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Impulsive differential and impulsive integral inequalities with integral jump conditions

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Abstract
In this article, we establish some impulsive differential and impulsive integral inequalities for integral jump conditions. The new jump conditions for impulse effects are related to the integral conditions of the past state. Two examples are given to illustrate the advantage of our results.
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1 Introduction

In [1], Lakshmikantham et al. developed a famous impulsive differential inequality given as Theorem A below.

Lakshmikantham et al. assume that $0 \le t_0 < t_1 < t_2 < \dots$, $\lim_{k\to\infty} t_k = \infty$, $R_+ = [0, +\infty)$ and $I \subset R$. They define $PC(R_+, I) = \{u: R_+ \to I;\ u(t)$ is continuous for $t \ne t_k$, and $u(0^+)$, $u(t_k^-)$, and $u(t_k^+)$ exist, and $u(t_k^-) = u(t_k), k = 1, 2, \dots\}$ and $PC^1(R_+, I) = \{u \in PC(R_+, I)$: $u'(t)$ is continuous everywhere for $t \ne t_k$, and $u'(0^+)$, $u'(t_k^+)$ and $u'(t_k^-)$ exist, and $u'(t_k^-) = u'(t_k), k = 1, 2, \dots\}$.

Theorem A. Assume that

(H_0) the sequence $\{t_k\}$ satisfies $0 \le t_0 < t_1 < t_2 < \dots$, $\lim_{k\to\infty} t_k = \infty$;

(H_1) $m \in PC^1[R_+, R]$ and $m(t)$ is left-continuous at t_k, $k = 1, 2, \dots$;

(H_2) for $k = 1, 2, \dots$, $t \ge t_0$,

$$m'(t) \le p(t)m(t) + q(t), \quad t \ne t_k, \tag{1.1}$$

$$m(t_k^+) \le d_k m(t_k) + b_k, \tag{1.2}$$

where $q, p \in C[R_+, R]$, $d_k \ge 0$ and b_k are constants.

Then,

$$\begin{aligned} m(t) \le{}& m(t_0) \prod_{t_0<t_k<t} d_k e^{\int_{t_0}^t p(s)ds} + \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} d_j e^{\int_{t_k}^t p(s)ds} \right) b_k \\ &+ \int_{t_0}^t \prod_{s<t_k<t} d_k e^{\int_s^t p(\sigma)d\sigma} q(s)ds, \quad t \ge t_0. \end{aligned} \tag{1.3}$$

Impulsive differential and impulsive integral inequalities play an important role in the study of the theory of impulsive differential equations (see [1-4]). In recent years, many authors have used impulsive (differential or integral) inequalities to investigate properties of solutions of various impulsive problems, such as existence, uniqueness, boundedness, stability, asymptotic behavior, and oscillation etc. (see, for example [5-39]). There are many good results on the impulsive differential and impulsive integral inequalities (see for example [40-48]). However, most of these articles deal with jump conditions at impulse point t_k depending on the left-hand limit $m(t_k)$ or a time-delay value, $m(t_k - \tau)$, $\tau > 0$. Our main goal is to extend the theory of impulsive differential and impulsive integral inequalities to include integral jump conditions.

In the present article, we will show that Theorem A can be generalized to obtain differential inequalities for integral jump conditions by replacing the inequality in (1.2) by the inequality in (1.4).

$$m(t_k^+) \leq d_k m(t_k) + c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m(s)ds + b_k, \quad k = 1, 2, \ldots, \tag{1.4}$$

where $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$. We note that if $c_k = 0$ for all $k = 1, 2, \ldots$, then condition (1.4) reduces to condition (1.2). If $d_k = 0$, $c_k \neq 0$ and $0 \leq \sigma_k < \tau_k \leq t_k - t_{k-1}$, $k = 1, 2, \ldots$, then condition (1.4) means that the bound of the jump condition at t_k is a functional of past states on the interval $(t_k - \tau_k, t_k - \sigma_k]$ before the impulse point t_k. Moreover, we establish some new impulsive integral inequalities with integral jump conditions.

At the end of this article, we will show some applications of our results to prove a maximum principle and the boundedness of solutions for impulsive problems.

2 Main results

Denote $l = \max\{k: t \geq t_k, k = 1, 2, \ldots\}$. Now we are in the position to state and prove our results.

Theorem 2.1. *Let* (H_0) *and* (H_1) *hold. Suppose that* $p, q \in C[R_+, R]$ *and for* $k = 1, 2, \ldots, t \geq t_0$,

$$m'(t) \leq p(t)m(t) + q(t), \quad t \neq t_k, \tag{2.1}$$

$$m(t_k^+) \leq d_k m(t_k) + c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m(s)ds + b_k, \tag{2.2}$$

where $c_k, d_k \geq 0$, $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$ *and* b_k *are constants.*

Then,

$$\begin{aligned} m(t) \leq & \Bigg\{ m(t_0) \prod_{t_0 < t_k < t} \left(d_k e^{\int_{t_{k-1}}^{t_k} p(\xi)d\xi} + c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} e^{\int_{t_{k-1}}^{s} p(\xi)d\xi} ds \right) \\ & + \sum_{t_0 < t_k < t} \Bigg[\prod_{t_k < t_j < t} \left(d_j e^{\int_{t_{j-1}}^{t_j} p(\xi)d\xi} + c_j \int_{t_j-\tau_j}^{t_j-\sigma_j} e^{\int_{t_{j-1}}^{s} p(\xi)d\xi} ds \right) \\ & \times \Bigg(d_k \int_{t_{k-1}}^{t_k} q(s) e^{\int_s^{t_k} p(\xi)d\xi} ds \\ & + c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \int_{t_{k-1}}^{s} q(r) e^{\int_r^s p(\xi)d\xi} dr ds + b_k \Bigg) \Bigg] \Bigg\} e^{\int_{t_l}^{t} p(\xi)d\xi} \\ & + \int_{t_l}^{t} q(s) e^{\int_s^t p(\xi)d\xi} ds, \quad t \geq t_0. \end{aligned} \tag{2.3}$$

Proof. From (2.1) we have that

$$\frac{d}{dt}\left[m(t)e^{-\int_{t_0}^{t}p(\xi)d\xi}\right]\le q(t)e^{-\int_{t_0}^{t}p(\xi)d\xi}, \tag{2.4}$$

for $t\in[t_0,t_1]$. Integrating (2.4) from t_0 to t for $t\in[t_0,t_1]$, we get

$$m(t)\le m(t_0)e^{\int_{t_0}^{t}p(\xi)d\xi}+\int_{t_0}^{t}q(s)e^{\int_{s}^{t}p(\xi)d\xi}ds. \tag{2.5}$$

Hence (2.3) is valid on $[t_0,t_1]$. Assume that (2.3) holds for $t\in[t_0,t_n]$ for some integer $n>1$. Then, for $t\in[t_n,t_{n+1}]$, it follow from (2.1) and (2.5) that

$$m(t)\le m(t_n^{+})e^{\int_{t_n}^{t}p(\xi)d\xi}+\int_{t_n}^{t}q(s)e^{\int_{s}^{t}p(\xi)d\xi}ds. \tag{2.6}$$

Now using (2.2) and (2.6), we have

$$m(t)\le\left(d_n m(t_n)+c_n\int_{t_n-\tau_n}^{t_n-\sigma_n}m(s)ds+b_n\right)e^{\int_{t_n}^{t}p(\xi)d\xi}+\int_{t_n}^{t}q(s)e^{\int_{s}^{t}p(\xi)d\xi}ds. \tag{2.7}$$

By the principle of mathematical induction, (2.7) can be expressed as

$$\begin{aligned}
m(t)\le&\left\{d_n\left(\left\{m(t_0)\prod_{t_0<t_k<t_n}\left(d_k e^{\int_{t_{k-1}}^{t_k}p(\xi)d\xi}+c_k\int_{t_k-\tau_k}^{t_k-\sigma_k}e^{\int_{t_{k-1}}^{s}p(\xi)d\xi}ds\right)\right.\right.\right.\\
&+\sum_{t_0<t_k<t_n}\left[\prod_{t_k<t_j<t_n}\left(d_j e^{\int_{t_{j-1}}^{t_j}p(\xi)d\xi}+c_j\int_{t_j-\tau_j}^{t_j-\sigma_j}e^{\int_{t_{j-1}}^{s}p(\xi)d\xi}ds\right)\right.\\
&\times\left(d_k\int_{t_{k-1}}^{t_k}q(s)e^{\int_{s}^{t_k}p(\xi)d\xi}ds\right.\\
&\left.\left.\left.+c_k\int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{t_{k-1}}^{s}q(r)e^{\int_{r}^{s}p(\xi)d\xi}drds+b_k\right)\right]\right\}e^{\int_{t_{n-1}}^{t_n}p(\xi)d\xi}\\
&\left.+\int_{t_{n-1}}^{t_n}q(s)e^{\int_{s}^{t_n}p(\xi)d\xi}ds\right)+c_n\int_{t_n-\tau_n}^{t_n-\sigma_n}\left\{\left\{m(t_0)\prod_{t_0<t_k<s}\left(d_k e^{\int_{t_{k-1}}^{t_k}p(\xi)d\xi}\right.\right.\right.\\
&\left.+c_k\int_{t_k-\tau_k}^{t_k-\sigma_k}e^{\int_{t_{k-1}}^{v}p(\xi)d\xi}dv\right)+\sum_{t_0<t_k<s}\left[\prod_{t_k<t_j<s}\left(d_j e^{\int_{t_{j-1}}^{t_j}p(\xi)d\xi}\right.\right.\\
&\left.+c_j\int_{t_j-\tau_j}^{t_j-\sigma_j}e^{\int_{t_{j-1}}^{v}p(\xi)d\xi}dv\right)\left(d_k\int_{t_{k-1}}^{t_k}q(v)e^{\int_{v}^{t_k}p(\xi)d\xi}dv\right.\\
&\left.\left.\left.+c_k\int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{t_{k-1}}^{v}q(r)e^{\int_{r}^{v}p(\xi)d\xi}drdv+b_k\right)\right]\right\}e^{\int_{t_{n-1}}^{s}p(\xi)d\xi}\\
&\left.\left.+\int_{t_{n-1}}^{s}q(v)e^{\int_{v}^{s}p(\xi)d\xi}dv\right\}ds+b_n\right\}e^{\int_{t_n}^{t}p(\xi)d\xi}+\int_{t_n}^{t}q(s)e^{\int_{s}^{t}p(\xi)d\xi}ds.
\end{aligned} \tag{2.8}$$

Set

$$E_k = d_k e^{\int_{t_{k-1}}^{t_k} p(\xi)d\xi} + c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} e^{\int_{t_{k-1}}^{s} p(\xi)d\xi} ds \tag{2.9}$$

$$G_k = d_k \int_{t_{k-1}}^{t_k} q(s) e^{\int_s^{t_k} p(\xi)d\xi} ds + c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \int_{t_{k-1}}^{s} q(r) e^{\int_r^{s} p(\xi)d\xi} dr ds + b_k. \tag{2.10}$$

Substituting (2.9), (2.10) into (2.8), we get that for $t \in [t_n, t_{n+1}]$

$$\begin{aligned}
m(t) \le & \Bigg\{ d_n \Bigg(\Bigg\{ m(t_0) \prod_{t_0<t_k<t_n} E_k + \sum_{t_0<t_k<t_n} \Bigg[\prod_{t_k<t_j<t_n} E_j G_k \Bigg] \Bigg\} e^{\int_{t_{n-1}}^{t_n} p(\xi)d\xi} \\
& + \int_{t_{n-1}}^{t_n} q(s) e^{\int_s^{t_n} p(\xi)d\xi} ds \Bigg) + c_n \int_{t_n-\tau_n}^{t_n-\sigma_n} \Bigg\{ \Bigg\{ m(t_0) \prod_{t_0<t_k<s} E_k \\
& + \sum_{t_0<t_k<s} \Bigg[\prod_{t_k<t_j<s} E_j G_k \Bigg] \Bigg\} e^{\int_{t_{n-1}}^{s} p(\xi)d\xi} \\
& + \int_{t_{n-1}}^{s} q(v) e^{\int_v^{s} p(\xi)d\xi} dv \Bigg\} ds + b_n \Bigg\} e^{\int_{t_n}^{t} p(\xi)d\xi} + \int_{t_n}^{t} q(s) e^{\int_s^{t} p(\xi)d\xi} ds \\
= & \Bigg\{ \Bigg(m(t_0) \prod_{t_0<t_k<t_n} E_k + \sum_{t_0<t_k<t_n} \Bigg[\prod_{t_k<t_j<t_n} E_j G_k \Bigg] \Bigg) d_n e^{\int_{t_{n-1}}^{t_n} p(\xi)d\xi} \\
& + d_n \int_{t_{n-1}}^{t_n} q(s) e^{\int_s^{t_n} p(\xi)d\xi} ds \\
& + \Bigg((m(t_0)) \prod_{t_0<t_k<t_n} E_k + \sum_{t_0<t_k<t_n} \Bigg[\prod_{t_k<t_j<t_n} E_j G_k \Bigg] \Bigg) c_n \int_{t_n-\tau_n}^{t_n-\sigma_n} e^{\int_{t_{n-1}}^{s} p(\xi)d\xi} ds \\
& + c_n \int_{t_n-\tau_n}^{t_n-\sigma_n} \int_{t_{n-1}}^{s} q(v) e^{\int_v^{s} p(\xi)d\xi} dv ds + b_n \Bigg\} e^{\int_{t_n}^{t} p(\xi)d\xi} + \int_{t_n}^{t} q(s) e^{\int_s^{t} p(\xi)d\xi} ds \\
= & \Bigg\{ \Bigg(m(t_0) \prod_{t_0<t_k<t_n} E_k + \sum_{t_0<t_k<t_n} \Bigg[\prod_{t_k<t_j<t_n} E_j G_k \Bigg] \Bigg) E_n + G_n \Bigg\} e^{\int_{t_n}^{t} p(\xi)d\xi} ds \\
& + \int_{t_n}^{t} q(s) e^{\int_s^{t} p(\xi)d\xi} ds \\
= & \Bigg\{ m(t_0) \prod_{t_0<t_k<t} E_k + \sum_{t_0<t_k<t} \Bigg[\prod_{t_k<t_j<t} E_j G_k \Bigg] \Bigg\} e^{\int_{t_n}^{t} p(\xi)d\xi} ds + \int_{t_n}^{t} q(s) e^{\int_s^{t} p(\xi)d\xi} ds.
\end{aligned}$$

Hence,

$$
\begin{aligned}
m(t) \le \Bigg\{ & m(t_0) \prod_{t_0<t_k<t} \left(d_k e^{\int_{t_{k-1}}^{t_k} p(\xi)d\xi} + c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} e^{\int_{t_{k-1}}^{s} p(\xi)d\xi} ds \right) \\
& + \sum_{t_0<t_k<t} \left[\prod_{t_k<t_j<t} \left(d_j e^{\int_{t_{j-1}}^{t_j} p(\xi)d\xi} + c_j \int_{t_j-\tau_j}^{t_j-\sigma_j} e^{\int_{t_{j-1}}^{s} p(\xi)d\xi} ds \right) \right. \\
& \times \left(d_k \int_{t_{k-1}}^{t_k} q(s) e^{\int_s^{t_k} p(\xi)d\xi} ds \right. \\
& \left.\left. + c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \int_{t_{k-1}}^{s} q(r) e^{\int_r^s p(\xi)d\xi} dr ds + b_k \right) \right] \Bigg\} e^{\int_{t_n}^{t} p(\xi)d\xi} \\
& + \int_{t_n}^{t} q(s) e^{\int_s^t p(\xi)d\xi} ds,
\end{aligned}
$$

for $t_n \le t \le t_{n+1}$. Therefore, the estimate (2.3) holds for $t_0 \le t \le t_{n+1}$. This completes the proof.

Remark 2.2. If $c_k = 0$ for all $k = 1, 2, \ldots$, then Theorem 2.1 reduces to Theorem A.

Corollary 2.3. *Let* (H_0) *and* (H_1) *hold. Suppose that* $p, q \in C[R_+, R]$ *and for* $k = 1, 2, \ldots$, $t \ge t_0$,

$$
m'(t) \le p(t)m(t) + q(t), \quad t \ne t_k, \tag{2.11}
$$

$$
m(t_k^+) \le c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m(s)ds + b_k, \tag{2.12}
$$

where $c_k \ge 0$, $0 \le \sigma_k \le \tau_k \le t_k - t_{k-1}$ *and* b_k *are constants.*

Then,

$$
\begin{aligned}
m(t) \le \Bigg\{ & m(t_0) \prod_{t_0<t_k<t} \left(c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} e^{\int_{t_{k-1}}^{s} p(\xi)d\xi} ds \right) \\
& + \sum_{t_0<t_k<t} \left[\prod_{t_k<t_j<t} \left(c_j \int_{t_j-\tau_j}^{t_j-\sigma_j} e^{\int_{t_{j-1}}^{s} p(\xi)d\xi} ds \right) \right. \\
& \left. \times \left(c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \int_{t_{k-1}}^{s} q(r) e^{\int_r^s p(\xi)d\xi} dr ds + b_k \right) \right] \Bigg\} e^{\int_{t_l}^{t} p(\xi)d\xi} \\
& + \int_{t_l}^{t} q(s) e^{\int_s^t p(\xi)d\xi} ds, \quad t \ge t_0.
\end{aligned} \tag{2.13}
$$

The following corollary will be used in our examples. For convenience, we set

$$
A_k = \frac{c_k}{p}\left(e^{-p\sigma_k} - e^{-p\tau_k}\right), \tag{2.14}
$$

$$B_k = \frac{c_k}{p}\left(e^{p(t_k-\sigma_k)}\int_{t_{k-1}}^{t_k-\sigma_k} q(r)e^{-pr}dr - e^{p(t_k-\tau_k)}\int_{t_{k-1}}^{t_k-\tau_k} q(r)e^{-pr}dr - \int_{t_k-\tau_k}^{t_k-\sigma_k} q(r)dr\right) + b_k. \tag{2.15}$$

Corollary 2.4. *Let* (H_0) *and* (H_1) *hold. Suppose that* $q \in C[R_+, R]$, *and for* $k = 1, 2,...,$ $t \geq t_0$,

$$m'(t) \leq pm(t) + q(t), \quad t \neq t_k, \tag{2.16}$$

$$m(t_k^+) \leq c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m(s)ds + b_k, \tag{2.17}$$

where $p \neq 0$, $c_k \geq 0$, $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$ *and* b_k *are constants.*
Then,

$$m(t) \leq m(t_0)\left(\prod_{t_0<t_k<t} A_k\right)e^{p(t-t_0)} + \sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t} A_j B_k e^{p(t-t_k)}\right) + \int_{t_l}^{t} q(s)e^{p(t-s)}ds, \tag{2.18}$$

for $t \geq t_0$ *where* A_k, B_k *are defined by* (2.14), (2.15), *respectively.*

Proof. By using Corollary 2.3 and reversing the order of double integration, we have the required result.

Corollary 2.5. *Let* (H_0) *and* (H_1) *hold. Suppose that* $q \in C[R_+, R]$, *and for* $k = 1, 2,...,t \geq t_0$,

$$m'(t) \leq q(t), \quad t \neq t_k, \tag{2.19}$$

$$\Delta m(t_k) \leq c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m(s)ds + b_k, \tag{2.20}$$

where $\Delta m(t_k) = m(t_k^+) - m(t_k)$, $c_k \geq 0$, $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$ *and* b_k *are constants.*
Then,

$$\begin{aligned} m(t) \leq{}& m(t_0)\left(\prod_{t_0<t_k<t}[1 + c_k(\tau_k - \sigma_k)]\right) + \sum_{t_0<t_k<t}\Bigg[\prod_{t_k<t_j<t}(1 + c_j(\tau_j - \sigma_j)) \\ &\times\Bigg([1 + c_k(\tau_k - \sigma_k)]\int_{t_{k-1}}^{t_k-\tau_k} q(s)ds + \int_{t_k-\tau_k}^{t_k-\sigma_k}[1 + c_k(t_k - \sigma_k - s)]q(s)ds \\ &+ \int_{t_k-\sigma_k}^{t_k} q(s)ds + b_k\Bigg)\Bigg] + \int_{t_l}^{t} q(s)ds, \quad t \geq t_0. \end{aligned} \tag{2.21}$$

Proof. By setting $p(t) \equiv 0$ and $d_k = 1(k = 1, 2,...)$ in Theorem 2.1 and reversing the order of double integration, we have the required result.

Next, we give an application of Theorem 2.1 to the determination of a bound for the solutions of impulsive integral inequalities with integral jump conditions.

Theorem 2.6. *Assume that* (H_0) *and* (H_1) *hold. Suppose that* $p \in C[R_+, R_+]$ *and for* $k = 1, 2, \ldots$

$$m(t) \leq C + \int_{t_0}^{t} p(s)m(s)ds + \sum_{t_0<t_k<t} \beta_k m(t_k) + \sum_{t_0<t_k<t} \alpha_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m(s)ds, \quad t \geq t_0, \tag{2.22}$$

where α_k, $\beta_k \geq 0$, $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$ *and C are constants. Then*

$$m(t) \leq C \prod_{t_0<t_k<t} \left[(1+\beta_k)e^{\int_{t_{k-1}}^{t_k} p(\xi)d\xi} + \alpha_k \int_{t_k-\tau_k}^{t_k-\sigma_k} e^{\int_{t_{k-1}}^{s} p(\xi)d\xi} ds \right] e^{\int_{t_l}^{t} p(\xi)d\xi}, \quad t \geq t_0. \tag{2.23}$$

Proof. Defining a function $v(t)$ by the right side of (2.22), we have

$$\begin{aligned} v'(t) &= p(t)m(t), \quad t \neq t_k, \quad v(t_0) = C, \\ v(t_k^+) &= v(t_k) + \beta_k m(t_k) + \alpha_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m(s)ds. \end{aligned}$$

Since $m(t) \leq v(t)$, we get

$$\begin{aligned} v'(t) &\leq p(t)v(t), \quad t \neq t_k, \quad v(t_0) = C, \\ v(t_k^+) &\leq (1+\beta_k)v(t_k) + \alpha_k \int_{t_k-\tau_k}^{t_k-\sigma_k} v(s)ds. \end{aligned}$$

Applying Theorem 2.1, we obtain

$$v(t) \leq C \prod_{t_0<t_k<t} \left[(1+\beta_k)e^{\int_{t_{k-1}}^{t_k} p(\xi)d\xi} + \alpha_k \int_{t_k-\tau_k}^{t_k-\sigma_k} e^{\int_{t_{k-1}}^{s} p(\xi)d\xi} ds \right] e^{\int_{t_l}^{t} p(\xi)d\xi}, \quad t \geq t_0,$$

which results in (2.23).

Theorem 2.7. *Assume that* (H_0) *and* (H_1) *hold. Suppose that* $p \in C[R_+, R_+]$, $h \in PC[R_+, R]$ *and for* $k = 1, 2, \ldots$

$$m(t) \leq h(t) + \int_{t_0}^{t} p(s)m(s)ds + \sum_{t_0<t_k<t} \beta_k m(t_k) + \sum_{t_0<t_k<t} \alpha_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m(s)ds, \quad t \geq t_0, \tag{2.24}$$

where α_k, $\beta_k \geq 0$ *and* $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$ *are constants.*

Then,

$$\begin{aligned} m(t) \leq h(t) + &\left\{ \sum_{t_0<t_k<t} \left[\prod_{t_k<t_j<t} \left((1+\beta_j)e^{\int_{t_{j-1}}^{t_j} p(\xi)d\xi} + \alpha_j \int_{t_j-\tau_j}^{t_j-\sigma_j} e^{\int_{t_{j-1}}^{s} p(\xi)d\xi} ds \right) \right.\right. \\ &\times \left((1+\beta_k) \int_{t_{k-1}}^{t_k} p(s)h(s)e^{\int_s^{t_k} p(\xi)d\xi} ds + \alpha_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \int_{t_{k-1}}^{s} p(r)h(r)e^{\int_r^{s} p(\xi)d\xi} drds \right. \\ &\left.\left.\left. + \beta_k h(t_k) + \alpha_k \int_{t_k-\tau_k}^{t_k-\sigma_k} h(s)ds \right) \right] \right\} e^{\int_{t_l}^{t} p(\xi)d\xi} + \int_{t_l}^{t} p(s)h(s)e^{\int_s^{t} p(\xi)d\xi} ds, \quad t \geq t_0. \end{aligned} \tag{2.25}$$

Proof. Setting

$$v(t) = \int_{t_0}^{t} p(s)m(s)ds + \sum_{t_0<t_k<t} \beta_k m(t_k) + \sum_{t_0<t_k<t} \alpha_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m(s)ds,$$

and from the fact that $m(t) \le h(t) + v(t)$, we obtain

$$v'(t) \le p(t)v(t) + p(t)h(t), \quad t \ne t_k, \quad v(t_0) = 0,$$

$$v(t_k^+) \le (1+\beta_k)v(t_k) + \alpha_k \int_{t_k-\tau_k}^{t_k-\sigma_k} v(s)ds + \beta_k h(t_k) + \alpha_k \int_{t_k-\tau_k}^{t_k-\sigma_k} h(s)ds.$$

Using Theorem 2.1 together with $m(t) \le h(t) + v(t)$, we then obtain the estimate (2.25).

Remark 2.8. If $\alpha_k = 0$ for all $k = 1, 2,...$, then Theorem 2.6 and Theorem 2.7 are reduced to the Theorems 1.5.1 and 1.5.2 in [1], respectively.

3 Some examples

In this section, two applications of impulsive differential and impulsive integral inequalities with integral jump conditions are given.

Corollary 3.1. Assume that $u \in PC^1[J, R]$ satisfies

$$\begin{cases} u'(t) - Mu(t) + a(t) \le 0, & t \ne t_k, \quad t \in J = [0, T], \\ u(t_k^+) \le c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} u(s)ds, & k = 1, \dots, n, \\ u(0) = u(T) + \lambda, \end{cases} \tag{3.1}$$

where $M > 0$, $a \in C[R_+, R_+]$, $0 < t_1 < t_2 < ... < t_n < T$. $c_k \ge 0$, $0 \le \sigma_k \le \tau_k \le t_k - t_{k-1}$, $k = 1, 2,..., n$.

Suppose in addition that

(D$_1$) $\prod_{k=1}^{n} \frac{c_k}{M}(e^{-\sigma_k M} - e^{-\tau_k M}) < e^{-MT}$,

(D_2)

$$e^{(t_k-\tau_k)M} \int_{t_{k-1}}^{t_k-\tau_k} a(r)e^{-Mr}dr + \int_{t_k-\tau_k}^{t_k-\sigma_k} a(r)dr \le e^{(t_k-\sigma_k)M} \int_{t_{k-1}}^{t_k-\sigma_k} a(r)e^{-Mr}dr, \quad k = 1, 2, \dots, n,$$

(D_3) $\lambda \le \int_{t_n}^{T} a(s)e^{M(T-s)}ds$.

Then $u(t) \le 0$ for $t \in [0, T]$.

Proof. By Corollary 2.4 for $t \in [0, T]$ we can write that

$$u(t) \le u(0) \left(\prod_{t_0<t_k<t} \overline{A}_k \right) e^{Mt} + \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} \overline{A}_j \right) \overline{B}_k e^{M(t-t_k)} - \int_{t_l}^{t} a(s)e^{M(t-s)}ds,$$

where

$$\overline{A}_k = \frac{c_k}{M}(e^{-\sigma_k M} - e^{-\tau_k M}) \ge 0,$$

and

$$\overline{B}_k = \frac{c_k}{M}\left(e^{(t_k-\tau_k)M}\int\limits_{t_{k-1}}^{t_k-\tau_k} a(r)e^{-Mr}dr + \int\limits_{t_k-\tau_k}^{t_k-\sigma_k} a(r)dr \right.$$
$$\left. - e^{(t_k-\sigma_k)M}\int\limits_{t_{k-1}}^{t_k-\sigma_k} a(r)e^{-Mr}dr\right), \quad k = 1, 2, \ldots, n.$$

Condition (D_2) implies that $\overline{B}_k \leq 0$ for k = 1, 2,..., n. Then, it is sufficient to show that $u(0) \leq 0$. For $t = T$ we have

$$u(T) \leq u(0)\left(\prod_{k=1}^{n}\overline{A}_k\right)e^{MT} + \sum_{t_0<t_k<T}\left(\prod_{t_k<t_j<T}\overline{A}_j\right)\overline{B}_k e^{M(T-t_k)} - \int\limits_{t_n}^{T} a(s)e^{M(T-s)}ds.$$

By the conditions (D_1) and (D_3), we see that

$$u(0)\left[1 - \left(\prod_{k=1}^{n}\overline{A}_k\right)e^{MT}\right] \leq \lambda + \sum_{t_0<t_k<T}\left(\prod_{t_k<t_j<T}\overline{A}_j\right)\overline{B}_k e^{M(T-t_k)} - \int\limits_{t_n}^{T} a(s)e^{M(T-s)}ds$$
$$\leq 0,$$

which implies that $u(0) \leq 0$.

Corollary 3.2. Let $v \in PC^1[R_+, R]$ such that

$$\begin{cases} v'(t) = f(t, v(t)), \quad t \neq t_k, \quad t \in [t_0, \infty), \\ \Delta v(t_k) = I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} v(s)ds\right), \quad k = 1, 2, \ldots, \\ v(t_0) = v_0, \end{cases} \tag{3.2}$$

where $f \in C(R \times R, R)$, $I_k \in C(R, R)$, $0 \leq t_0 < t_1 < t_2 < \ldots$, $\lim_{k\to\infty} t_k = \infty$, $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$, k = 1, 2,.... Assume that

(D_4) there exists a constant $N > 0$, such that

$$|f(t, v(t))| \leq N|v(t)| \quad \text{for} \quad t \geq t_0,$$

(D_5) there exist constants $L_k \geq 0$ such that

$$|I_k(x)| \leq L_k|x|, \quad x \in R, \quad k = 1, 2, \ldots.$$

Then the following inequality is valid

$$|v(t)| \leq |v_0|\prod_{t_0<t_k<t}\left[1 + \frac{L_k}{N}(e^{-\sigma_k N} - e^{-\tau_k N})\right]e^{(t-t_0)N}, \quad t \geq t_0. \tag{3.3}$$

Proof. The solution $v(t)$ of problem (3.2) satisfies the equation

$$v(t) = v(t_0) + \int\limits_{t_0}^{t} f(s, v(s))ds + \sum_{t_0<t_k<t} I_k\left(\int\limits_{t_k-\tau_k}^{t_k-\sigma_k} v(s)ds\right).$$

From the hypothesis (D_4), (D_5) it follows for $t \geq t_0$ that

$$\begin{aligned}\left|v(t)\right| &\leq |v_0| + \int_{t_0}^{t} \left|f(s, v(s))\right| ds + \sum_{t_0<t_k<t} \left| I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} v(s)ds\right)\right| \\ &\leq |v_0| + \int_{t_0}^{t} N\left|v(s)\right| ds + \sum_{t_0<t_k<t} L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \left|v(s)\right| ds.\end{aligned}$$

Hence Theorem 2.6 yields the estimate

$$\left|v(t)\right| \leq |v_0| \prod_{t_0<t_k<t} \left[e^{(t_k-t_{k-1})N} + L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} e^{(s-t_{k-1})N} ds\right] e^{(t-t_l)N}.$$

Therefore, the inequality (3.3) holds for $t \geq t_0$ and the proof is complete.

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