Journal of Inequalities and Applications
a SpringerOpen Journal

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Boundedness of Littlewood-Paley operators and their commutators on Herz-Morrey spaces with variable exponent

Lijuan Wang and Shuangping Tao*

*Correspondence:
taosp@nwnu.edu.cn
College of Mathematics and Statistics Science, Northwest Normal University, Lanzhou, Gansu 730070, P.R. China

Abstract

The aim of this paper is to establish the vector-valued inequalities for Littlewood-Paley operators, including the Lusin area integrals, the Littlewood-Paley g-functions and g_μ^*-functions, and their commutators on the Herz-Morrey spaces with variable exponent $M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$. By applying the properties of $L^{p(\cdot)}(\mathbb{R}^n)$ spaces and the vector-valued inequalities for Littlewood-Paley operators and their commutators generated by BMO function on $L^{p(\cdot)}(\mathbb{R}^n)$, the boundedness of the vector-valued Littlewood-Paley operators and their commutators is obtained on $M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$.

MSC: 42B20; 42B25; 42B35

Keywords: vector-valued inequality; Littlewood-Paley operators; Herz-Morrey spaces with variable exponent; BMO spaces

1 Introduction

Let $\psi \in L^1(\mathbb{R}^n)$ and satisfies

(i) $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$,

(ii) $|\psi(x)| \le C(1+|x|)^{-n-\varepsilon}$,

(iii) $|\psi(x+y)-\psi(x)| \le C|y|^\gamma(1+|x|)^{-n-\gamma-\varepsilon}$, $|x| \ge 2|y|$,

where C, ε, γ are all positive constants. Denote $\psi_t(x) = t^{-n}\psi(x/t)$ with $t > 0$ and $x \in \mathbb{R}^n$. Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Lusin area integral of f is defined by

$$S_{\psi,a}(f)(x) = \left(\int_{\Gamma_a(x)} \left|\psi_t * f(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2},$$

where $\Gamma_a(x)$ denote the usual cone of aperture one

$$\Gamma_a(x) = \left\{(t,y) \in \mathbb{R}^{n+1}_+ : |y-x| < at, a \ge 1\right\}.$$

As $a = 1$, we denote $S_{\psi,a}(f)$ as $S_\psi(f)$.

Now let us turn to introduce the other two Littlewood-Paley operators. It is well known that the Littlewood-Paley operators include also the Littlewood-Paley g-functions and the Littlewood-Paley g_μ^*-functions besides the Lusin area integrals. The Littlewood-Paley g-functions, which can be viewed as a 'zero-aperture' version of S_ψ, and g_μ^*-functions, which can be viewed as a 'infinite-aperture' version of S_ψ, are defined, respectively, by

$$g(f)(x) = \left(\int_0^\infty \left|\psi_t * f(y)\right|^2 \frac{dt}{t}\right)^{1/2},$$

$$g_\mu^*(f)(x) = \left(\int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{\mu n} \left|\psi_t * f(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2}, \quad \mu > 0.$$

If we take ψ to be the poisson kernel, then the functions defined above are the classical Littlewood-Paley operators.

Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $m \geq 1$, the corresponding m-order commutators of Littlewood-Paley operators above generated by a function b are defined by

$$\left[b^m, g_\psi\right](f)(x) = \left(\int_0^\infty \left|\int_{\mathbb{R}^n} \left[b(x)-b(y)\right]^m \psi_t(x-y) f(y)\,dy\right|^2 \frac{dt}{t}\right)^{1/2},$$

$$\left[b^m, S_{\psi,a}\right](f)(x) = \left(\int_{\Gamma_a(x)} \left|\int_{\mathbb{R}^n} \left[b(x)-b(z)\right]^m \psi_t(y-z) f(z)\,dz\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2}$$

and

$$\left[b^m, g_\mu^*\right](f)(x) = \left(\int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{\mu n} \left|\int_{\mathbb{R}^n} \left[b(x)-b(z)\right]^m \psi_t(y-z) f(z)\,dz\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

$$\mu > 0.$$

The Littlewood-Paley operators are very important objects in the study of harmonic analysis. They play very important roles in harmonic analysis and PDE (see [1–3]), so it is natural and meaningful to consider the boundedness of Littlewood-Paley operators and their commutators. Lu and Yang investigated the behavior of Littlewood-Paley operators in the space $\mathrm{CBMO}_p(\mathbb{R}^n)$ in [4]. In 2005, Zhang and Liu proved the commutator $[b, g_\psi]$ is bounded on $L^p(\omega)$ (see [5]). In 2009, Xue and Ding gave the weighted estimate for Littlewood-Paley operators and their commutators (see [6]). There are some other results about Littlewood-Paley operators in [7–9].

On the other hand, Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ become one of the important function spaces due to the fundamental paper [10] by Kováčik and Rákosník. In the past 20 years, the theory of these spaces has made progress rapidly, and the study of it has many applications in fluid dynamics, elasticity, calculus of variations and differential equations with non-standard growth conditions (see [11–15]). In [16], Cruz-Uribe *et al.* proved the extrapolation theorem which leads the boundedness of some classical operators including the commutator on $L^{p(\cdot)}(\mathbb{R}^n)$. Karlovich and Lerner also independently obtained the boundedness of the singular integral commutators in [17]. Recently, Izuki considered the boundedness of vector-valued sub-linear operators and fractional integrals on Herz-Morrey spaces with variable exponent in [18] and [19], respectively.

Inspired by the above works, in this paper we will consider the vector-valued inequalities of the Littlewood-Paley operators and their m-order commutators on Herz-Morrey spaces with variable exponent. To do this, we need recall some definitions about the spaces with variable exponent.

Let E be a Lebesgue measurable set in $\mathbb{R}^n$ with measure $|E| > 0$.

Definition 1.1 [10] Let $p(\cdot): E \to [1,\infty)$ be a measurable function.

The Lebesgue space with variable exponent $L^{p(\cdot)}(E)$ is defined by

$$L^{p(\cdot)}(E) = \left\{f \text{ is measurable}: \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx < \infty \text{ for some constant } \eta > 0\right\}.$$

The space $L^{p(\cdot)}_{\mathrm{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(E) = \left\{f \text{ is measurable}: f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset E\right\}.$$

The Lebesgue space $L^{p(\cdot)}(E)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{\eta > 0: \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx \le 1\right\}.$$

We denote

$$p_- := \operatorname{ess\,inf}\left\{p(x): x \in E\right\}, \qquad p_+ := \operatorname{ess\,sup}\left\{p(x): x \in E\right\}.$$

Then $\mathcal{P}(E)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

Let M be the Hardy-Littlewood maximal operator. We denote $\mathcal{B}(E)$ to be the set of all functions $p(\cdot) \in \mathcal{P}(E)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(E)$.

Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $C_k = B_k \backslash B_{k-1}$, $k \in \mathbb{Z}$, $\chi_k = \chi_{C_k}$.

Definition 1.2 [18] Let $\alpha \in \mathbb{R}^n$, $0 \le \lambda < \infty$, $0 < p < \infty$, and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz-Morrey spaces with variable exponent $M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ is defined by

$$M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n) = \left\{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

Remark 1.1 Comparing the homogeneous Herz-Morrey spaces with variable exponent $M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ with the homogeneous Herz spaces with variable exponent $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ (see [20]), where $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \left\{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{\infty} 2^{\alpha kp} \|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{1/p} < \infty\right\}.$$

Obviously, $M\dot{K}^{\alpha,0}_{p,q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

We now make some conventions. Throughout this paper, given a function f, we denote the mean value of f on E by $f_E =: \frac{1}{|E|}\int_E f(x)\,dx$. $p'(\cdot)$ means the conjugate exponent of $p(\cdot)$, namely $1/p(x) + 1/p'(x) = 1$ holds. C always means a positive constant independent of the main parameters and may change from one occurrence to another.

2 Preliminary lemmas

In this section, we need some conclusions which will be used in the proofs of our main results.

Lemma 2.1 [10] (Generalized Hölder's Inequality) *Let* $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

(1) *For any* $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $g \in L^{p'(\cdot)}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \left|f(x)g(x)\right| dx \le C_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $C_p = 1 + 1/p_- - 1/p_+$.

(2) *For any* $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$, $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$, *when* $1/p(x) = 1/p_1(x) + 1/p_2(x)$, *we have*

$$\left\|f(x)g(x)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C_{p_1,p_2} \|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p_2(\cdot)}(\mathbb{R}^n)},$$

where $C_{p_1,p_2} = (1 + 1/p_{1_-} - 1/p_{1_+})^{1/p_-}$.

Lemma 2.2 [17] *If* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, *then there exist constants* $\delta_1, \delta_2, C > 0$, *such that for all balls* $B \subset \mathbb{R}^n$ *and all measurable subsets* $S \subset B$,

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\frac{|B|}{|S|}, \qquad \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \qquad \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2}.$$

Lemma 2.3 [18] *If* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, *then there exist constants* $C > 0$, *such that for all balls* $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C.$$

Lemma 2.4 [20] *Let* $b \in BMO(\mathbb{R}^n)$, *m is a positive integer, there exist constants* $C > 0$, *such that for any* $k, j \in \mathbb{Z}$ *with* $k > j$,

(1) $C^{-1}\|b\|_*^m \le \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_B)^m \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*^m$;

(2) $\|(b - b_{B_j})^m \chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(k - j)^m \|b\|_*^m \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Lemma 2.5 [21] *Given an open set* $\Omega \subset \mathbb{R}^n$, *suppose that* $p(\cdot) \in \mathcal{P}(\Omega)$ *satisfies*

$$\left|p(x) - p(y)\right| \le \frac{-C}{\ln(|x - y|)}, \quad x, y \in \Omega, |x - y| \le 1/2, \tag{2.1}$$

$$\left|p(x) - p(y)\right| \le \frac{C}{\ln(e + |x|)}, \quad x, y \in \Omega, |y| \ge |x|. \tag{2.2}$$

Then $p(\cdot) \in \mathcal{B}(\Omega)$, *that is, the Hardy-Littlewood maximal operator is bounded on* $L^{p(\cdot)}(\Omega)$.

Let A_p be the classical Muckenhoupt weighted class.

Lemma 2.6 [16]

(1) *Given a family* $\mathcal{F}$ *and open set* $\Omega \subset \mathbb{R}^n$, *assume that for some* $p_0 : 1 < p_0 < \infty$, *and for every* $\omega \in A_{p_0}$,

$$\int_\Omega f(x)^{p_0} \omega(x)\, dx \le C \int_\Omega g(x)^{p_0} \omega(x)\, dx, \quad (f_j, g_j) \in \mathcal{F}.$$

Let $p(\cdot) \in \mathcal{P}(\Omega)$ be such that there exists $1 < p_1 < p_-$, with $(p(\cdot)/p_1)' \in \mathcal{B}(\Omega)$. Then for every $1 < q < \infty$ and sequence $\{(f_j, g_j)\}_j \in \mathcal{F}$,

$$\left\| \left(\sum_j (f_j)^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\Omega)} \leq C \left\| \left(\sum_j (g_j)^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\Omega)}.$$

(2) *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then for all $1 < q < \infty$,*

$$\left\| \left(\sum_j (Mf_j)^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \left\| \left(\sum_j (|f_j|)^q \right)^{1/q} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.7 [6] *Let $\psi \in L^1(\mathbb{R}^n)$ satisfies* (i)-(iii), $\mu > 2$, $0 < \gamma < \min\{(\mu - 2)n/2, \varepsilon\}$, *there exists a constant C and for all bounded functions f with compact support*:

(1) *If $0 < p < \infty$, $\omega \in A_\infty$, then*

$$\int_{\mathbb{R}^n} \left[g_\mu^*(f)(x) \right]^p \omega(x)\, dx \leq C[\omega]_{A_\infty}^p \int_{\mathbb{R}^n} \left[M(f)(x) \right]^p \omega(x)\, dx.$$

(2) *If $1 < p < \infty$, $\omega \in A_p$, $b \in BMO$, then*

$$\int_{\mathbb{R}^n} \left(\left[b^m, g_\mu^* \right](f)(x) \right)^p \omega(x)\, dx \leq C \int_{\mathbb{R}^n} \left| (f)(x) \right|^p \omega(x)\, dx.$$

Remark 2.1

(1) If we replace g_μ^* with $S_{\psi,a}$, then the results of Lemma 2.7 also hold. That is due to for any $x \in \mathbb{R}^n$, $\mu \geq 1$, the inequalities $S_{\psi,a}(f)(x) \leq C g_\mu^*(f)(x)$ and $[b^m, S_{\psi,a}](f)(x) \leq C[b^m, g_\mu^*](f)(x)$ hold.

(2) According to the argument in [6], we may deduce that Lemma 2.7 is also suit to g_ψ.

Combining Lemmas 2.6-2.7 and Remark 2.1, we get the following conclusions.

Lemma 2.8 *Let $\psi \in L^1(\mathbb{R}^n)$ satisfies* (i)-(iii). *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $1 < q < \infty$, then for all bounded compactly support functions f_j such that $\{f_j\}_{j=1}^\infty \in L^{p(\cdot)}(l^q)$, that is $\|(\sum_j (|f_j|)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty$, the below vector-valued inequalities hold*:

(1) $\|(\sum_j (S_{\psi,a}(f)_j)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|(\sum_j (|f_j|)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

(2) $\|(\sum_j (g_\psi(f)_j)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|(\sum_j (|f_j|)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

(3) $\|(\sum_j (g_\mu^*(f)_j)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|(\sum_j (|f_j|)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

where $\mu > 2$, $0 < \gamma < \min\{(\mu - 2)n/2, \varepsilon\}$.

Lemma 2.9 *Let $b \in BMO$, $\psi \in L^1(\mathbb{R}^n)$ satisfies* (i)-(iii), $m \in \mathbb{N}\backslash\{0\}$. *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $1 < q < \infty$, then for all bounded compactly support functions f_j such that $\{f_j\}_{j=1}^\infty \in L^{p(\cdot)}(l^q)$, that is $\|(\sum_j (|f_j|)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty$, the following vector-valued inequalities hold*:

(1) $\|(\sum_j ([b^m, S_{\psi,a}](f)_j)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|(\sum_j (|f_j|)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

(2) $\|(\sum_j ([b^m, g_\psi](f)_j)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|(\sum_j (|f_j|)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

(3) $\|(\sum_j ([b^m, g_\mu^*](f)_j)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|(\sum_j (|f_j|)^q)^{1/q}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$,

where $\mu > 2$, $0 < \gamma < \min\{(\mu - 2)n/2, \varepsilon\}$.

3 Main results and their proofs

In this section, we will establish the vector-valued inequalities of the Littlewood-Paley operators, $S_{\psi,a}, g_\psi, g_\mu^*$, and their m-order commutators on Herz-Morrey spaces with variable exponent, respectively.

We begin with the study of the vector-valued inequalities of the Littlewood-Paley operators on $M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$.

Theorem 3.1 *Suppose that* $\psi \in L^1(\mathbb{R}^n)$ *satisfies* (i)-(iii), $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. *Let* $0 < p < \infty$, $1 < r < \infty$, $\lambda - n\delta_2 < \alpha < \lambda + n\delta_1$, *where* δ_1, δ_2 *is the constant in Lemma* 2.2. *Then for all function sequences* $\{f_h\}_{h=1}^\infty : \|\{\sum_h |f_h|^r\}^{1/r}\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} < \infty$, *the following vector-valued inequalities hold*:

(1) $\|\{\sum_h |S_{\psi,a}(f_h)|^r\}^{1/r}\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} \le C\|\{\sum_h |f_h|^r\}^{1/r}\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}$,

(2) $\|\{\sum_h |g_\psi(f_h)|^r\}^{1/r}\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} \le C\|\{\sum_h |f_h|^r\}^{1/r}\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}$,

(3) $\|\{\sum_h |g_\mu^*(f_h)|^r\}^{1/r}\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} \le C\|\{\sum_h |f_h|^r\}^{1/r}\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}$,

where $\mu > 3 + 2(\varepsilon + \gamma)/n$, $0 < \gamma < \min\{(\mu - 2)n/2, \varepsilon\}$, *and the constant C is independent of* $\{f_h\}_{h=1}^\infty$.

Proof Firstly, we consider the inequality (1). For any function sequence $\{f_h\}_h$ satisfies $\|\|\{f_h\}_h\|_{l^r}\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} < \infty$, we write

$$f_h(x) = \sum_{j=-\infty}^{\infty} f_h(x)\chi_j(x) \triangleq \sum_{j=-\infty}^{\infty} f_h^j(x).$$

Thus

$$\begin{aligned}
&\left\|\left\{\sum_h \left|S_{\psi,a}(f_h)\right|^r\right\}^{1/r}\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} \\
&\quad = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left\|\left\{\sum_h \left|\sum_{j=-\infty}^{\infty} S_{\psi,a}\left(f_h^j\right)\right|^r\right\}^{1/r}\chi_k\right\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} \\
&\quad \le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=-\infty}^{\infty}\left\|\left\{\sum_h \left|S_{\psi,a}\left(f_h^j\right)\right|^r\right\}^{1/r}\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\quad \le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}\left\|\left\{\sum_h \left|S_{\psi,a}\left(f_h^j\right)\right|^r\right\}^{1/r}\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\qquad + C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k-2}^{k+2}\left\|\left\{\sum_h \left|S_{\psi,a}\left(f_h^j\right)\right|^r\right\}^{1/r}\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\qquad + C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+3}^{\infty}\left\|\left\{\sum_h \left|S_{\psi,a}\left(f_h^j\right)\right|^r\right\}^{1/r}\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\quad \triangleq D_1 + D_2 + D_3.
\end{aligned}$$

For the term D_2, noting that $\operatorname{supp} f_h^j \subset C_j$, we can easily obtain, by Lemma 2.8,

$$\begin{aligned} D_2 &\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{j=k-2}^{k+2} \left\| \left\{ \sum_h |f_h^j|^r \right\}^{1/r} \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\ &\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left\| \left\{ \sum_h |f_h|^r \right\}^{1/r} \chi_k \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} \\ &= C \left\| \left\{ \sum_h |f_h|^r \right\}^{1/r} \right\|_{M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}. \end{aligned}$$

We now turn to estimate D_1. Let $\Gamma' = \{y \in \mathbb{R}^n : |y-x| < at, y \le 2^{j+1}\}$, $\Gamma'' = \{y \in \mathbb{R}^n : |y-x| < at, y > 2^{j+1}\}$. We write

$$\begin{aligned} S_{\psi,a}(f_h^j)(x) &= \left(\int_{\Gamma_a(x)} \left| \int_{\mathbb{R}^n} t^{-n} \psi\left(\frac{y-z}{t}\right) f_h^j(z)\,dz \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2} \\ &\le \left(\int_0^\infty \int_{\Gamma'} \left| \int_{\mathbb{R}^n} t^{-n} \psi\left(\frac{y-z}{t}\right) f_h^j(z)\,dz \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2} \\ &\quad + \left(\int_0^\infty \int_{\Gamma''} \left| \int_{\mathbb{R}^n} t^{-n} \psi\left(\frac{y-z}{t}\right) f_h^j(z)\,dz \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2} \\ &\triangleq I + II. \end{aligned}$$

For I, observe that as $a \ge 1$, $x \in C_k$, $j \le k-3$, $z \in C_j$, then we have $t > \frac{|x-y|}{a} \ge \frac{|x|-|y|}{a} > \frac{2^{k-1}-2^{j+1}}{a} \ge \frac{2^{j+1}}{a}$, and $t + |y-z| \ge \frac{|x-y|}{a} + |y-z| \ge \frac{|x-y|+|y-z|}{a} \ge \frac{|x|-|z|}{a} \ge \frac{3|x|}{4a}$. Hence, it follows from the condition (ii) that

$$\begin{aligned} I &\le C \left\{ \int_{\frac{2^{j+1}}{a}}^\infty \int_{\Gamma'} \left(\int_{\mathbb{R}^n} |f_h^j(z)| t^{-n} \left(1 + \frac{|y-z|}{t}\right)^{-n-\varepsilon} dz \right)^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2} \\ &\le C \left\{ \int_{\frac{2^{j+1}}{a}}^\infty \int_{\Gamma'} \left(\int_{\mathbb{R}^n} |f_h^j(z)| \frac{t^\varepsilon}{(t+|y-z|)^{n+\varepsilon}}\,dz \right)^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2} \\ &\le C \left(\frac{4a}{3|x|} \right)^n \left\{ \int_{\frac{2^{j+1}}{a}}^\infty t^{-n-1}\,dt \int_{|y|<2^{j+1}} dy \right\}^{1/2} \|f_h^j\|_{L^1(\mathbb{R}^n)} \\ &\le Ca^{3n/2} |x|^{-n} \|f_h^j\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

For the estimate of II, denote

$$\begin{aligned} II &\le C \left\{ \int_0^\infty \int_{\Gamma''} \left(\int_{\mathbb{R}^n} |f_h^j(z)| t^{-n} \left| \psi\left(\frac{y-z}{t}\right) - \psi\left(\frac{y}{t}\right) \right| dz \right)^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2} \\ &\quad + C \left\{ \int_0^\infty \int_{\Gamma''} \left(\int_{\mathbb{R}^n} |f_h^j(z)| t^{-n} \left| \psi\left(\frac{y}{t}\right) \right| dz \right)^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2} \\ &\triangleq II_1 + II_2. \end{aligned}$$

Noting that $y > 2^{j+1} \geq 2|z|$, by the condition (iii), we have

$$\begin{aligned}
II_1 &\leq C\left\{\int_0^\infty \int_{\Gamma''} \left(\int_{\mathbb{R}^n} \left|f_h^j(z)\right| t^{-n}\left(\frac{|z|}{t}\right)^\gamma \left(1+\frac{|y|}{t}\right)^{-n-\gamma-\varepsilon} dz\right)^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2} \\
&\leq C\left\{\int_0^\infty \int_{\Gamma''} \frac{t^{2\varepsilon-n-1}}{(t+|y|)^{2(n+\gamma+\varepsilon)}}\,dy\,dt\right\}^{1/2} \int_{\mathbb{R}^n} \left|f_h^j(z)\right| \cdot |z|^\gamma\,dz \\
&\leq C2^{j\gamma}\left\{\int_0^{|x|} \int_{\Gamma''} \frac{t^{2\varepsilon-n-1}}{(t+|y|)^{2(n+\gamma+\varepsilon)}}\,dy\,dt \right. \\
&\quad \left. + \int_{|x|}^\infty \int_{\Gamma''} \frac{t^{2\varepsilon-n-1}}{(t+|y|)^{2(n+\gamma+\varepsilon)}}\,dy\,dt\right\}^{1/2} \left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

Since $t+|y| > \frac{|x-y|}{a} + |y| \geq \frac{|x-y|+|y|}{a} \geq \frac{|x|}{a}$, then we obtain

$$\begin{aligned}
&\int_0^{|x|} \int_{\Gamma''} \frac{t^{2\varepsilon-n-1}}{(t+|y|)^{2(n+\gamma+\delta)}}\,dy\,dt \\
&\quad \leq C\left(\frac{a}{|x|}\right)^{2(n+\gamma+\varepsilon)} \int_0^{|x|} \int_{|y-x|<at} t^{2\varepsilon-n-1}\,dy\,dt \\
&\quad \leq C\left(\frac{a}{|x|}\right)^{2(n+\gamma+\varepsilon)} \int_0^{|x|} t^{2\delta-n-1} a^n t^n\,dt \\
&\quad \leq Ca^{3n+2\gamma+2\varepsilon}|x|^{-2(n+\gamma)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{|x|}^\infty \int_{\Gamma''} \frac{t^{2\varepsilon-n-1}}{(t+|y|)^{2(n+\gamma+\delta)}}\,dy\,dt \\
&\quad \leq C\int_{|x|}^\infty \int_{|y-x|<at} t^{-3n-2\gamma-1}\,dy\,dt \\
&\quad \leq Ca^n \int_{|x|}^\infty t^{-2n-2\gamma-1}\,dy\,dt \leq Ca^n|x|^{-2(n+\gamma)}.
\end{aligned}$$

Thus, we have

$$II_1 \leq Ca^{3n/2+\gamma+\varepsilon}|x|^{-(n+\gamma)}2^{j\gamma}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)} \leq Ca^{3n/2+\gamma+\varepsilon}|x|^{-n}\left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}.$$

On the other hand, by the condition (ii) and $t+|y| > \frac{|x|}{a}$, we deduce

$$\begin{aligned}
II_2 &\leq C\left\{\int_0^\infty \int_{\Gamma''} \left(\int_{\mathbb{R}^n} \left|f_h^j(z)\right| t^{-n}\left(1+\frac{|y|}{t}\right)^{-n-\varepsilon} dz\right)^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2} \\
&\leq C\left\{\int_0^\infty \int_{\Gamma''} \frac{t^{2\varepsilon-n-1}}{(t+|y|)^{2(n+\varepsilon)}}\,dy\,dt\right\}^{1/2} \left\|f_h^j\right\|_{L^1(\mathbb{R}^n)} \\
&\leq C\left\{\int_0^{|x|} \int_{\Gamma''} \frac{t^{2\varepsilon-n-1}}{(t+|y|)^{2(n+\varepsilon)}}\,dy\,dt \right. \\
&\quad \left. + \int_{|x|}^{|\infty|} \int_{\Gamma''} \frac{t^{2\varepsilon-n-1}}{(t+|y|)^{2(n+\varepsilon)}}\,dy\,dt\right\}^{1/2} \left\|f_h^j\right\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

Thus, similar to II_1, we get

$$II_2 \le Ca^{3n/2+\varepsilon}|x|^{-n}\|f_h^j\|_{L^1(\mathbb{R}^n)}.$$

Combining the estimates of I, II, we obtain

$$S_{\psi,a}(f_h^j)(x) \le Ca^{3n/2+\gamma+\varepsilon}|x|^{-n}\|f_h^j\|_{L^1(\mathbb{R}^n)}.$$

Therefore, by using the above inequality and Minkowski's inequality, we have

$$\begin{aligned}
D_1 &\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\right.\\
&\quad\left.\times\left(\sum_{j=-\infty}^{k-3}\left\|\left\{\sum_h\left(a^{3n/2+\gamma+\varepsilon}|x|^{-n}\|f_h^j\|_{L^1(\mathbb{R}^n)}\right)^r\right\}^{1/r}\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\
&\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{(\alpha-n)kp}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\left(\sum_{j=-\infty}^{k-3}\left\{\sum_h\|f_h^j\|_{L^1(\mathbb{R}^n)}^r\right\}^{1/r}\right)^p\right\}^{1/p}\\
&\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{(\alpha-n)kp}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\left(\sum_{j=-\infty}^{k-3}\left\|\left\{\sum_h|f_h^j|^r\right\}^{1/r}\right\|_{L^1(\mathbb{R}^n)}\right)^p\right\}^{1/p}.
\end{aligned}$$

It follows from Hölder's inequality and Lemmas 2.2-2.3 that

$$\begin{aligned}
D_1 &\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{(\alpha-n)kp}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right.\\
&\quad\left.\times\left(\sum_{j=-\infty}^{k-3}\left\|\left\{\sum_h|f_h^j|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\
&\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{(\alpha-n)kp}\right.\\
&\quad\left.\times\left\{\sum_{j=-\infty}^{k-3}\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}|B_k|\left\|\left\{\sum_h|f_h^j|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^p\right\}^{1/p}\\
&\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{\alpha kp}\left\{\sum_{j=-\infty}^{k-3}\left(\frac{|B_j|}{|B_k|}\right)^{\delta_1}\left\|\left\{\sum_h|f_h^j|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^p\right\}^{1/p}.
\end{aligned}$$

Noting that $2^{j\alpha}\|\{\sum_h|f_h^j|^r\}^{1/r}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le \{\sum_{l=-\infty}^{j}2^{\alpha lp}\|\{\sum_h|f_h^l|^r\}^{1/r}\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\}^{1/p}$, thus, when $\alpha < \lambda + n\delta_1$, we get

$$\begin{aligned}
D_1 &\le C\sup_{k_0\in\mathbb{Z}}\left\{\sum_{k=-\infty}^{k_0} 2^{\lambda(k-k_0)p}\left\{\sum_{j=-\infty}^{k-3} 2^{(j-k)(n\delta_1+\lambda-\alpha)}\right.\right.\\
&\quad\left.\left.\times 2^{-j\lambda}\left(\sum_{l=-\infty}^{j}2^{\alpha lp}\left\|\left\{\sum_h|f_h^l|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right)^{1/p}\right\}^p\right\}^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&\le C\sup_{k_0\in\mathbb{Z}}\left\{\sum_{k=-\infty}^{k_0}2^{\lambda(k-k_0)p}\left(\sum_{j=-\infty}^{k-3}2^{(j-k)(n\delta_1+\lambda-\alpha)}\right)^p\right\}^{1/p}\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}\\
&\le C\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Finally, let us to estimate D_3. By using Minkowski's inequality, we obtain

$$
\begin{aligned}
S_{\psi,a}(f_h^j)(x) &= \left(\int_{\Gamma_a(x)}\left|\int_{\mathbb{R}^n}t^{-n}\psi\left(\frac{y-z}{t}\right)f_h^j(z)\,dz\right|^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2}\\
&\le \int_{\mathbb{R}^n}|f_h^j(z)|\left(\int_{\Gamma_a(x)}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)\right|^2dy\,dt\right)^{1/2}dz\\
&\le \int_{\mathbb{R}^n}|f_h^j(z)|\left(\int_0^\infty\int_{\Gamma'}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)\right|^2dy\,dt\right)^{1/2}dz\\
&\quad+\int_{\mathbb{R}^n}|f_h^j(z)|\left(\int_0^\infty\int_{\Gamma''}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)\right|^2dy\,dt\right)^{1/2}dz.
\end{aligned}
$$

Let $a\ge 1$, $x\in C_k$, $j\ge k+3$, $z\in C_j\subset B_j$. Then we have $t+|y-z|\ge\frac{|x-y|}{a}+|y-z|\ge\frac{|x-y|+|y-z|}{a}\ge\frac{|z|-|x|}{a}\ge\frac{3|z|}{4a}$. Hence, it follows from the condition (ii) that

$$
\begin{aligned}
&\int_0^\infty\int_{\Gamma'}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)\right|^2dy\,dt\\
&\quad\le\int_0^\infty\int_{\Gamma'}\frac{t^{2\varepsilon-n-1}}{(t+|y-z|)^{2(n+\varepsilon)}}\,dy\,dt\\
&\quad\le\int_0^{2^{j+1}}\int_{\Gamma'}\frac{t^{2\varepsilon-n-1}}{(t+|y-z|)^{2(n+\varepsilon)}}\,dy\,dt+\int_{2^{j+1}}^\infty\int_{\Gamma'}\frac{t^{2\varepsilon-n-1}}{(t+|y-z|)^{2(n+\varepsilon)}}\,dy\,dt\\
&\quad\le C\frac{a^{2(n+\varepsilon)}}{|z|^{2(n+\varepsilon)}}\int_0^{2^{j+1}}\int_{|x-y|<at}t^{2\varepsilon-n-1}\,dy\,dt+\int_{2^{j+1}}^\infty\int_{y\le 2^{j+1}}t^{-3n-1}\,dy\,dt\\
&\quad\le Ca^{3n+2\varepsilon}2^{-2jn}.
\end{aligned}
$$

If $y>2^{j+1}$, then $t>\frac{|x-y|}{a}\ge\frac{|x|-|y|}{a}>\frac{2^{j+1}-2^k}{a}\ge\frac{2^j}{a}$. Applying the condition (ii), we obtain

$$
\begin{aligned}
&\int_0^\infty\int_{\Gamma''}t^{-3n-1}\left|\psi\left(\frac{y-z}{t}\right)\right|^2dy\,dt\\
&\quad\le C\int_{\frac{2^j}{a}}^\infty\int_{\Gamma''}t^{-3n-1}\left(1+\frac{|y-z|}{t}\right)^{-2(n+\varepsilon)}dy\,dt\\
&\quad\le C\int_{\frac{2^j}{a}}^\infty\int_{|x-y|<at}\frac{t^{2\varepsilon-n-1}}{(t+|y-z|)^{-2(n+\varepsilon)}}\,dy\,dt\\
&\quad\le Ca^{3n}2^{-2jn}.
\end{aligned}
$$

Thus

$$
S_{\psi,a}(f_h^j)(x)\le\int_{\mathbb{R}^n}|f_h^j(z)|\left(Ca^{3n+2\varepsilon}2^{-2jn}+Ca^{3n}2^{-2jn}\right)^{1/2}dz\le Ca^{3n/2+\varepsilon}2^{-jn}\|f_h^j\|_{L^1(\mathbb{R}^n)}.
$$

Therefore, similar to D_1, as $\lambda - n\delta_2 < \alpha$, by applying Minkowski's inequality, Lemmas 2.1-2.3, and $2^{j\alpha}\|\{\sum_h |f_h^j|^r\}^{1/r}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq \{\sum_{l=-\infty}^{j} 2^{\alpha lp}\|\{\sum_h |f_h^l|^r\}^{1/r}\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\}^{1/p}$, we have

$$
\begin{aligned}
D_3 &\leq C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+3}^{\infty}\left\|\left\{\sum_h \left(a^{3n/2+\varepsilon}2^{-jn}\|f_h^j\|_{L^1(\mathbb{R}^n)}\right)^r\right\}^{1/r}\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\leq C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{\alpha kp}\|\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\sum_{j=k+3}^{\infty} 2^{-jn}\left\|\left\{\sum_h |f_h^j|^r\right\}^{1/r}\right\|_{L^1(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\leq C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{\alpha kp}\left\{\sum_{j=k+3}^{\infty} 2^{-jn}|B_j|\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\left\|\left\{\sum_h |f_h^j|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^p\right\}^{1/p} \\
&\leq C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{\alpha kp}\left\{\sum_{j=k+3}^{\infty}\left(\frac{|B_k|}{|B_j|}\right)^{\delta_2}\left\|\left\{\sum_h |f_h^j|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^p\right\}^{1/p} \\
&\leq C \sup_{k_0\in\mathbb{Z}} \left\{\sum_{k=-\infty}^{k_0} 2^{\lambda(k-k_0)p}\left\{\sum_{j=k+3}^{\infty} 2^{(j-k)(-n\delta_2+\lambda-\alpha)}\right.\right. \\
&\qquad \left.\left.\times 2^{-j\lambda}\left(\sum_{l=-\infty}^{j} 2^{\alpha lp}\left\|\left\{\sum_h |f_h^l|^r\right\}^{1/r}\right\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{1/p}\right\}^p\right\}^{1/p} \\
&\leq C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Adding up the results of D_1, D_2, D_3, we have

$$
\left\|\left\{\sum_h |S_{\psi,a}(f_h)|^r\right\}^{1/r}\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} \leq C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}.
$$

That is, the inequality (1) in Theorem 3.1 holds.

Next we show the other two vector-valued inequalities also hold.

We consider g_ψ first. Similar to $S_{\psi,a}$, via calculation, we shall get the following conclusions (also see [22]):

(1) If $x \in C_k$, $j \leq k-3$, $\operatorname{supp} f_h^j \subset C_j$, then $g_\psi(f_h^j)(x) \leq C|x|^{-n}\|f_h^j\|_{L^1(\mathbb{R}^n)}$;

(2) If $x \in C_k$, $j \geq k+3$, $\operatorname{supp} f_h^j \subset C_j$, then $g_\psi(f_h^j)(x) \leq C2^{-jn}\|f_h^j\|_{L^1(\mathbb{R}^n)}$.

Thus, with a similar argument in the proof of the inequality (1) in Theorem 3.1, by Lemma 2.8 and the above two conclusions, we shall get the following inequality immediately:

$$
\left\|\left\{\sum_h |g_\psi(f_h)|^r\right\}^{1/r}\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} \leq C\left\|\left\{\sum_h |f_h|^r\right\}^{1/r}\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}.
$$

For g_μ^*, by the definition of $S_{\psi,a}$ and g_μ^*, we have

$$
\begin{aligned}
g_\mu^*(f)(x) &= \left(\int_0^\infty \int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\mu n}\left|\psi_t * f(y)\right|^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2} \\
&\leq \left(\int_0^\infty \int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{\mu n}\left|\psi_t * f(y)\right|^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2}
\end{aligned}
$$

$$+\sum_{l=1}^{\infty}\left(\int_0^{\infty}\int_{2^{l-1}t\le|x-y|<2^l t}\left(\frac{t}{t+|x-y|}\right)^{\mu n}\left|\psi_t*f(y)\right|^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2}$$
$$\le S_{\psi,a}\big(f_h^j\big)(x)+\sum_{l=1}^{\infty}\big(1+2^{l-1}\big)^{-\mu n/2}S_{\psi,2^l a}\big(f_h^j\big)(x).$$

By the above estimate of $S_{\psi,a}$, we know that, as $x\in C_k$, $j\le k-3$, $\operatorname{supp} f_h^j\subset C_j$,

$$S_{\psi,a}\big(f_h^j\big)(x)\le Ca^{3n/2+\varepsilon+\gamma}|x|^{-n}\big\|f_h^j\big\|_{L^1(\mathbb{R}^n)}.$$

Thus, when $\mu>3+2(\varepsilon+\gamma)/n$, we have

$$g_\mu^*\big(f_h^j\big)(x)\le Ca^{3n/2+\varepsilon+\gamma}\left(1+\sum_{l=1}^{\infty}2^{(3n/2+\varepsilon+\gamma-\mu n/2)}\right)|x|^{-n}\big\|f_h^j\big\|_{L^1(\mathbb{R}^n)}$$
$$\le Ca^{3n/2+\varepsilon+\gamma}|x|^{-n}\big\|f_h^j\big\|_{L^1(\mathbb{R}^n)}.$$

On the other hand, as $x\in C_k$, $j\ge k+3$, $\operatorname{supp} f_h^j\subset C_j$, we have

$$S_{\psi,a}\big(f_h^j\big)(x)\le Ca^{3n/2+\varepsilon}2^{-jn}\big\|f_h^j\big\|_{L^1(\mathbb{R}^n)}.$$

Furthermore, when $\mu>3+2(\varepsilon+\gamma)/n$, we obtain

$$g_\mu^*\big(f_h^j\big)(x)\le Ca^{3n/2+\varepsilon}\left(1+\sum_{l=1}^{\infty}2^{(3n/2+\varepsilon-\mu n/2)}\right)2^{-jn}\big\|f_h^j\big\|_{L^1(\mathbb{R}^n)}$$
$$\le Ca^{3n/2+\varepsilon}2^{-jn}\big\|f_h^j\big\|_{L^1(\mathbb{R}^n)}.$$

Hence, similar to the proof of the inequality (1) too, by Lemma 2.8 and the above two conclusions about g_μ^*, we get the following inequality immediately:

$$\left\|\left\{\sum_h\big|g_\mu^*(f_h)\big|^r\right\}^{1/r}\right\|_{M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}\le C\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\right\|_{M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}.$$

This completes the proof of Theorem 3.1. □

Now, let us to establish the vector-valued inequalities of the commutators generated by the Littlewood-Paley operators with BMO functions on $M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$.

Theorem 3.2 *Suppose that* $\psi\in L^1(\mathbb{R}^n)$ *satisfies* (i)-(iii), $b\in BMO(\mathbb{R}^n)$, $q(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $m\in\mathbb{N}\backslash\{0\}$. *Let* $0<p<\infty$, $1<r<\infty$, $\lambda-n\delta_2<\alpha<\lambda+n\delta_1$, *where* δ_1, δ_2 *is the constant in Lemma* 2.2. *Then for all function sequence* $\{f_h\}_{h=1}^{\infty}:\|\{\sum_h|f_h|^r\}^{1/r}\|_{M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}<\infty$, *the following vector-valued inequalities hold*:

(1) $\|\{\sum_h|[b^mS_{\psi,a}](f_h)|^r\}^{1/r}\|_{M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}\le C\|b\|_*^m\|\{\sum_h|f_h|^r\}^{1/r}\|_{M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}$,
(2) $\|\{\sum_h|[b^m,g_\psi](f_h)|^r\}^{1/r}\|_{M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}\le C\|b\|_*\|\{\sum_h|f_h|^r\}^{1/r}\|_{M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}$,
(3) $\|\{\sum_h|[b^m,g_\mu^*](f_h)|^r\}^{1/r}\|_{M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}\le C\|b\|_*^m\|\{\sum_h|f_h|^r\}^{1/r}\|_{M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}$,

where $\mu > 3 + 2(\varepsilon + \gamma)/n$, $0 < \gamma < \min\{(\mu - 2)n/2, \varepsilon\}$, *and the constant* C *is independent of* $\{f_h\}_{h=1}^{\infty}$.

Proof Firstly, we consider the inequality (1). Let $b \in BMO(\mathbb{R}^n)$. For any function sequence $\{f_h\}_h$ satisfies $\| \|\{f_h\}_h\|_{l^r} \|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} < \infty$. We write

$$f_h(x) = \sum_{j=-\infty}^{\infty} f_h(x)\chi_j(x) \triangleq \sum_{j=-\infty}^{\infty} f_h^j(x).$$

Thus,

$$\begin{aligned}
&\left\| \left\{ \sum_h \left| [b^m, S_{\psi,a}](f_h) \right|^r \right\}^{1/r} \right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} \\
&\quad = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left\| \left\{ \sum_h \left| [b^m, S_{\psi,a}](f_h) \right|^r \right\}^{1/r} \chi_k \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} \\
&\quad \le \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{j=-\infty}^{\infty} \left\| \left\{ \sum_h \left| [b^m, S_{\psi,a}](f_h^j) \right|^r \right\}^{1/r} \chi_k \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
&\quad \le C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} \left\| \left\{ \sum_h \left| [b^m, S_{\psi,a}](f_h^j) \right|^r \right\}^{1/r} \chi_k \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
&\qquad + C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{j=k-2}^{k+2} \left\| \left\{ \sum_h \left| [b^m, S_{\psi,a}](f_h^j) \right|^r \right\}^{1/r} \chi_k \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
&\qquad + C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{j=k+3}^{\infty} \left\| \left\{ \sum_h \left| [b^m, S_{\psi,a}](f_h^j) \right|^r \right\}^{1/r} \chi_k \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
&\quad \triangleq E_1 + E_2 + E_3.
\end{aligned}$$

We are now going to estimate each term, respectively. For the term E_2, noting that $\mathrm{supp} f_h^j \subset C_j$, we can easily obtain by Lemma 2.9

$$\begin{aligned}
E_2 &\le C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{j=k-2}^{k+2} \left\| \left\{ \sum_h |f_h^j|^r \right\}^{1/r} \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
&\le C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left\| \left\{ \sum_h |f_h|^r \right\}^{1/r} \chi_k \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} \\
&= C \left\| \left\{ \sum_h |f_h|^r \right\}^{1/r} \right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For the term E_1, observe that as $x \in C_k$, $j \le k - 3$, $\mathrm{supp} f_h^j \subset C_j$. With the same argument as in the estimate of D_1, we have

$$S_{\psi,a}(f_h^j)(x) \le C a^{3n/2+\varepsilon+\gamma} |x|^{-n} \|f_h^j\|_{L^1(\mathbb{R}^n)}.$$

Thus,

$$\begin{aligned}\left|\left[b^m, S_{\psi,a}\right]\left(f_h^j\right)(x)\right| &= \left|S_{\psi,a}\left[\left(b(x)-b\right)^m f\right](x)\right| \\ &\le Ca^{3n/2+\varepsilon+\gamma}|x|^{-n}\left\|\left(b(\cdot)-b\right)^m f_h^j\right\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Hence, by Minkowski's inequality, we get

$$\begin{aligned}&\left\|\left\{\sum_h\left|\left[b^m, S_{\psi,a}\right]\left(f_h^j\right)\right|^r\right\}^{1/r}\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad\le C\left\||x|^{-n}\left\{\sum_h\left\|\left(b(\cdot)-b\right)^m f_h^j\right\|_{L^1(\mathbb{R}^n)}^r\right\}^{1/r}\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad\le C\left\||x|^{-n}\left\|\left\{\sum_h\left|\left(b(\cdot)-b\right)^m f_h^j\right|^r\right\}^{1/r}\right\|_{L^1(\mathbb{R}^n)}\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad\le C2^{-nk}\left\|(b-b_j)^m\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^1(\mathbb{R}^n)} \\ &\qquad+2^{-nk}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\left\{\sum_h\left|(b-b_j)^m f_h^j\right|^r\right\}^{1/r}\right\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Furthermore, by Hölder's inequality and Lemmas 2.2-2.4, we obtain

$$\begin{aligned}&\left\|\left\{\sum_h\left|\left[b^m, S_{\psi,a}\right]\left(f_h^j\right)\right|^r\right\}^{1/r}\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad\le C2^{-nk}(k-j)^m\|b\|_*^m\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\qquad+2^{-nk}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|(b-b_j)^m\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\quad\le C2^{-nk}\left[(k-j)^m+1\right]\|b\|_*^m\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\quad\le C2^{-nk}(k-j)^m\|b\|_*^m|B_k|\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad\le C(k-j)^m\|b\|_*^m\frac{|B_j|}{|B_k|}\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad\le C2^{(j-k)n\delta_1}(k-j)^m\|b\|_*^m\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Therefore, as $\alpha < \lambda + n\delta_1$, noting that

$$2^{j\alpha}\left\|\left\{\sum_h\left|f_h^j\right|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le \left\{\sum_{l=-\infty}^{j}2^{\alpha lp}\left\|\left\{\sum_h\left|f_h^l\right|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right\}^{1/p},$$

then we have

$$\begin{aligned}
E_1 &\leq C\|b\|_*^m \sup_{k_0\in\mathbb{Z}} 2^{-\lambda k_0} \left\{ \sum_{k=-\infty}^{k_0} 2^{\alpha kp} \left(\sum_{j=-\infty}^{k-3} 2^{(j-k)n\delta_1} (k-j)^m \left\| \left\{ \sum_h |f_h^j|^r \right\}^{1/r} \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^p \right\}^{1/p} \\
&\leq C\|b\|_*^m \sup_{k_0\in\mathbb{Z}} \left\{ \sum_{k=-\infty}^{k_0} 2^{\lambda(k-k_0)p} \left(\sum_{j=-\infty}^{k-3} (k-j)^m 2^{(j-k)(n\delta_1+\lambda-\alpha)} \right.\right. \\
&\quad \times 2^{-j\lambda} \left.\left. \left\{ \sum_{l=-\infty}^{j} 2^{\alpha lp} \left\| \left\{ \sum_h |f_h^l|^r \right\}^{1/r} \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p} \right)^p \right\}^{1/p} \\
&\leq C\|b\|_*^m \left\| \left\{ \sum_h |f_h|^r \right\}^{1/r} \right\|_{M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}.
\end{aligned}$$

Now we turn to estimate E_3. Observe that as $x \in C_k$, $j \geq k+3$, $\operatorname{supp} f_h^j \subset C_j$. With the same argument as in the estimate of D_3, then we have

$$S_{\psi,a}(f_h^j)(x) \leq Ca^{3n/2+\varepsilon} 2^{-jn} \|f_h^j\|_{L^1(\mathbb{R}^n)}.$$

Thus,

$$\left|[b^m, S_{\psi,a}](f_h^j)(x)\right| \leq Ca^{3n/2+\varepsilon} 2^{-jn} \left\|(b(\cdot)-b)^m f_h^j\right\|_{L^1(\mathbb{R}^n)}.$$

Hence, by Minkowski's inequality, we get

$$\begin{aligned}
&\left\| \left\{ \sum_h \left|[b^m, S_{\psi,a}](f_h^j)\right|^r \right\}^{1/r} \chi_k \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad \leq C \left\| 2^{-jn} \left\{ \sum_h \left\|(b(\cdot)-b)^m f_h^j\right\|_{L^1(\mathbb{R}^n)}^r \right\}^{1/r} \chi_k \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad \leq C \left\| 2^{-jn} \left\| \left\{ \sum_h \left|(b(\cdot)-b)^m f_h^j\right|^r \right\}^{1/r} \right\|_{L^1(\mathbb{R}^n)} \chi_k \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad \leq C 2^{-jn} \left\|(b-b_k)^m \chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left\| \left\{ \sum_h |f_h^j|^r \right\}^{1/r} \right\|_{L^1(\mathbb{R}^n)} \\
&\quad\quad + 2^{-jn} \|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left\| \left\{ \sum_h \left|(b-b_k)^m f_h^j\right|^r \right\}^{1/r} \right\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

Furthermore, by Hölder's inequality and Lemmas 2.2-2.4, we obtain

$$\begin{aligned}
&\left\| \left\{ \sum_h \left|[b^m, S_{\psi,a}](f_h^j)\right|^r \right\}^{1/r} \chi_k \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad \leq C 2^{-jn} \|b\|_*^m \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left\| \left\{ \sum_h |f_h^j|^r \right\}^{1/r} \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\quad\quad + 2^{-jn} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left\| \left\{ \sum_h |f_h^j|^r \right\}^{1/r} \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \left\|(b-b_k)^m \chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

$$\begin{aligned}
&\leq C2^{-jn}\big[(k-j)^m+1\big]\|b\|_*^m\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\left\{\sum_h\big|f_h^j\big|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\
&\leq C2^{-jn}(k-j)^m\|b\|_*^m|B_j|\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\left\|\left\{\sum_h\big|f_h^j\big|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C(k-j)^m\|b\|_*^m\frac{|B_k|}{|B_j|}\left\|\left\{\sum_h\big|f_h^j\big|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C2^{(k-j)n\delta_2}(j-k)^m\|b\|_*^m\left\|\left\{\sum_h\big|f_h^j\big|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, similar to E_1, as $\alpha > \lambda - n\delta_2$, noting that

$$2^{j\alpha}\left\|\left\{\sum_h\big|f_h^j\big|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq \left\{\sum_{l=-\infty}^{j}2^{\alpha lp}\left\|\left\{\sum_h\big|f_h^l\big|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right\}^{1/p},$$

then we have

$$\begin{aligned}
E_3 &\leq C\|b\|_*^m\sup_{k_0\in\mathbb{Z}}2^{-\lambda k_0}\left\{\sum_{k=-\infty}^{k_0}2^{\alpha kp}\left(\sum_{j=k+3}^{\infty}2^{(k-j)n\delta_2}(j-k)^m\left\|\left\{\sum_h\big|f_h^j\big|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\
&\leq C\|b\|_*^m\sup_{k_0\in\mathbb{Z}}\left\{\sum_{k=-\infty}^{k_0}2^{\lambda(k-k_0)p}\left(\sum_{j=k+3}^{\infty}(j-k)^m2^{(j-k)(-n\delta_2+\lambda-\alpha)}\right.\right.\\
&\quad\left.\left.\times 2^{-j\lambda}\left\{\sum_{l=-\infty}^{j}2^{\alpha lp}\left\|\left\{\sum_h\big|f_h^l\big|^r\right\}^{1/r}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right\}^{1/p}\right)^p\right\}^{1/p}\\
&\leq C\|b\|_*^m\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Adding up the results of E_1, E_2, E_3, we have

$$\left\|\left\{\sum_h\big|\big[b^m S_{\psi,a}\big](f_h)\big|^r\right\}^{1/r}\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_*^m\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}.$$

That is, the inequality (1) in Theorem 3.2 holds.

For the proofs about the vector-valued inequalities of $[b^m, g_\psi]$ and $[b^m, g_\mu^*]$, with a similar arguments in the proof of the vector-valued inequality (1) in Theorem 3.2, by Lemma 2.9 and the estimates of g_ψ, and g_μ^* in Theorem 3.1, it is not difficult to deduce

$$\left\|\left\{\sum_h\big|\big[b^m, g_\psi\big](f_h)\big|^r\right\}^{1/r}\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_*\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}$$

and, if $\mu > 3 + 2(\varepsilon+\gamma)/n$ and $0 < \gamma < \min\{(\mu-2)n/2, \varepsilon\}$,

$$\left\|\left\{\sum_h\big|\big[b^m, g_\mu^*\big](f_h)\big|^r\right\}^{1/r}\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_*^m\left\|\left\{\sum_h|f_h|^r\right\}^{1/r}\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)}.$$

We complete the proof of Theorem 3.2. □

Remark 3.1 By Remark 1.1, we can easily to see that the results in Theorems 3.1-3.2 are also suitable for Herz spaces with variable exponent $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
Shuangping Tao is supported by National Natural Foundation of China (Grant No. 11161042 and 11071250).

Received: 25 December 2013 Accepted: 23 May 2014 Published: 03 Jun 2014

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10.1186/1029-242X-2014-227
Cite this article as: Wang and Tao: **Boundedness of Littlewood-Paley operators and their commutators on Herz-Morrey spaces with variable exponent.** *Journal of Inequalities and Applications* 2014, 2014:227