Scientific Research and Essays Vol. 6 (28), pp. 5846-5854, 23 November, 2011
Available online at http://www.academicjournals.org/SRE
DOI: 10.5897/SRE10.819
ISSN 1992-2248 ©2011 Academic Journals

Full Length Research Paper

Sensitivity of Schur stability of systems of linear difference equations with constant coefficients

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Accepted 17 February, 2011

In this work, sensitivity of the Schur stability and the ω^*–Schur stability of linear difference equation systems with constant coefficients have been investigated, and new results on the sensitivity problem have been given. The results have applied the scalar-linear difference equations with order k and supported with numerical examples and also compared with the existing ones in the literature.

Key words: Schur stability, difference equations, sensitivity, perturbation systems.

INTRODUCTION

Let A be a matrix of dimensions $N \times N$ and $x(n)$ be a vector of dimension N, and consider the following difference equation system:

$$x(n+1) = A\,x(n), \quad n \in Z. \tag{1}$$

The asymptotic stability of the system (1) is equivalent to the asymptotic stability of the coefficient matrix A. It is well-known that with respect to Lyapunov, a matrix A is discrete-asymptotically stable if and only if the discrete-Lyapunov matrix equation $A^*XA – X + C = 0,\ C = C^* > 0$ has a solution matrix X which is positive definite matrix, that is, $X = X^* > 0$. Moreover, this solution is given by $X = \sum_{k=0}^{\infty}(A^*)^k C A^k$. And also according to the spectral criteria, a matrix A is discrete-asymptotically stable if and only if the eigenvalues of the coefficient matrix A lay in the unit disc, that is, $|\lambda_i(A)| < 1$ for all $i = 1,2,\ldots,N$, where λ_i $(i = 1,2,\ldots,N)$ stands for the eigenvalues of the coefficient matrix A (Elaydi, 1996; Akın and Bulgak, 1998; Godunov, 1998; Bulgak, 1999). Such systems are also called as *Schur stable* (Wang and Michel, 1993; Rohn, 1994; Voicu and Pastravanu, 2006). Throughout the study, we focus our attention to the concept of *Schur stability*.

In the literature, some restrictions on the perturbation matrix B are assumed to study the Schur stability of the following system

$$y(n+1) = (A+B)\,y(n), \quad n \in Z, \tag{2}$$

where A is the coefficient matrix of the Schur stable system (1).

So called continuation are used to study the sensitivity of the Schur stability and the ω^*–Schur stabilitiy of the system (1) (Van Loan, 1984; Akın and Bulgak, 1998; Bulgak, 1999; Aydın et al., 2001).

In this work, some results on the sensitivity of the Schur stability and the ω^*–Schur stabilitiy of the difference equation system (1) were presented, and these results were compared with the existing results in the literature. We have also applied the results to the delay difference equations.

SENSITIVITY of SCHUR STABILITY of SYSTEMS of LINEAR DIFFERENCE EQUATIONS

In this study, we give some results in the literature on the sensitivity of the Schur stability of the systems with constant and periodic coefficients.

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Sensitivity of Schur stability of systems with constant coefficients

Let's start with the parameter $\omega(A)$ that shows the quality of Schur stability of the system (1) and holds and an important place in the theory of stability.

Schur stability parameter $\omega(A)$ is defined as follows:

$$\omega(A) = \|H\|; \quad H = \sum_{k=0}^{\infty} (A^*)^k A^k,$$
$$H = H^* > 0, \quad A^*HA - H + I = 0,$$

where I is unit matrix, A^* is adjoint of the matrix A, $\|A\| = \max_{\|x\|=1} \|Ax\|$ is the spectral norm of the matrix A, furthermore, the norm $\|x\|$ is Euclidean norm for the vector $x = (x_1, x_2, \ldots, x_N)^T$. Linear difference system (1) is Schur stable if and only if $\omega(A) < \infty$ holds, and so it is clear that the perturbed linear difference system (2) is Schur stable if and only if $\omega(A+B) = \|\widetilde{H}\| < \infty$ holds, where the matrix $\widetilde{H} = \sum_{k=0}^{\infty} (A^* + B^*)^k (A+B)^k$ is positive definite solution of the discrete-Lyapunov matrix equation $(A+B)^* \widetilde{H} (A+B) - \widetilde{H} + I = 0$.

Moreover, let ω^* be the practical Schur stability parameter of the system (1), then the matrix A is called as *practically Schur stable* (ω^*–Schur stable) provided that $\omega^* > 1$ and $\omega(A) \leq \omega^*$ hold. If $\omega(A) > \omega^*$ holds, then the matrix A is called as ω^*–Schur unstable matrix (Bulgakov and Bulgak, 1980; Akın and Bulgak, 1998; Bulgak, 1999).

Lemma 1 (Bulgak, 1999; Lemma 5.2)

Let A be a Schur stable matrix (system (1) is Schur stable) ($\omega(A) < \infty$). We have $\|A\|^2 \leq \omega(A)$ and $\omega(A) \geq 1$. Moreover, if $\|A\| < 1$ then $\omega(A) \leq \frac{1}{1 - \|A\|^2}$.

It is important to know the distance from Schur stable matrices to the Schur unstable matrices. The distance is usually investigated with the help of the theorems which are known as continuity theorems in the literature. Now, we briefly introduce the theorems which shows the sensitivity of the Schur stability of the system (1).

Theorem 1 (Bulgak, 1999; Theorem 5.1)

Let A be a Schur stable matrix (system (1) is Schur stable) ($\omega(A) < \infty$). If $\|B\| \leq \delta_1$ then the matrix $A+B$ is Schur stable, and

$$|\omega(A+B) - \omega(A)| \leq 5\omega^{\frac{5}{2}}(A)\|B\|,$$

holds, where $\delta_1 = \frac{1}{20\omega^{\frac{3}{2}}(A)}$.

Theorem 2 (**Bulgak, 1999; Theorem 6.1)**

Let A be a Schur stable matrix ($\omega(A) < \infty$). If $\|B\| \leq \delta_2$ then the matrix $A+B$ is Schur stable. Moreover, if $(2\|B\|\|A\| + \|B\|^2)\omega(A) < 0.5$ then the inequality

$$|\omega(A+B) - \omega(A)| \leq 2\omega^2(A)(2\|A\| + \|B\|)\|B\|,$$

holds, where $\delta_2 = \frac{1}{6\pi\omega(A)}$.

Sensitivity of Schur stability of systems with periodic coefficients

For convenience, we give some results on the sensitivity of the Schur stability of the system with periodic coefficients. Now, let us introduce the Schur stability concept for *system with periodic coefficients*.

Let $A(n)$ be a matrix of $N \times N$ dimensions and T–periodic ($T > 0$) $N \times N$ and let $x(n)$ be a vector of N dimensions. Consider the following system

$$x(n+1) = A(n)\,x(n), \quad n \in Z. \tag{3}$$

The parameter $\omega_1(A,T)$ is used as Schur stability parameter, and is defined by

$$\omega_1(A,T) = \|F\|;$$
$$F = \sum_{k=0}^{\infty} \big(X^*(T)\big)^k \big(X(T)\big)^k, \quad X(T) = \prod_{j=0}^{T-1} A(j),$$

where the matrix $X(T)$ is the monodromy matrix of the system (3) (Akın and Bulgak, 1998; Aydın et al., 2000; Aydın et al., 2001). The system (3) is Schur stable if and only if $\omega_1(A,T) < \infty$. And note that, in the case $T = 1$, we have $\omega_1(A,T) = \omega(A)$ (Aydın et al., 2000).

Let $B(n+T) = B(n)$ and consider the perturbed system of the system (3):

$$y(n+1) = (A(n) + B(n))\,y(n), \quad n \in Z. \tag{4}$$

Let $Y(T)$ be the monodromy matrix of (4), we state below, the so called the continuity theorems for the system (3) with periodic coefficients, which also shows the sensitivity of the Schur stability.

Theorem 3

Let the system (3) be Schur stable and $X(T)$ and $Y(T)$ be the monodromy matrices of the systems (3) and (4) respectively. If the matrix $B(n)$ satisfies:

$$\|Y(T)-X(T)\| < \sqrt{\|X(T)\|^2 + \frac{1}{\omega_1(A,T)}} - \|X(T)\|,$$

then the system (4) is Schur stable (Aydın et al., 2001; Theorem 2). Moreover, the inequality

$$\|\tilde{F}-F\| \leq \frac{1-\alpha}{\alpha}\omega_1(A,T) \ ,$$

holds, where
$\alpha = 1-(2\|X(T)\|\|Y(T)-X(T)\| + \|Y(T)-X(T)\|^2)\|F\| > 0$
and $\tilde{F} = \sum_{k=0}^{\infty}(Y^*(T))^k(Y(T))^k$ (Aydın et al., 2001; Theorem 3).

SOME RESULTS on SENSITIVITY of SCHUR STABILITY and ω^*–SCHUR STABILITY of SYSTEMS with CONSTANT COEFFICIENTS

In this section, some results on the sensitivity of the Schur stability are stated.

Sensitivity of Schur stability

Now, we give the Corollary 1 which is a result of Theorem 3.

Corollary 1

Suppose that A is a Schur stable matrix, that is $\omega(A) < \infty$. If the matrix B satisfies $\|B\| < \delta_3$, then $A+B$ is Schur stable. Moreover, the inequality

$$|\omega(A+B)-\omega(A)| \leq \frac{(2\|A\|+\|B\|)\|B\|\omega^2(A)}{1-(2\|A\|+\|B\|)\|B\|\omega(A)},$$

holds. where $\delta_3 = \sqrt{\|A\|^2 + \frac{1}{\omega(A)}} - \|A\|$.

Proof

In the case $T=1$, the proof of Corollary 1 is clear since
$A(n)=A, \quad B(n)=B, \quad X(1)=A, \quad Y(1)=A+B,$
$\|Y(1)-X(1)\| = \|(A+B)-A\| = \|B\|$,

$F = \sum_{k=0}^{\infty}(A^*)^k A^k = H$, $\omega_1(A,1) = \|H\| = \omega(A)$,
$\alpha = 1-(2\|A\|\|B\| + \|B\|^2)\omega(A)$,

$\tilde{F} = \sum_{k=0}^{\infty}(A^*+B^*)^k(A+B)^k = \widetilde{H}, \qquad \omega(A+B) = \|\widetilde{H}\|$,
$|\omega(A+B)-\omega(A)| \leq \|\widetilde{H}-H\|$.

Remark 1

Corollary 1 occurs also as a result of Theorem 3.1 in Aydın et al. (2002), in the case *T*=1. Furthermore the first part of Corollary 1 exist in Aydın et al. (2002).

Remark 2

The inequality $\|B\| < \delta_3$ in Corollary 1 is equivalent to

$$(2\|B\|\|A\| + \|B\|^2)\omega(A) < 1.$$

This inequality has been taken in Theorem 2 as

$$(2\|B\|\|A\| + \|B\|^2)\omega(A) < 0.5,$$

and therefore, obtained the following inequality

$$|\omega(A+B)-\omega(A)| < 2\omega^2(A)(2\|A\| + \|B\|)\|B\|.$$

The inequality $(2\|B\|\|A\| + \|B\|^2)\omega(A) < 0.5$ is also equivalent to $\|B\| < \overline{\delta_2}$, therefore it is clear that $\overline{\delta_2} < \delta_3$, where $\overline{\delta_2} = \sqrt{\|A\|^2 + \frac{1}{2\omega(A)}} - \|A\|$. Furhermore, we note that Theorem 2 does not guarantee the Schur stability while $\delta_2 < \overline{\delta_2}$.

Lemma 2

The following statements are true, where the parameters δ_1, δ_2 and δ_3 are as defined in Theorem 1, Theorem 2 and Corollary 1, respectively, and $\lambda = \max_i\{|\lambda_i|\}$, λ_i – eigenvalues of the matrix A;

1. If $\|A\| < 1$, then $\frac{(1-\|A\|^2)^{3/2}}{20} \leq \delta_1 \leq \frac{1}{20}$.
2. If $\|A\| < 1$, then $\frac{1-\|A\|^2}{6\pi} \leq \delta_2 \leq \frac{1}{6\pi}$.
3. If $\|A\| < 1$, then $1-\|A\| \leq \delta_3 \leq \sqrt{\|A\|^2+1} - \|A\|$.
4. For the all Schur stable matrices A, $\delta_1 < \delta_2$.
5. If $\|A\| < 0.95$, then $\delta_1 < \delta_3$.
6. If the diagonal matrix A which satisfies $\|A\| < 1$, then $\delta_2 < \delta_3$.
7. If $\|A\| < 1 - \frac{1}{6\pi}$, then $\delta_2 < \delta_3$.

Proof

1. Let $\|A\| < 1$. It is clear that $1 \leq \omega(A) \leq \frac{1}{1-\|A\|^2}$ from Lemma 1 and the definition of Schur stability parameter $\omega(A)$. Thus the inequalities

$$(1-\|A\|^2)^{3/2} \leq \frac{1}{\omega^{3/2}(A)} \leq 1 \ , \quad \frac{(1-\|A\|^2)^{3/2}}{20} \leq \delta_1 \leq \frac{1}{20}$$

occurs. This completes the proof.
2. Let $\|A\| < 1$, and so $1 \leq \omega(A) \leq \frac{1}{1-\|A\|^2}$. Thus the inequalities

$$1-\|A\|^2 \leq \frac{1}{\omega(A)} \leq 1 \ , \quad \frac{1-\|A\|^2}{6\pi} \leq \frac{1}{6\pi\omega(A)} \leq \frac{1}{6\pi}$$

Table 1. Comparison of δ_1, δ_2 and δ_3 for the matrix $A(\alpha)$.

α	$\|A(\alpha)\|$	$\omega(A(\alpha))$	δ_1	δ_2	δ_3
0	0.5	1.33333	$0.32476e-1$	$0.397888e-1$	0.5
1	1.207106	4.5425	$0.516449e-2$	$0.116789e-1$	$0.8798e-1$
9.3	9.326804	257.866	$0.120747e-4$	$0.205733e-3$	$0.207892e-3$
9.4	9.42652	263.407	$0.116957e-4$	$0.201405e-3$	$0.201366e-3$
9.5	9.526243	269.007	$0.113324e-4$	$0.197212e-3$	$0.195111e-3$
10	10.024937	297.896	$0.972462e-5$	$0.178087e-3$	$0.167424e-3$

occurs. This completes the proof.

3. For $\|A\| < 1$, from the inequality $1 - \|A\|^2 \le \frac{1}{\omega(A)} \le 1$ and the definition of the parameter δ_3 , it is clear that

$$1 \le \|A\|^2 + \frac{1}{\omega(A)} \le \|A\|^2 + 1,$$

and so $1 \le \sqrt{\|A\|^2 + \frac{1}{\omega(A)}} \le \sqrt{\|A\|^2 + 1}$,

$$1 - \|A\| \le \sqrt{\|A\|^2 + \frac{1}{\omega(A)}} - \|A\| \le \sqrt{\|A\|^2 + 1} - \|A\|.$$

This completes the proof.

4. For any Schur stable matrix A, from the inequality $\frac{1}{\omega(A)} \le 1$ it is clear that $\delta_1 < \delta_2$.

5. Let $\|A\| < 1 - \frac{1}{20} = 0.95$. Therefore $\frac{1}{20} < 1 - \|A\|$, and so the inequality $\delta_1 < \delta_3$ is obtained from the inequalities in Lemma 2-1 and Lemma 2-3.

6. In case the diagonal matrix $A = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ which satisfies $\|A\| < 1$, since $\|A\| = \lambda = \max_i |\lambda_i|$ the matrix A is Schur stable and the parameter of Schur stability $\omega(A)$ is equal to $\frac{1}{1-\lambda^2}$, that is, $\omega(A) = \frac{1}{1-\lambda^2}$. .Therefore,

$$\delta_2 = \frac{1}{6\pi\omega(A)} = \frac{1-\lambda^2}{6\pi}; \quad \delta_3 = \sqrt{\|A\|^2 + \frac{1}{\omega(A)}} - \|A\| = 1 - \lambda.$$

Since $\frac{1+\lambda}{6\pi} < 1$, we conclude that $\delta_2 < \delta_3$.

7. Let $\|A\| < 1 - \frac{1}{6\pi}$. Therefore $\frac{1}{6\pi} < 1 - \|A\|$, and so the inequality $\delta_2 < \delta_3$ is obtained from the inequalities in Lemma 2-2 and Lemma 2-3.

Proposition 1

The inequality $f(x) > g(x)$ is true, where $f(x) = \sqrt{x^2+1} - x$, $g(x) = \frac{1-x^2}{6\pi}$ for $x \in [0,1]$. Moreover, the inequality $\min_x f(x) > \max_x g(x)$ is also true.

Proof

Let $f(x) = \sqrt{x^2+1} - x$ and $g(x) = \frac{1-x^2}{6\pi}$ for $x \in [0,1]$. The functions $f(x)$ and $g(x)$ are monotone decreasing functions since $f'(x) < 0$ and $g'(x) \le 0, x \in [0,1]$. Therefore, it is seen that $f(1) = \sqrt{2} - 1 > g(0) = 1/6\pi$. This completes the proof.

Note 1

In case $\|A\| < 1$, we do not have the possibility give the range such that $\delta_2 > \delta_3$ from Proposition 1, similarly $\delta_1 > \delta_3$. This sentence does not imply the inequality $\delta_2 > \delta_3$ ($\delta_1 > \delta_3$) will not be. But the numerical examples supports the inequality $\delta_2 > \delta_3$ ($\delta_1 > \delta_3$) may not be in this case.

Consider the parameters δ_1, δ_2 and δ_3, for example, for the matrix $A(\alpha) = \begin{pmatrix} 0.5 & \alpha \\ 0 & 0.5 \end{pmatrix}$. For the different values of α we give the values of norm $\|A(\alpha)\|$, $\omega(A(\alpha))$, δ_1, δ_2, δ_3.

As is also seen from Table 1, δ_1, δ_2 and δ_3 decreases toward zero, while $\omega(A(\alpha))$ increases. δ_2 can be greater than δ_3 while δ_2 and δ_3 decreases toward zero (δ_2 and δ_3 are very small).

Now, considering Theorem 2 and Corollary 1, we give the continuity theorem *which allows the greater perturbe than others* without disturbing the Schur stability for the linear difference equation systems with constant coefficients.

Theorem 4

Suppose that A is a Schur stable matrix, that is $\omega(A) < \infty$. If the matrix B satisfies $\|B\| < \gamma$, then $A + B$ is Schur stable. Moreover, if $\|B\| < \delta_3$, then the following inequalities

$$\omega(A+B) \le \frac{\omega(A)}{1 - (2\|A\| + \|B\|)\|B\|\omega(A)},$$

$$|\omega(A+B) - \omega(A)| \le \frac{(2\|A\| + \|B\|)\|B\|\omega^2(A)}{1 - (2\|A\| + \|B\|)\|B\|\omega(A)}.$$

holds, where $\gamma = \max\{\delta_2, \delta_3\}$, $\delta_2 = \frac{1}{6\pi\omega(A)}$ and $\delta_3 =$

$\sqrt{\|A\|^2+\frac{1}{\omega(A)}}-\|A\|.$

Proof

The proof of the first part of the theorem and the second inequality in the second part of the theorem is clear from Corollary 1 and Theorem 2. The first inequality in the second part of the theorem follows the arrangement of the second inequality.

Now, we give the Corollary 2 which is a special case of Corollary 1 and Theorem 4.

Corollary 2

Let $\|A\|<1$. If the matrix B satisfies $\|A\|+\|B\|<1$ then the matrix $A+B$ is Schur stable. Moreover, the following inequalities

$$\omega(A+B)\le\frac{1}{1-(\|A\|+\|B\|)^2},$$
$$|\omega(A+B)-\omega(A)|\le\frac{\|B\|}{1-\|A\|}\frac{1}{1-(\|A\|+\|B\|)^2},$$

holds.

Proof

Using the inequality $1-\|A\|^2\le\frac{1}{\omega(A)}$ in Lemma 1 and the inequalities in Theorem 4, it is obtained that

1. $\|B\|<1-\|A\|\le\delta_3\le\max\{\delta_2,\ \delta_3\}$ and so $\|A\|+\|B\|<1$
2. $|\omega(A+B)-\omega(A)|\le\frac{1}{1-\|A\|^2}\frac{(2\|A\|+\|B\|)\|B\|}{1-(\|A\|^2+2\|A\|\|B\|+\|B\|^2)}\le\frac{1}{1-\|A\|}\frac{\|B\|}{1-(\|A\|+\|B\|)^2}$
3. $\omega(A+B)\le\frac{1}{1-\|A\|^2-(2\|A\|+\|B\|)\|B\|}=\frac{1}{1-(\|A\|+\|B\|)^2}$.

Remark 3

Since $\|A+B\|\le\|A\|+\|B\|<1$, the inequality $\omega(A+$ $B\le 11-A+B2$occurs as a direct result of Lemma 1. Thus, we can conclude that the results support each other.

Sensitivity of ω^*Schur stability

Theorem 5

Let A be a ω^*–Schur stable matrix ($\omega(A)\le\omega^*$). If the matrix B satisfies $\|B\|\le\delta_3^*$, then $A+B$ is ω^*–Schur stable, where $\delta_3^*=\sqrt{\|A\|^2+\frac{\omega^*-\omega(A)}{\omega^*\omega(A)}}-\|A\|.$

Proof

Let A be a ω^*–Schur stable. Let us $\|B\|\le\delta_3^*$. From this inequality

$$\|B\|\le\sqrt{\|A\|^2+\frac{\omega^*-\omega(A)}{\omega^*\omega(A)}}-\|A\|$$

$$\Rightarrow\|B\|+\|A\|\le\sqrt{\|A\|^2+\frac{\omega^*-\omega(A)}{\omega^*\omega(A)}}$$

$$\Rightarrow(\|B\|+\|A\|)^2\le\|A\|^2+\frac{\omega^*-\omega(A)}{\omega^*\omega(A)},$$

$$\Rightarrow(\|B\|+\|A\|)^2-\|A\|^2\le\frac{\omega^*-\omega(A)}{\omega^*\omega(A)},$$

$$\Rightarrow(2\|A\|+\|B\|)\|B\|\omega(A)\omega^*\le\omega^*-\omega(A),$$

$$\Rightarrow\omega(A)\le\omega^*[1-(2\|A\|+\|B\|)\|B\|\omega(A)],$$

and therefore the inequality

$$\frac{\omega(A)}{1-(2\|A\|+\|B\|)\|B\|\omega(A)}\le\omega^*,$$

is obtained. Since $\omega(A+B)\le\frac{\omega(A)}{1-(2\|A\|+\|B\|)\|B\|\omega(A)}$ is valid from Theorem 4, the inequality $\omega(A+B)\le\omega^*$ is found. This completes the proof.

APPLICATION to SCALAR-LINEAR DIFFERENCE EQUATIONS with ORDER k of THE RESULTS on THE SENSITIVITY

Consider the scalar-linear difference equations with order k as follows

$$x(n+1)-a_0x(n)-\ldots-a_{k-1}x(n-k+1)=0, n\ge 0 \qquad (5)$$

By taking $x(n-k+1)=y_1(n)$, $x(n-k+2)=y_2(n)$,…, $x(n)=y_k(n)$, the equation (5) can be written as

$$y(n+1)=Cy(n),\ n\ge 0 \qquad (6)$$

in matrix-vector form where the matrix C is companion matrix as follows

$$C=\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ a_{k-1} & a_{k-2} & a_{k-3} & \ldots & a_0 \end{pmatrix},$$

Thus, the results on the sensitivity of Schur stability which are given for the system (1) can easily be used for the sensitivity of Schur stability of the scalar-linear difference equations with order k (5).

Consider the perturbation of the Equation (5), and so, of the system (6)

$$z(n+1) = (C+D)z(n), \tag{7}$$

and the set B_γ called as the nD–ball, that is, the n–dimensional ball (Roger and Charles, 1999), where

$$D = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ d_{k-1} & d_{k-2} & d_{k-3} & \dots & d_0 \end{pmatrix};$$

$$d = (d_{k-1}, d_{k-2}, \dots, d_1, d_0)$$

$$B_\gamma = \{x = (x_1, x_2, \dots, x_n) | \|x\| < \gamma\}$$

For the system (6), the variant of Theorem 4 is as follows.

Theorem 6

Let the system (6) be a Schur stable (the companian matrix C is Schur stable). If the k-tuple $d \in B_{\gamma(C)}$, then the perturbed system (7) is a Schur stable,

where $\gamma(C) = \max\left\{\frac{1}{6\pi\omega(C)}, \sqrt{\|C\|^2 + \frac{1}{\omega(C)}} - \|C\|\right\}$.

Proof

Let the system (6) be a Schur stable (the companian matrix C is Schur stable). While the matrix C is Schur stable, the condition $\|D\| < \max\{\delta_2, \delta_3\}$ in the first part of the Theorem 4 quarantees the Schur stability of the matrix $(C+D)$. Therefore, it obtained the inequality $\|d\| < \gamma(C)$ since $\|D\| = \sqrt{d_{k-1}^2 + d_{k-2}^2 + \dots + d_1^2 + d_0^2} = \|d\|$, $\max\{\delta_2, \delta_3\} = \gamma(C)$ for the matrix C and $d \in B_{\gamma(C)}$.

Note 2

The kD–ball $B_{\gamma(C)}$ occurs as a region of Schur stability for the perturbation matrix D. The 1D–ball $B_{\gamma(C)}$ is a interval, the 2D–ball $B_{\gamma(C)}$ is a disc and the 3D–ball $B_{\gamma(C)}$ is also the interior of a sphere, that is, a solid ball.

Now, we give the Theorem 7 that is a result of Theorem 5 for the system (6).

Theorem 7

Let the system (6) be ω^*–Schur stable (the companian matrix C is ω^*–Schur stable). If the k-tuple $d \in B_{\delta_3^*}$, then the perturbed system (7) is also ω^*–Schur stable, where

$$\delta_3^* = \sqrt{\|A\|^2 + \frac{\omega^* - \omega(A)}{\omega^*\omega(A)}} - \|A\|$$

Proof

Proof is clear from Theorem 5 and Theorem 6.

Remark 4

If the perturbation matrix B in the perturbed system (2) only occurs from a row, Theorem 7 is valid for the system (2) too, where $B = (r_1, r_2, \dots, r_k)^t$, $r_i - i\,th$ row vector, for any $i \neq i_0$, $r_i = \mathbf{0}$, $r_{i_0} \neq \mathbf{0}$, $\mathbf{0}$ –zero vector.

NUMERICAL RESULTS

Firstly, we compare the results of Theorem 1, Theorem 2, Corollary 1 and Theorem 4. For simplicity in the comparison, we let the perturbation matrix B with upper bounds of $\|B\|$ as follows:

$\delta_1 = \frac{1}{20\omega^{\frac{3}{2}}(A)}$ (Theorem 1),

$\delta_2 = \frac{1}{6\pi\omega(A)}$, $\overline{\delta_2} = \sqrt{\|A\|^2 + \frac{1}{2\omega(A)}} - \|A\|$ (Theorem 2),

$\delta_3 = \sqrt{\|A\|^2 + \frac{1}{\omega(A)}} - \|A\|$ (Corollary 1),

$\gamma = \max\{\delta_2, \delta_3\}$ (Theorem 4),

and let us denote the perturbation matrices of A_i by B_i^k ($i = 1,2,3$; $k = 1,2$) and let

$\Delta_1 = 5\omega^{\frac{5}{2}}(A)\|B\|$ (Theorem 1),

$\Delta_2 = 2\omega^2(A)(2\|A\| + \|B\|)\|B\|$ (Theorem 2),

$\Delta_3 = \frac{(2\|A\|+\|B\|)\|B\|\omega^2(A)}{1-(2\|A\|+\|B\|)\|B\|\omega(A)}$ (Corollary 1, Theorem 4),

be the upper bounds of $\Delta = |\omega(A+B) - \omega(A)|$.

Example 1. Let

1. $A_1 = \begin{pmatrix} 1 & -0.5 \\ 0.75 & -1 \end{pmatrix}$, $B_1^1 = \begin{pmatrix} 0.3e-2 & 0 \\ 0 & -0.3e-2 \end{pmatrix}$, $B_1^2 = \begin{pmatrix} 0.1e-1 & 0 \\ 0 & -0.1e-1 \end{pmatrix}$,

2. $A_2 = \begin{pmatrix} 0.1 & 9 \\ 0 & -0.2 \end{pmatrix}$, $B_2^1 = \begin{pmatrix} 0 & 0.6e-4 \\ -0.6e-4 & 0 \end{pmatrix}$,

Table 2. The values of $\omega(A)$, δ_1, δ_2, δ_3, $\overline{\delta_2}$, γ, $\omega(A+B)$, $\|B\|$, Δ, Δ_1, Δ_2 and Δ_3 according to given datas in Example 1.

A	A_1	A_2	A_3
$\omega(A)$	6.01596	82.9364	545.141
δ_1	$0.338853e-2$	$0.661991e-4$	$0.392831\ e-5$
δ_2	$0.881848e-2$	$0.639666e-3$	$0.973172e-4$
$\overline{\delta_2}$	$0.252557e-1$	$0.334819e-3$	$0.456911e-4$
δ_3	$0.501325e-1$	$0.669625e-3$	$0.91382e-4$
γ	$0.501325e-1$	$0.669625e-3$	$0.973172e-4$

B	B_1^1	B_1^2	B_2^1	B_2^2	B_3^1	B_3^2
$\omega(A+B)$	6.10758	6.33469	82.9348	82.9586	545.139	545.313
$\|B\|$	$0.3e-2$	$0.1e-1$	$0.6e-4$	$0.334e-3$	$0.3e-5$	$0.95e-4$
Δ	$0.9162e-1$	0.31873	$0.16e-2$	$0.222e-1$	$0.2e-2$	0.172
Δ_1	1.331538	$\star$	18.792484	$\star$	104.079063	$\star$
Δ_2	$\star$	$\star$	14.862078	82.733494	35.792938	$\star$
Δ_3	0.377132	1.476424	8.162382	82.531580	18.503936	$\star$

$B_2^2 = \begin{pmatrix} 0 & 0.334e-3 \\ 0.334e-3 & 0 \end{pmatrix}$,

3. $A_3 = \begin{pmatrix} 0.5 & 0 \\ 10 & 0.7 \end{pmatrix}$, $B_3^1 = \begin{pmatrix} 0.3e-5 & 0 \\ 0 & -0.3e-5 \end{pmatrix}$,
$B_3^2 = \begin{pmatrix} 0.95e-4 & 0 \\ 0 & 0.1e-4 \end{pmatrix}$.

Now, let us give in Table 2, the values of $\omega(A)$, δ_1, δ_2, $\overline{\delta_2}$, δ_3, γ, $\omega(A+B)$, $\|B\|$, Δ, Δ_1, Δ_2 and Δ_3 according to given datas:

As can be seen from Table 2, δ_3 is bigger than δ_2 for relatively large values of perturbation, so is bigger than δ_1. δ_3 is smaller than δ_2 for relatively small values of perturbation. However, γ is the biggest of the Schur stability perturbation bounds δ_1, δ_2, δ_3 and γ, so in any case γ is biggest bound of perturbation.

For the perturbation matrices B, while $\|B\| < \delta_1$ (Theorem 1), $\|B\| \le \overline{\delta_2} \le \delta_2$ (Theorem 2) (Schur stability is not guaranteed by Theorem 2 while $\delta_2 < \overline{\delta_2}$), $\|B\| < \delta_3$ (Corollary 1 and Theorem 4), the values Δ_1, Δ_2 and Δ_3 have been calculated, respectively. Otherwise, the symbol $\star$ has been used for the values Δ_1, Δ_2 and Δ_3. As can clearly be seen from Table 2, the upper bound Δ_3 is closer to the accured value Δ than others.

Secondly, we give the examples of Theorem 5, Theorem 6 and Theorem 7.

Example 2

Let $A = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.1 \end{pmatrix}$ and $\omega^* = 100$.

Schur stability parameter $\omega(A)$ is 4/3, i.e. $\omega(A) = \frac{4}{3}$.
Schur stability boundary γ is 0.5 , i.e. $\gamma = 0.5$ (Theorem 4).
100– Schur stability boundary δ_3^* is 0.494987, that is, $\delta_3^* = 0.494987$ (Theorem 5).

Since $\omega(A) \le 100$ holds, the matrix A is 100-Schur stable. Also it follows from Theorem 5, for any perturbation matrix satisfying $\|B\| \le 0.494987$, we know that matrix $A+B$ is 100–Schur stable.

Let us the perturbation matrix $B_1 = \begin{pmatrix} 0.494987 & 0 \\ 0 & 0.494987 \end{pmatrix}$ with $\|B_1\| = 0.494987$. We have $A+B_1$, thus we see that $\omega(A+B_1) = 99.9913 < 100$ holds, therefore the matrix $A+B_1$ is 100–Schur stable matrix.

Let us the perturbation matrix $B_2 = \begin{pmatrix} 0.49499 & 0 \\ 0 & 0.49499 \end{pmatrix}$ with $\|B_2\| = 0.49499 > \delta_3^*$. We have $A+B_2$, thus we see that $\omega(A+B_2) = 100.051 > 100$ holds, therefore the matrix $A+B_2$ is 100–Schur unstable matrix.

Example 3

Consider the delay difference equation

$$x_{n+1} - x_n = -\frac{21}{100}x_{n-1} - \frac{1}{100}x_{n-2},\ n > 0, \qquad (8)$$

in Driver et al. (1992) and analyze it. For the companion matrix C, it is easy to check that $\omega(C) = \|H\| = 10.0889$. Since $\omega(C) < \infty$ the equation (8) is Schur stable. Let $\omega_1^* = 15$, $\omega_2^* = 60$.

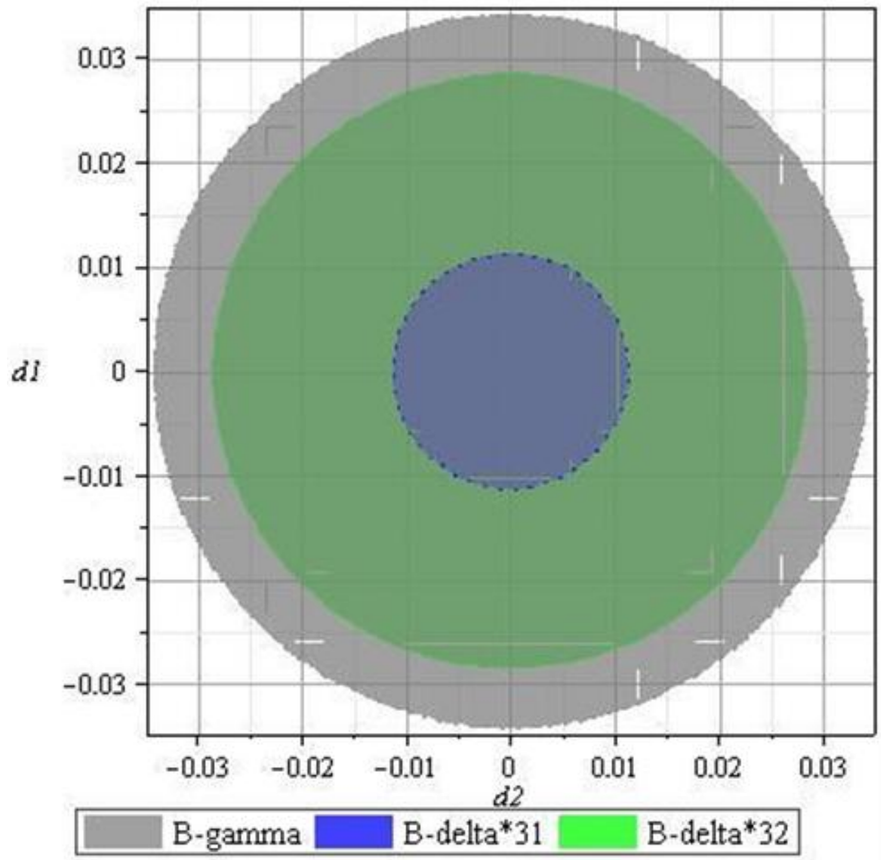

Figure 1. The regions of B_γ, $B_{\delta^*_{31}}$ and $B_{\delta^*_{32}}$.

- $\gamma = \gamma(C) = \max\{0.525789e-2,\ 0.342496e-1\} = 0.342497e{-}1$
- $\delta^*_{31} = 0.113009e - 1$, $\delta^*_{32} = 0.285463e - 1$,
- $d = (d_2, d_1, 0) \Rightarrow \|d\| = \sqrt{d_2^2 + d_1^2} < \gamma(C) \Rightarrow$ $d_2^2 + d_1^2 < \gamma^2$
- $B_\gamma = \{(d_2, d_1) \,|\, \|d\| < 0.342496e - 1\}$

$B_{\delta^*_{31}} = \{(d_2, d_1) \,|\, \|d\| < 0.113009e - 1\}$,
$B_{\delta^*_{32}} = \{(d_2, d_1) \,|\, \|d\| < 0.285463e - 1\}$

Consider the perturbed equation

$$y_{n+1} - y_n = \left(-\frac{21}{100} + d_1\right) y_{n-1} + \left(-\frac{1}{100} + d_2\right) y_{n-2}, \qquad n > 0, \tag{9}$$

- The Equation (9) for all elements of the set B_γ is Schur stable.
- The Equation (9) for all elements of the set $B_{\delta^*_{31}}$ is 15 −Schur stable.
- The Equation (9) for all elements of the set $B_{\delta^*_{32}}$ is 60−Schur stable.

Schur stability region B_γ, 15 −Schur stability region $B_{\delta^*_{31}}$ and 60 −Schur stability region $B_{\delta^*_{32}}$ of the Equation (8) have been given with Figure 1. As is clearly seen from Figure 1, $B_{\delta^*_{31}} \subset B_{\delta^*_{32}} \subset B_\gamma$.

Remark 5

For matrix $A = \mathbf{0}$, we have $\omega(A) = 1$, which has the best quality of the Schur stability. In this case, the values $\delta_1 = 1/20$, $\delta_2 = \frac{1}{6\pi}$, $\overline{\delta_2} = \frac{1}{\sqrt{2}}$ and $\gamma = \delta_3 = 1$. In view of the spectral criterion, we see that the value $\gamma = 1$ for the Schur stability of the system is the upper bound of the largest perturbation allowed at the same time.

Remark 6

As is shown in Example 2, the inequality $\|B\| \le \delta^*_3$ for perturbation of ω^*–Schur stability are the *very sharp* inequality.

Note 3

The numerical examples have been computed by using matrix vector calculator MVC (Bulgak and Eminov, 2001) and Maple 12.

Conclusıons

Firstly, the perturbe bounds in Theorem 1, Theorem 2 and Corollary 1 have been compared in Lemma 2 and in Table 1. As a result of this comparison, we have given the continuity theorem (Theorem 4) *which allows the greater perturbe than others* without disturbing the Schur stability, and then the theorem (Theorem 5) which gives the perturbe bounds without disturbing the ω^*–Schur Stability.

Secondly, we have applied the scalar-linear difference equations with order k the obtained results and obtained the new useful results (Theorem 6 and 7) which gives the regions of Schur stability and ω^*–Schur stability.

Then, all the results have been supported with numerical examples. We have seen from examples too that the bound of perturbation of Schur stability in Theorem 4 is *greater than others* (Example 1), the bound of perturbation of ω^*–Schur stability in Theorem 5 is very sharp (Example 2), the region of Schur stability in Theorem 6 and the region of ω^*–Schur stability in Theorem 7 can easily be calculated (Example 3).

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