φ is defined by

$$uC_\varphi(f) = u \cdot (f \circ \varphi), \quad f \in H(\mathbb{D}).$$

As for $u \equiv 1$, the weighted composition operator is the usual composition operator C_φ. When φ is the identity mapping I, the operator uC_I is the multiplication operator M_u.

The differentiation operator D is defined by

$$Df = f', \quad f \in H(\mathbb{D}).$$

The products of differentiation and composition operators DC_φ and $C_\varphi D^m$ are defined, respectively, as follows:

$$DC_\varphi f(z) = f'(\varphi(z))\varphi'(z), \qquad C_\varphi D^m f = f^{(m)} \circ \varphi, \quad f \in H(\mathbb{D}), m \in \mathbb{N}.$$

The essential norm of a continuous linear operator T between two normed linear spaces X and Y is its distance from the compact operators. That is,

$$\|T\|_{e,X\to Y} = \inf\big\{\|T-K\| : K \text{ is compact}\big\},$$

where $\|\cdot\|$ denotes the operator norm. Notice that $\|T\|_{e,X\to Y} = 0$ if and only if T is compact, so the estimate on $\|T\|_{e,X\to Y}$ will lead to the condition for the operator T to be compact.

Throughout this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next. The notations $A \asymp B$, $A \preceq B$, $A \succeq B$ mean that there maybe different positive constants C such that $B/C \le A \le CB$, $A \le CB$, $A \ge CB$.

For convenience of the reader we list the results related with our conclusions in this paper.

Theorem A [9, Theorem 1] *Let $0 < \alpha, \beta < \infty$, m be a nonnegative integer and φ be a holomorphic self-map of the unit disk $\mathbb{D}$. Then $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded if and only if*

$$\sup_{n\in\mathbb{N}} n^{\alpha-1} \big\| C_\varphi D^m I_n(z) \big\|_\beta < \infty.$$

Theorem B [9, Theorem 2] *Let $0 < \alpha, \beta < \infty$, m be a nonnegative integer and φ be a holomorphic self-map of the unit disk $\mathbb{D}$. Suppose that $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. Then the estimate for the essential norm of $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is*

$$\big\| C_\varphi D^m \big\|_e \asymp \limsup_{n\to\infty} n^{\alpha-1} \big\| C_\varphi D^m I_n(z) \big\|_\beta,$$

where $I_n(z) = z^n$, $z \in \mathbb{D}$, $n \in \mathbb{N}$.

Theorem C [10, Theorem 2.3] *Let $0 < \alpha, \beta < \infty$, and $\varphi \in S(\mathbb{D})$. Then $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded if and only if*

$$\sup_{n\ge1} n^\alpha \big\| I_{\varphi'}\big(\varphi^n\big) \big\|_\beta < \infty \quad \textit{and} \quad \sup_{n\ge1} n^\alpha \big\| J_{\varphi'}\big(\varphi^{n-1}\big) \big\|_\beta < \infty.$$

Theorem D [10, Theorem 3.5] *Let $0 < \alpha, \beta < \infty$ and $\varphi \in S(\mathbb{D})$. Suppose that $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. Then the estimate for the essential norm of $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is*

$$\max\left\{ \frac{A}{3\cdot 2^{\alpha+1}}, \frac{B}{2^{\alpha+1}(3\alpha+4)} \right\} \le \|DC_\varphi\|_e \preceq (A + B),$$

where $A := (\frac{e}{2(\alpha+1)})^{\alpha+1} \limsup_{n\to\infty} n^\alpha \|I_{\varphi'}(\varphi^n)\|_\beta$ and $B := (\frac{e}{2\alpha})^\alpha \limsup_{n\to\infty} n^\alpha \|J_{\varphi'}(\varphi^{n-1})\|_\beta$. The definitions of $I_{\varphi'}(\varphi^n)$ and $J_{\varphi'}(\varphi^{n-1})$ can be found in Section 4.

We would like to point out that the first author and Zhou got the above four theorems by using complex calculations and intricate discussions. In this paper, we will use a brief way to give other equivalent characterizations for the boundedness and the essential norm of $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ on the unit disk in Section 3. In addition, using this method we will show new proofs of Theorem C and Theorem D in Section 4.

2 Lemmas

In this section we quote some lemmas for our further application. The first lemma is a well-known characterization for $\mathcal{B}^\alpha$ $(0 < \alpha < \infty)$.

Lemma 2.1 *For $f \in H(\mathbb{D})$, $m \in \mathbb{N}$ and $\alpha > 0$. Then*

$$f \in \mathcal{B}^\alpha \quad \Leftrightarrow \quad \|f\|_\alpha \asymp \sum_{j=0}^{m-1} \big|f^{(j)}(0)\big| + \sup_{z\in\mathbb{D}} \big(1 - |z|^2\big)^{\alpha+m-1} \big|f^{(m)}(z)\big| < \infty.$$

So for $f \in \mathcal{B}^\alpha$, the above lemma implies that $f' \in H^\infty_{v_\alpha}$ and more general $f^{(m+1)} \in H^\infty_{v_{\alpha+m}}$. Therefore, theories of the weighted composition operator $uC_\varphi : H^\infty_v \to H^\infty_w$ play a key role in the proof of our main results. Here we list some lemmas which will be used later.

Lemma 2.2 [23, Proposition 3.1] *Let v and w be weights. Then the weighted composition operator $uC_\varphi : H^\infty_v \to H^\infty_w$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{w(z)|u(z)|}{\tilde{v}(\varphi(z))} < \infty.$$

Moreover, the following holds:

$$\|uC_\varphi\|_{H^\infty_v \to H^\infty_w} = \sup_{z\in\mathbb{D}} \frac{w(z)|u(z)|}{\tilde{v}(\varphi(z))}.$$

Lemma 2.3 [23, Theorem 4.4] *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Suppose $uC_\varphi : H^\infty_v \to H^\infty_w$ is bounded. Then*

$$\|uC_\varphi\|_{e,H^\infty_v \to H^\infty_w} \asymp \lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{w(z)|u(z)|}{\tilde{v}(\varphi(z))}.$$

Lemma 2.4 [24, Theorem 2.4] *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then*

(a) *$uC_\varphi : H^\infty_v \to H^\infty_w$ is bounded if and only if*

$$\sup_{n\geq 0} \frac{\|u\varphi^n\|_w}{\|z^n\|_v} < \infty$$

with the norm comparable to the above supremum.

(b) $\|uC_\varphi\|_{e,H^\infty_v \to H^\infty_w} = \limsup_{n\to\infty} \frac{\|u\varphi^n\|_w}{\|z^n\|_v}$.

Lemma 2.5 [22, Lemma 2.1] *For $\alpha > 0$, we have $\lim_{n\to\infty}(n+1)^\alpha \|z^n\|_{v_\alpha} = (\frac{2\alpha}{e})^\alpha$.*

The following criterion for compactness follows from an easy modification of [25, Proposition 3.11]. Hence we omit the details.

Lemma 2.6 *Let $0 < \alpha, \beta < \infty$ and T be a linear operator from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$. Then $T : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if $T : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded and for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{B}^\alpha$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, $\|Tf_k\|_{\mathcal{B}^\beta} \to 0$ as $k \to \infty$.*

3 Boundedness and essential norm of $C_\varphi D^m$

In this section, we give other equivalent characterizations for the boundedness and the essential norm of the operator $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ with $0 < \alpha, \beta < \infty$.

Theorem 3.1 *Let $0 < \alpha, \beta < \infty$, $m \in \mathbb{N}$, and $\varphi \in S(\mathbb{D})$. Then the following statements are equivalent*:

(a) *$C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded.*

(b)

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}} < \infty. \tag{3.1}$$

(c)

$$\sup_{n\geq 1} n^{\alpha+m} \|\varphi'\varphi^{n-1}\|_{v_\beta} < \infty. \tag{3.2}$$

Proof (a) $\Rightarrow$ (b). Suppose that $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. Choose $f_1(z) = z^{m+1}$ and

$$f_w(z) = \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^\alpha}, \quad w \in \mathbb{D}.$$

It is easy to verify that $f_1 \in \mathcal{B}^\alpha$ and $f_w \in \mathcal{B}^\alpha$ for $w \in \mathbb{D}$. By $\|C_\varphi D^m f\|_\beta \preceq \|f\|_\alpha$ for $f \in \mathcal{B}^\alpha$, we obtain

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |\varphi'(z)| < \infty,$$

and

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)||\varphi(z)|^{m+1}}{(1-|\varphi(z)|^2)^{\alpha+m}} < \infty.$$

Then it follows that

$$\sup_{|\varphi(z)|\leq\frac{1}{2}} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}} \preceq \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |\varphi'(z)| < \infty,$$

and

$$\sup_{|\varphi(z)|>\frac{1}{2}} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}} \preceq \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)||\varphi(z)|^{m+1}}{(1-|\varphi(z)|^2)^{\alpha+m}} < \infty.$$

That is, (b) holds.

(b) $\Leftrightarrow$ (c). From Lemma 2.2, the condition (b) is a necessary and sufficient condition for the boundedness of weighted composition operator $\varphi' C_\varphi : H^\infty_{v_{\alpha+m}} \to H^\infty_{v_\beta}$. Further by Lemma 2.4(a) and Lemma 2.5, the boundedness of the weighted composition operator $\varphi' C_\varphi : H^\infty_{v_{\alpha+m}} \to H^\infty_{v_\beta}$ is equivalent to the following:

$$\begin{aligned} \sup_{n\geq 1} \frac{\|\varphi'\varphi^{n-1}\|_{v_\beta}}{\|z^{n-1}\|_{v_{\alpha+m}}} &= \sup_{n\geq 1} \frac{n^{\alpha+m}\|\varphi'\varphi^{n-1}\|_{v_\beta}}{n^{\alpha+m}\|z^{n-1}\|_{v_{\alpha+m}}} \\ &\asymp \sup_{n\geq 1} n^{\alpha+m} \|\varphi'\varphi^{n-1}\|_{v_\beta} < \infty. \end{aligned}$$

(b) $\Rightarrow$ (a). Suppose (b) holds. For every $f \in \mathcal{B}^\alpha$, then it follows from Lemma 2.1 that

$$\|C_\varphi D^m f\|_\beta = \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |f^{(m+1)}(\varphi(z))\varphi'(z)| \leq \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}} \|f\|_{\mathcal{B}^\alpha} < \infty.$$

Moreover, $|C_\varphi D^m f(0)| = |f^{(m)}(\varphi(0))| \preceq \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|\varphi(0)|^2)^{\alpha+m-1}}$. Thus $\|C_\varphi D^m f\|_{\mathcal{B}^\beta} < \infty$, and hence (a) holds. □

Remark 3.2

(1) The relation (a) ⇔ (b) was essentially proved in a very general result in [18]. For convenience of the reader, we sketch the proof in [18].

(2) One can easily see that

$$\begin{aligned}\sup_{n\in\mathbb{N}} n^{\alpha-1}\big\|C_\varphi D^m I_n(z)\big\|_\beta &= \sup_{n\geq m+1} n^{\alpha-1}n(n-1)\cdots(n-m)\big\|\varphi'\varphi^{n-m-1}\big\|_{v_\beta}\\ &= \sup_{k\geq 1}(k+m)^\alpha(k+m-1)\cdots k\big\|\varphi'\varphi^{k-1}\big\|_{v_\beta}\\ &\asymp \sup_{k\geq 1} k^{\alpha+m}\big\|\varphi'\varphi^{k-1}\big\|_{v_\beta} = \sup_{n\geq 1} n^{\alpha+m}\big\|\varphi'\varphi^{n-1}\big\|_{v_\beta}.\end{aligned}$$

Therefore, the characterizations for the boundedness of the operator $C_\varphi D^m$ in Theorem 3.1 are equivalent to that in Theorem A.

As an application of Theorem 3.1, we present an example of the bounded operator $C_\varphi D^m$, according to either (3.1) or (3.2).

Example 3.3 Let $\varphi(z) = z^2$ for $z \in \mathbb{D}$ and $\beta = \alpha + m$. Then we study the boundedness of $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^{\alpha+m}$. Firstly, by (3.1), it is clear that

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\alpha+m}|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}} = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\alpha+m}|2z|}{(1-|z|^4)^{\alpha+m}} < \infty.$$

Secondly, by (3.2) we obtain

$$\begin{aligned}\sup_{n\geq 1} n^{\alpha+m}\big\|\varphi'\varphi^{n-1}\big\|_{v_\beta} &= \sup_{n\geq 1} n^{\alpha+m}\big\|2zz^{2(n-1)}\big\|_{v_\beta}\\ &= \sup_{n\geq 1} n^{\alpha+m}\sup_{z\in\mathbb{D}}\big(1-|z|^2\big)^{\alpha+m}\big|2zz^{2(n-1)}\big|\\ &\preceq \sup_{n\geq 1} n^{\alpha+m}\sup_{x\in[0,1[}(1-x)^{\alpha+m}x^{n-\frac{1}{2}}\\ &= \sup_{n\geq 1} n^{\alpha+m}\left(1-\frac{n-\frac{1}{2}}{\beta+n-\frac{1}{2}}\right)^{\alpha+m}\left(\frac{n-\frac{1}{2}}{\beta+n-\frac{1}{2}}\right)^{n-\frac{1}{2}}\\ &= \sup_{n\geq 1}\left(\frac{\beta n}{\beta+n-\frac{1}{2}}\right)^{\alpha+m}\left(\frac{n-\frac{1}{2}}{\beta+n-\frac{1}{2}}\right)^{n-\frac{1}{2}} < \infty.\end{aligned}$$

From each of these conditions, one sees that $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^{\alpha+m}$ is bounded.

Next we estimate the essential norm of the operator $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ for all $0 < \alpha, \beta < \infty$. Denote $\tilde{\mathcal{B}}^\alpha = \{f \in \mathcal{B}^\alpha : f(0) = 0\}$. Let $D_{m+1} : \mathcal{B}^\alpha \to H^\infty_{v_{\alpha+m}}$ be defined by $D_{m+1}f = f^{(m+1)}(z)$. Then we have $\|D_{m+1}f\|_{v_{m+\alpha}} \asymp \|f\|_{\mathcal{B}^\alpha}$ for $f \in \tilde{\mathcal{B}}^\alpha$. Since $f^{(m+1)} \in H^\infty_{v_{\alpha+m}}$ when $f \in \mathcal{B}^\alpha$, and further by the equality $(C_\varphi D^m f)' = \varphi' f^{(m+1)}(\varphi)$ for all $f \in \mathcal{B}^\alpha$, it follows that

$$\big\|C_\varphi D^m\big\|_{e,\tilde{\mathcal{B}}^\alpha\to\mathcal{B}^\beta} \preceq \big\|\varphi' C_\varphi\big\|_{e,H^\infty_{v_{\alpha+m}}\to H^\infty_{v_\beta}}. \tag{3.3}$$

Thus we only need to estimate $\|\varphi' C_\varphi\|_{e,H^\infty_{v_{\alpha+m}}\to H^\infty_{v_\beta}}$ for *the upper bound* of the essential norm of $C_\varphi D^m$. It is obvious that every compact operator $T\in\mathcal{K}(\tilde{\mathcal{B}}^\alpha,\mathcal{B}^\beta)$ can be extended to a compact operator $K\in\mathcal{K}(\mathcal{B}^\alpha,\mathcal{B}^\beta)$. In fact, for every $f\in\mathcal{B}^\alpha$, $f-f(0)\in\tilde{\mathcal{B}}^\alpha$, and we can define $K(f):=T(f-f(0))+f(0)$, which is a *compact* operator from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$, due to $K(f_k)$ has convergent subsequence when $\{f_k\}$ is a bounded sequence. In the following lemma we will use the compact operator K_r defined on the space $\mathcal{B}^\alpha$ by $K_rf(z)=f(rz)$.

Lemma 3.4 *If $0<\alpha,\beta<\infty$ and $C_\varphi D^m$ is a bounded operator from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$, then*

$$\|C_\varphi D^m\|_{e,\tilde{\mathcal{B}}^\alpha\to\mathcal{B}^\beta}=\|C_\varphi D^m\|_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta}.$$

Proof Although the proof is similar to [20, Lemma 3.1], we will give all the details for convenience of the reader. It is obvious that

$$\|C_\varphi D^m\|_{e,\tilde{\mathcal{B}}^\alpha\to\mathcal{B}^\beta}\le\|C_\varphi D^m\|_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta}.$$

Conversely, let $T\in K(\mathcal{B}^\alpha,\mathcal{B}^\beta)$ be given. Choose an increasing sequence $(r_n)_n$ in $(0,1)$ converging to 1. We denote by $\mathcal{A}$ the closed subspace of $\mathcal{B}^\alpha$ consisting of all constant functions. Then we have

$$\begin{aligned}\|C_\varphi D^m-T\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta}&=\sup_{\|f\|_{\mathcal{B}^\alpha}\le1}\|C_\varphi D^m(f)-T(f)\|_{\mathcal{B}^\beta}\\&\le\sup_{\|f\|_{\mathcal{B}^\alpha}\le1}\|C_\varphi D^m(f-f(0))-T|_{\tilde{\mathcal{B}}^\alpha}(f-f(0))\|_{\mathcal{B}^\beta}\\&\quad+\sup_{\|f\|_{\mathcal{B}^\alpha}\le1}\|C_\varphi D^m(f(0))-T(f(0))\|_{\mathcal{B}^\beta}\\&\le\sup_{g\in\tilde{\mathcal{B}}^\alpha}\|C_\varphi D^m(g)-T|_{\tilde{\mathcal{B}}^\alpha}(g)\|_{\mathcal{B}^\beta}+\sup_{h\in\mathcal{A}}\|C_\varphi D^m(h)-T|_{\mathcal{A}}(h)\|_{\mathcal{B}^\beta}.\end{aligned}$$

Hence

$$\begin{aligned}\inf_{T\in\mathcal{K}(\mathcal{B}^\alpha,\mathcal{B}^\beta)}\|C_\varphi D^m-T\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta}&\le\inf_{T\in\mathcal{K}(\mathcal{B}^\alpha,\mathcal{B}^\beta)}\|C_\varphi D^m-T|_{\tilde{\mathcal{B}}^\alpha}\|_{\tilde{\mathcal{B}}^\alpha\to\mathcal{B}^\beta}\\&\quad+\inf_{T\in\mathcal{K}(\mathcal{B}^\alpha,\mathcal{B}^\beta)}\|C_\varphi D^m-T|_{\mathcal{A}}\|_{\mathcal{A}\to\mathcal{B}^\beta}\\&\le\|C_\varphi D^m\|_{e,\tilde{\mathcal{B}}^\alpha\to\mathcal{B}^\beta}+\lim_{n\to\infty}\|C_\varphi D^m(I-K_{r_n})\|_{\mathcal{A}\to\mathcal{B}^\beta}.\end{aligned}$$

Since $C_\varphi D^m:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is bounded, it follows that

$$\lim_{n\to\infty}\|C_\varphi D^m(I-K_{r_n})\|_{\mathcal{A}\to\mathcal{B}^\beta}\le C\lim_{n\to\infty}\|I-K_{r_n}\|_{\mathcal{A}\to\mathcal{B}^\beta}=0.$$

Thus we obtain $\|C_\varphi D^m\|_{e,\tilde{\mathcal{B}}^\alpha\to\mathcal{B}^\beta}\ge\|C_\varphi D^m\|_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta}$. The proof is finished. □

Thus by Lemma 3.4 and (3.3) it follows that

$$\|C_\varphi D^m\|_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta}\preceq\|\varphi' C_\varphi\|_{e,H^\infty_{v_{\alpha+m}}\to H^\infty_{v_\beta}}.\tag{3.4}$$

Theorem 3.5 *Let* $0 < \alpha, \beta < \infty$, $m \in \mathbb{N}$, *and* $\varphi \in S(\mathbb{D})$. *Suppose that* $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is bounded. Then*

$$\begin{aligned}\|C_\varphi D^m\|_{e,\mathcal{B}^\alpha \to \mathcal{B}^\beta} &\asymp \limsup_{n\to\infty} n^{\alpha+m} \|\varphi' \varphi^{n-1}\|_{v_\beta} \\ &\asymp \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}}. \end{aligned} \tag{3.5}$$

Proof If $\|\varphi\|_\infty < 1$, then by [26, Lemma 3.1], the operator $uC_\varphi : \mathcal{B}^\alpha \to H_\mu^\infty$ is compact. The boundedness (compactness) of $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is equivalent to the boundedness (compactness) of $\varphi' C_\varphi : \mathcal{B}^{\alpha+m} \to H_\beta^\infty$. In this case, all items in (3.5) are zero.

If $\|\varphi\|_\infty = 1$, since $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded, then the boundedness of $\varphi' C_\varphi : H_{v_{\alpha+m}}^\infty \to H_{v_\beta}^\infty$ follows from the proof in Theorem 3.1. Thus by (3.4), Lemma 2.4(b), and Lemma 2.5,

$$\begin{aligned}\|C_\varphi D^m\|_{e,\mathcal{B}^\alpha \to \mathcal{B}^\beta} \preceq \|\varphi' C_\varphi\|_{e,H_{v_{\alpha+m}}^\infty \to H_{v_\beta}^\infty} &= \limsup_{n\to\infty} \frac{\|\varphi'\varphi^{n-1}\|_{v_\beta}}{\|z^{n-1}\|_{v_{\alpha+m}}} \\ &= \limsup_{n\to\infty} \frac{n^{\alpha+m}\|\varphi'\varphi^{n-1}\|_{v_\beta}}{\|z^{n-1}\|_{v_{\alpha+m}} n^{\alpha+m}} \asymp \limsup_{n\to\infty} n^{\alpha+m}\|\varphi'\varphi^{n-1}\|_{v_\beta}.\end{aligned}$$

Since $\|\varphi\|_\infty = 1$, we may choose a sequence $\{z_k\}_{k\in\mathbb{N}} \subset \mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Define

$$f_k(z) = \frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^\alpha}, \quad k \in \mathbb{N}.$$

It is easy to show that $f_k \in \mathcal{B}^\alpha$ and converges to zero uniformly on the compact subsets of $\mathbb{D}$ as $k \to \infty$. Moreover,

$$f_k^{(m+1)}\bigl(\varphi(z_k)\bigr) = \frac{\alpha(\alpha+1)\cdots(\alpha+m)(\overline{\varphi(z_k)})^{m+1}}{(1-|\varphi(z_k)|^2)^{\alpha+m}}.$$

Then for every compact operator $T : \mathcal{B}^\alpha \to \mathcal{B}^\beta$, by Lemma 2.6, it follows that $\lim_{k\to\infty} \|Tf_k\|_\beta = 0$. Thus

$$\begin{aligned}\|C_\varphi D^m - T\|_{\mathcal{B}^\alpha \to \mathcal{B}^\beta} &\succeq \limsup_{k\to\infty} \|C_\varphi D^m(f_k)\|_\beta - \limsup_{k\to\infty} \|Tf_k\|_\beta \\ &= \limsup_{k\to\infty} \|C_\varphi D^m(f_k)\|_\beta \\ &\geq \limsup_{k\to\infty} \bigl(1-|z_k|^2\bigr)^\beta \bigl|f_k^{(m+1)}\bigl(\varphi(z_k)\bigr)\varphi'(z_k)\bigr| \\ &\succeq \limsup_{k\to\infty} \bigl(1-|z_k|^2\bigr)^\beta \frac{|\varphi'(z_k)||\varphi(z_k)|^{m+1}}{(1-|\varphi(z_k)|^2)^{\alpha+m}} \\ &= \limsup_{k\to\infty} \frac{(1-|z_k|^2)^\beta |\varphi'(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha+m}}.\end{aligned}$$

Consequently,

$$\|C_\varphi D^m\|_{e,\mathcal{B}^\alpha \to \mathcal{B}^\beta} \succeq \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}}.$$

Since the operator $\varphi' C_\varphi : H^\infty_{v_{\alpha+m}} \to H^\infty_{v_\beta}$ is bounded, then applying Lemma 2.3, Lemma 2.4(b), and Lemma 2.5, we get

$$\begin{aligned}\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}} &\asymp \|\varphi' C_\varphi\|_{e, H^\infty_{v_{\alpha+m}} \to H^\infty_{v_\beta}} \\ &= \limsup_{n\to\infty} \frac{\|\varphi'\varphi^{n-1}\|_{v_\beta}}{\|z^{n-1}\|_{v_{\alpha+m}}} \asymp \limsup_{n\to\infty} n^{\alpha+m} \|\varphi'\varphi^{n-1}\|_{v_\beta}.\end{aligned}$$

Thus

$$\begin{aligned}\limsup_{n\to\infty} n^{\alpha+m} \|\varphi'\varphi^{n-1}\|_{v_\beta} &\succeq \|C_\varphi D^m\|_{e, \mathcal{B}^\alpha \to \mathcal{B}^\beta} \\ &\succeq \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}} \\ &\asymp \limsup_{n\to\infty} n^{\alpha+m} \|\varphi'\varphi^{n-1}\|_{v_\beta}.\end{aligned}$$

Hence

$$\begin{aligned}\|C_\varphi D^m\|_{e, \mathcal{B}^\alpha \to \mathcal{B}^\beta} &\asymp \limsup_{n\to\infty} n^{\alpha+m} \|\varphi'\varphi^{n-1}\|_{v_\beta} \\ &\asymp \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}}.\end{aligned}$$

This completes the proof. □

Remark 3.6

(1) The relation $\|C_\varphi D^m\|_{e, \mathcal{B}^\alpha \to \mathcal{B}^\beta} \asymp \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}}$ can be proved similarly to [26, Theorem 3.2]. Here we give a complete proof for the reader's convenience.

(2) Similar to Remark 3.2, one can get

$$\limsup_{n\to\infty} n^{\alpha+m} \|\varphi'\varphi^{n-1}\|_{v_\beta} \asymp \limsup_{n\to\infty} n^{\alpha-1} \|C_\varphi D^m I_n(z)\|_\beta.$$

Therefore, the characterizations for the essential norms of the operator $C_\varphi D^m$ in Theorem 3.5 are equivalent to that in Theorem B.

The following corollary is an immediate consequence of Theorem 3.5.

Corollary 3.7 *Let* $0 < \alpha, \beta < \infty$, $m \in \mathbb{N}$, *and* $\varphi \in S(\mathbb{D})$. *Then the following statements are equivalent*:

(a) $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is compact.*

(b) $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is bounded and*

$$\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}} = 0.$$

(c) $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is bounded and*

$$\limsup_{n\to\infty} n^{\alpha+m} \|\varphi'\varphi^{n-1}\|_{v_\beta} = 0.$$

4 Boundedness and essential norm of DC_φ

In this section, the corresponding problems for the operator $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ are considered. Let $u \in H(\mathbb{D})$, then for every $f \in H(\mathbb{D})$, define

$$I_u f(z) = \int_0^z f'(\zeta)u(\zeta)\,d\zeta, \qquad J_u f(z) = \int_0^z f(\zeta)u'(\zeta)\,d\zeta.$$

Then it follows that

$$I_{\varphi'}(\varphi^n)(z) = \int_0^z (\varphi^n)'(\zeta)\varphi'(\zeta)\,d\zeta, \qquad J_{\varphi'}(\varphi^{n-1})(z) = \int_0^z \varphi^{n-1}(\zeta)\varphi''(\zeta)\,d\zeta.$$

By an easy calculation, one can get

$$\bigl(I_{\varphi'}(\varphi^n)(z)\bigr)' = n\varphi(z)^{n-1}\bigl(\varphi'(z)\bigr)^2 \tag{4.1}$$

and

$$\bigl(J_{\varphi'}(\varphi^{n-1})(z)\bigr)' = \varphi(z)^{n-1}\varphi''(z). \tag{4.2}$$

In 2007, S Li and S Stević gave the following characterizations for the boundedness and compactness of the operator $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$.

Lemma 4.1 *Let $\alpha, \beta > 0$ and $\varphi \in S(\mathbb{D})$. Then the following statements hold*:

(a) [4, Theorem 1] *$DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} < \infty \quad \textit{and} \quad \sup_{z\in\mathbb{D}} \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha}} < \infty. \tag{4.3}$$

(b) [4, Theorem 2] *$DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded,*

$$\lim_{|\varphi(z)|\to 1} \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} = 0 \quad \textit{and} \quad \lim_{|\varphi(z)|\to 1} \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha}} = 0.$$

First, we will give a brief proof of Theorem C as regards the bounded operator $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ for all $0 < \alpha, \beta < \infty$.

Theorem 4.2 *Let $0 < \alpha, \beta < \infty$ and $\varphi \in S(\mathbb{D})$. Then $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded if and only if*

$$\sup_{n\geq 1} n^\alpha \bigl\| I_{\varphi'}(\varphi^n)\bigr\|_\beta < \infty \quad \textit{and} \quad \sup_{n\geq 1} n^\alpha \bigl\| J_{\varphi'}(\varphi^{n-1})\bigr\|_\beta < \infty.$$

Proof Lemma 4.1 shows that DC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$ if and only if (4.3) holds. On the other hand, Lemma 2.2 shows that (4.3) holds if and only if the weighted composition operators $(\varphi')^2 C_\varphi$ maps $H^\infty_{v_{\alpha+1}}$ boundedly into $H^\infty_{v_\beta}$ and $\varphi'' C_\varphi$ maps $H^\infty_{v_\alpha}$ boundedly into $H^\infty_{v_\beta}$, and hence it follows from Lemma 2.4(a) that (4.3) is equivalent to

$$\sup_{n\geq 1} \frac{\|(\varphi')^2\varphi^{n-1}\|_{v_\beta}}{\|z^{n-1}\|_{v_{\alpha+1}}} < \infty \quad \text{and} \quad \sup_{n\geq 1} \frac{\|\varphi''\varphi^{n-1}\|_{v_\beta}}{\|z^{n-1}\|_{v_\alpha}} < \infty.$$

Using Lemma 2.5, (4.1) and (4.2), then the boundedness of $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is equivalent to

$$\sup_{n\ge1} \frac{\|(\varphi')^2\varphi^{n-1}\|_{v_\beta} n^{\alpha+1}}{n^{\alpha+1}\|z^{n-1}\|_{v_{\alpha+1}}} \asymp \sup_{n\ge1} \|(\varphi')^2\varphi^{n-1}\|_{v_\beta} n^{\alpha+1} = \sup_{n\ge1} n^\alpha \|I_{\varphi'}(\varphi^n)\|_\beta < \infty$$

and

$$\sup_{n\ge1} \frac{n^\alpha\|\varphi''\varphi^{n-1}\|_{v_\beta}}{n^\alpha\|z^{n-1}\|_{v_\alpha}} \asymp \sup_{n\ge1} n^\alpha \|\varphi''\varphi^{n-1}\|_{v_\beta} = \sup_{n\ge1} n^\alpha \|J_{\varphi'}(\varphi^{n-1})\|_\beta < \infty.$$

This completes the proof. □

Now, we give a new proof of Theorem D about the essential norm of $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ for $0 < \alpha, \beta < \infty$. We denote $\tilde{\mathcal{B}}^\alpha = \{f \in \mathcal{B}^\alpha : f(0) = 0\}$. Let $D_\alpha : \mathcal{B}^\alpha \to H^\infty_{v_\alpha}$ and $S_\alpha : \mathcal{B}^\alpha \to H^\infty_{v_{\alpha+1}}$ be the first-order derivative operator and the second-order derivative operator, respectively. That is,

$$D_\alpha(f) = f', \qquad S_\alpha(f) = f''.$$

By Lemma 2.1 we have

$$\|D_\alpha f\|_{v_\alpha} = \|f\|_{\mathcal{B}^\alpha} \quad \text{and} \quad \|S_\alpha f\|_{v_{\alpha+1}} \asymp \|f\|_{\mathcal{B}^\alpha} \quad \text{for } f \in \tilde{\mathcal{B}}^\alpha.$$

For $f \in \mathcal{B}^\alpha$, by Lemma 2.1, $f'' \in H^\infty_{v_{\alpha+1}}$, and $f' \in H^\infty_{v_\alpha}$. Then by the equation $(DC_\varphi f)' = f''(\varphi)(\varphi')^2 + f'(\varphi)\varphi''$, it follows that

$$\|DC_\varphi\|_{e,\tilde{\mathcal{B}}^\alpha \to \mathcal{B}^\beta} \preceq \|(\varphi')^2 C_\varphi\|_{e,H^\infty_{v_{\alpha+1}} \to H^\infty_{v_\beta}} + \|\varphi'' C_\varphi\|_{e,H^\infty_{v_\alpha} \to H^\infty_{v_\beta}}. \tag{4.4}$$

Moreover, every compact operator $T \in \mathcal{K}(\tilde{\mathcal{B}}^\alpha, \mathcal{B}^\beta)$ can be extended to a compact operator $K \in \mathcal{K}(\mathcal{B}^\alpha, \mathcal{B}^\beta)$. Then similar to Lemma 3.4, one can easily get

$$\|DC_\varphi\|_{e,\tilde{\mathcal{B}}^\alpha \to \mathcal{B}^\beta} = \|DC_\varphi\|_{e,\mathcal{B}^\alpha \to \mathcal{B}^\beta}.$$

Thus combining the above equation with (4.4), we obtain

$$\|DC_\varphi\|_{e,\mathcal{B}^\alpha \to \mathcal{B}^\beta} \preceq \|(\varphi')^2 C_\varphi\|_{e,H^\infty_{v_{\alpha+1}} \to H^\infty_{v_\beta}} + \|\varphi'' C_\varphi\|_{e,H^\infty_{v_\alpha} \to H^\infty_{v_\beta}}. \tag{4.5}$$

According to (4.5), we only need to estimate the right two essential norms for *the upper bound* of the essential norm of $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$.

Theorem 4.3 *Let* $0 < \alpha, \beta < \infty$ *and* $\varphi \in S(\mathbb{D})$. *Suppose that* $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is bounded. Then*

$$\|DC_\varphi\|_{e,\mathcal{B}^\alpha \to \mathcal{B}^\beta} \asymp \max\Big\{\limsup_{n\to\infty} n^\alpha \|I_{\varphi'}(\varphi^n)\|_\beta, \limsup_{n\to\infty} n^\alpha \|J_{\varphi'}(\varphi^{n-1})\|_\beta\Big\}.$$

Proof By Lemma 4.1(a) and Lemma 2.2, the boundedness of $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is equivalent to $(\varphi')^2 C_\varphi : H^\infty_{v_{\alpha+1}} \to H^\infty_{v_\beta}$ and $\varphi'' C_\varphi : H^\infty_{v_\alpha} \to H^\infty_{v_\beta}$ are bounded weighted composition operators.

The upper estimate. From Lemma 2.4(b) and Lemma 2.5, we obtain

$$\begin{aligned}
\big\|(\varphi')^2 C_\varphi\big\|_{e,H^\infty_{v_{\alpha+1}}\to H^\infty_{v_\beta}} &= \limsup_{n\to\infty} \frac{\|(\varphi')^2\varphi^{n-1}\|_{v_\beta}}{\|z^{n-1}\|_{v_{\alpha+1}}} = \limsup_{n\to\infty} \frac{n^{\alpha+1}\|(\varphi')^2\varphi^{n-1}\|_{v_\beta}}{n^{\alpha+1}\|z^{n-1}\|_{v_{\alpha+1}}} \\
&\asymp \limsup_{n\to\infty} n^{\alpha+1} \big\|(\varphi')^2\varphi^{n-1}\big\|_{v_\beta} = \limsup_{n\to\infty} n^\alpha \big\|I_{\varphi'}(\varphi^n)\big\|_\beta, \\
\big\|\varphi'' C_\varphi\big\|_{e,H^\infty_{v_\alpha}\to H_{v^\infty_\beta}} &= \limsup_{n\to\infty} \frac{\|\varphi''\varphi^{n-1}\|_{v_\beta}}{\|z^{n-1}\|_{v_\alpha}} = \limsup_{n\to\infty} \frac{\|\varphi''\varphi^{n-1}\|_{v_\beta} n^\alpha}{\|z^{n-1}\|_{v_\alpha} n^\alpha} \\
&\asymp \limsup_{n\to\infty} n^\alpha \big\|\varphi''\varphi^{n-1}\big\|_{v_\beta} = \limsup_{n\to\infty} n^\alpha \big\|J_{\varphi'}(\varphi^{n-1})\big\|_\beta.
\end{aligned}$$

Then it follows from (4.5) that

$$\|DC_\varphi\|_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta} \preceq \max\Big\{\limsup_{n\to\infty} n^\alpha \big\|I_{\varphi'}(\varphi^n)\big\|_\beta, \limsup_{n\to\infty} n^\alpha \big\|J_{\varphi'}(\varphi^{n-1})\big\|_\beta\Big\}.$$

The lower estimate. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Define

$$\begin{aligned}
f_k(z) &= \frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^\alpha} - \frac{\alpha}{\alpha+1}\frac{(1-|\varphi(z_k)|^2)^2}{(1-\overline{\varphi(z_k)}z)^{\alpha+1}}, \\
g_k(z) &= \frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^\alpha} - \frac{\alpha}{\alpha+2}\frac{(1-|\varphi(z_k)|^2)^2}{(1-\overline{\varphi(z_k)}z)^{\alpha+1}}.
\end{aligned}$$

We can easily show both f_k and g_k belong to $\mathcal{B}^\alpha$ and converge to zero uniformly on the compact subsets of $\mathbb{D}$ as $k \to \infty$. Moreover,

$$\begin{aligned}
&f_k'\big(\varphi(z_k)\big) = 0, \qquad f_k''\big(\varphi(z_k)\big) = \frac{-\alpha(\overline{\varphi(z_k)})^2}{(1-|\varphi(z_k)|^2)^{\alpha+1}}; \\
&g_k'\big(\varphi(z_k)\big) = \frac{\alpha\overline{\varphi(z_k)}}{(\alpha+2)(1-|\varphi(z_k)|^2)^\alpha}, \qquad g_k''\big(\varphi(z_k)\big) = 0.
\end{aligned}$$

Then for every compact operator $T : \mathcal{B}^\alpha \to \mathcal{B}^\beta$, by Lemma 2.6 we obtain

$$\begin{aligned}
\|DC_\varphi - T\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta} &\succeq \limsup_{k\to\infty}\big\|DC_\varphi(f_k)\big\|_\beta \geq \limsup_{k\to\infty}\big(1-|z_k|^2\big)^\beta \left|\frac{-\alpha(\varphi'(z_k))^2(\overline{\varphi(z_k)})^2}{(1-|\varphi(z_k)|^2)^{\alpha+1}}\right|, \\
\|DC_\varphi - T\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta} &\succeq \limsup_{k\to\infty}\big\|DC_\varphi(g_k)\big\|_\beta \geq \limsup_{k\to\infty}\big(1-|z_k|^2\big)^\beta \left|\frac{\alpha\varphi''(z_k)\overline{\varphi(z_k)}}{(\alpha+2)(1-|\varphi(z_k)|^2)^\alpha}\right|.
\end{aligned}$$

Since the weighted composition operators $(\varphi')^2C_\varphi : H^\infty_{v_{\alpha+1}} \to H^\infty_{v_\beta}$ and $\varphi''C_\varphi : H^\infty_{v_\alpha} \to H^\infty_{v_\beta}$ are bounded. Then applying Lemma 2.3, Lemma 2.4(b), and Lemma 2.5, it follows that

$$\begin{aligned}
\|DC_\varphi\|_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta} &\succeq \limsup_{|\varphi(z)|\to 1}\bigl(1-|z|^2\bigr)^\beta \left|\frac{(\varphi'(z))^2}{(1-|\varphi(z)|^2)^{\alpha+1}}\right| \\
&\asymp \left\|\left(\varphi'\right)^2 C_\varphi\right\|_{e,H^\infty_{v_{\alpha+1}}\to H^\infty_{v_\beta}} = \limsup_{n\to\infty}\frac{\|(\varphi')^2\varphi^{n-1}\|_{v_\beta}}{\|z^{n-1}\|_{v_{\alpha+1}}} \\
&\asymp \limsup_{n\to\infty} n^{\alpha+1}\left\|\left(\varphi'\right)^2\varphi^{n-1}\right\|_{v_\beta} = \limsup_{n\to\infty} n^\alpha \left\|I_{\varphi'}\left(\varphi^n\right)\right\|_\beta, \\
\|DC_\varphi\|_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta} &\succeq \limsup_{|\varphi(z)|\to 1}\bigl(1-|z|^2\bigr)^\beta \left|\frac{\varphi''(z)}{(1-|\varphi(z)|^2)^{\alpha}}\right| \\
&\asymp \left\|\varphi'' C_\varphi\right\|_{e,H^\infty_{v_\alpha}\to H^\infty_{v_\beta}} = \limsup_{n\to\infty}\frac{\|\varphi''\varphi^{n-1}\|_{v_\beta}}{\|z^{n-1}\|_{v_\alpha}} \\
&\asymp \limsup_{n\to\infty} n^{\alpha}\left\|\varphi''\varphi^{n-1}\right\|_{v_\beta} = \limsup_{n\to\infty} n^\alpha \left\|J_{\varphi'}\left(\varphi^{n-1}\right)\right\|_\beta.
\end{aligned}$$

Hence

$$\|DC_\varphi\|_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta} \succeq \max\left\{\limsup_{n\to\infty} n^\alpha \left\|I_{\varphi'}\left(\varphi^n\right)\right\|_\beta, \limsup_{n\to\infty} n^\alpha \left\|J_{\varphi'}\left(\varphi^{n-1}\right)\right\|_\beta\right\}.$$

This completes the proof. □

The following result is an immediate consequence of Theorem 4.3 and Lemma 4.1(b).

Corollary 4.4 *Let $\alpha, \beta > 0$ and $\varphi \in S(\mathbb{D})$. Then the following statements are equivalent*:

(a) *$DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact.*

(b) *$DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded,*

$$\limsup_{n\to\infty} n^\alpha \left\|I_{\varphi'}\left(\varphi^n\right)\right\|_\beta = 0 \quad \textit{and} \quad \limsup_{n\to\infty} n^\alpha \left\|J_{\varphi'}\left(\varphi^{n-1}\right)\right\|_\beta = 0.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived and drafted the manuscript, and read and approved the final manuscript.

Author details
[1]School of Mathematical Sciences, Tianjin Normal University, Tianjin, 300387, P.R. China. [2]Department of Mathematics, Tianjin University, Tianjin, 300072, P.R. China.

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