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Dynamic integral inequalities on time scales with 'maxima'

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Abstract

In this paper, some new types of integral inequalities on time scales with 'maxima' are established, which can be used as a handy tool in the investigation of making estimates for bounds of solutions of dynamic equations on time scales with 'maxima'. The theoretical results are illustrated by an example at the end of this paper.

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1 Introduction

Integral inequalities which provide explicit bounds of the unknown functions play a fundamental role in the development of the theory of differential and integral equations. In the past few years, a number of integral inequalities have been established by many researchers, which are motivated by certain applications such as existence, uniqueness, continuous dependence, comparison, boundedness and stability of solutions of differential and integral equations.

Many integral inequalities have been established on time scales, which have been designed in order to unify continuous and discrete analysis; see, for example, [1–15]. The development of the theory of time scales was initiated by Hilger [16].

Differential equations with 'maxima' are a special type of differential equations that contain the maximum of the unknown function over a previous interval. Several integral inequalities have been established in the case when maxima of the unknown scalar function are involved in the integral; see [17–21] and references cited therein.

To the best of our knowledge, there are not papers in the literature dealing with inequalities on time scales with 'maxima'. To fill this gap, we initiate in this paper the study of integral inequalities on time scales with 'maxima'. Some new inequalities are established and some applications for them are presented. The significance of our work lies in the fact that 'maxima' are taken on intervals $[\beta t, t]$ which have non-constant length, where $0 < \beta < 1$. The most papers take the 'maxima' on $[t - h, t]$, where $h > 0$ is a given constant.

2 Preliminaries

In this section, we list the following well-known definitions and some lemmas which can be found in [22] and the references therein.

Definition 2.1 A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$.

The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}_+$ are defined, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} \mid s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} \mid s < t\}, \qquad \mu(t) := \sigma(t) - t,$$

for all $t \in \mathbb{T}$. If $\sigma(t) > t$, t is said to be right-scattered, and if $\rho(t) < t$, t is said to be left-scattered; if $\sigma(t) = t$, t is said to be right-dense, and if $\rho(t) = t$, t is said to be left-dense. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k = \mathbb{T} - \{M\}$; otherwise set $\mathbb{T}^k = \mathbb{T}$.

Definition 2.2 A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous (rd-continuous is short for right-dense continuous) provided it is continuous at each right-dense point in $\mathbb{T}$ and has a left-sided limit at each left-dense point in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

Definition 2.3 For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, the delta derivative of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left|f\bigl(\sigma(t)\bigr) - f(s) - f^{\Delta}(t)\bigl(\sigma(t) - s\bigr)\right| \le \epsilon\left|\sigma(t) - s\right|$$

for all $s \in U$.

Definition 2.4 For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by a Banach space), the (delta) derivative is defined at point t by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}$$

if f is continuous at t and t is right-scattered. If t is not right-scattered, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

provided this limit exists.

Definition 2.5 If $F^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

Lemma 2.1 ([22]) *Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\nu^{\Delta}(t)\Delta t = \int_{\nu(a)}^{\nu(b)} \bigl(f \circ \nu^{-1}\bigr)(s)\widetilde{\Delta}s.$$

Definition 2.6 We say that a function $p:\mathbb{T}\to\mathbb{R}$ is regressive provided $1+\mu(t)p(t)\neq 0$ for all $t\in\mathbb{T}^\kappa$ holds. The set of all regressive and rd-continuous functions $p:\mathbb{T}\to\mathbb{R}$ will be denoted by $\mathcal{R}(\mathbb{T},\mathbb{R})$. We also define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by

$$\mathcal{R}^+=\mathcal{R}(\mathbb{T},\mathbb{R})=\{p\in\mathcal{R}:1+\mu(t)p(t)>0 \text{ for all } t\in\mathbb{T}\}.$$

Definition 2.7 If $p\in\mathcal{R}(\mathbb{T},\mathbb{R})$, then we define the generalized exponential function $e_p(t,s)$ by

$$e_p(t,s)=\exp\biggl(\int_s^t \xi_{\mu(\tau)}\bigl(p(\tau)\bigr)\Delta\tau\biggr) \quad \text{for all } s,t\in\mathbb{T},$$

where

$$\xi_h(z)=\begin{cases}\frac{1}{h}\operatorname{Log}(1+zh), & h>0,\\ z, & h=0\end{cases}$$

and Log is the principal logarithm function.

Lemma 2.2 ([22]) (Gronwall's inequality) *Suppose $u\in C_{\mathrm{rd}}(\mathbb{T})$, $p\in\mathcal{R}^+$, $p\geq 0$ and $\alpha\in\mathbb{R}$. Then*

$$u(t)\leq\alpha+\int_{t_0}^t p(s)u(s)\Delta s, \quad t\in\mathbb{T},$$

implies

$$u(t)\leq\alpha e_p(t,t_0), \quad t\in\mathbb{T}.$$

Lemma 2.3 ([23]) *Assume that $a\geq 0$, $p\geq q\geq 0$, and $p\neq 0$. Then*

$$a^{\frac{q}{p}}\leq\biggl(\frac{q}{p}k^{\frac{q-p}{p}}a+\frac{p-q}{p}k^{\frac{q}{p}}\biggr) \quad \textit{for any } k>0.$$

3 Main results

For convenience of notation, we let throughout $t_0\in\mathbb{T}$, $t_0\geq 0$, $\mathbb{T}_0=[t_0,\infty)\cap\mathbb{T}$ and an interval $[\gamma,\eta]_{\mathbb{T}}=[\gamma,\eta]\cap\mathbb{T}$. In addition, for a strictly increasing function $\alpha:\mathbb{T}\to\mathbb{R}$, $\widetilde{\mathbb{T}}=\alpha(\mathbb{T})$ is a time scale such that $\widetilde{\mathbb{T}}\subseteq\mathbb{T}$. For $f\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, we define a notation of the composition of two functions on time scales by

$$f(\gamma)\circ\alpha^{-1}(s)=f\bigl(\alpha^{-1}(s)\bigr), \quad \gamma\in\mathbb{T}, s\in\widetilde{\mathbb{T}}.$$

Example 3.1 Let $f(t)=5^{t^2}$ for $t\in\mathbb{T}:\mathbb{N}_0^{\frac{1}{2}}=\{\sqrt{n}:n\in\mathbb{N}_0\}$ and $\alpha(t)=t^2$ for $t\in\mathbb{T}$. Then we have $\alpha^{-1}(t)=\sqrt{t}$ for $t\in\widetilde{\mathbb{T}}=\mathbb{N}_0$ and

$$f(\gamma)\circ\alpha^{-1}(s)=\bigl(5^{\gamma^2}\bigr)\circ\sqrt{s}=5^s, \quad s\in\widetilde{\mathbb{T}}.$$

Theorem 3.1 *Let the following conditions be satisfied*:

(i) *The function* $\alpha \in C_{\mathrm{rd}}(\mathbb{T}_0, \mathbb{R}_+)$ *is strictly increasing.*

(ii) *The functions* a, b, p *and* $q \in C_{\mathrm{rd}}(\mathbb{T}_0, \mathbb{R}_+)$.

(iii) *The function* $\phi \in C_{\mathrm{rd}}([\beta\tau, t_0]_{\mathbb{T}}, \mathbb{R}_+)$, *where* $0 < \beta < 1$ *and* $\tau = \min\{t_0, \alpha(t_0)\}$.

(iv) *The functions* $f, g \in C_{\mathrm{rd}}(\mathbb{T}_0, [1, \infty))$ *are nondecreasing.*

(v) *The function* $u \in C_{\mathrm{rd}}([\beta\tau, \infty)_{\mathbb{T}}, \mathbb{R}_+)$ *and satisfies the inequalities*

$$\begin{aligned} u(t) &\le k + f(t) \int_{t_0}^{t} \Big[p(s)u(s) + q(s) \max_{\xi \in [\beta s, s]_{\mathbb{T}}} u(\xi) \Big] \Delta s \\ &\quad + g(t) \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(\gamma)u(\gamma) + b(\gamma) \max_{\xi \in [\beta\gamma, \gamma]_{\mathbb{T}}} u(\xi) \Big] \circ \alpha^{-1}(s) \widetilde{\Delta} s, \quad t \in \mathbb{T}_0, \end{aligned} \tag{3.1}$$

$$u(t) \le \phi(t), \quad t \in [\beta\tau, t_0]_{\mathbb{T}}, \tag{3.2}$$

where $k \ge 0$.

Then

$$u(t) \le M f(t) g(t) e_A(t, t_0), \quad t \in \mathbb{T}_0, \tag{3.3}$$

holds, where

$$M = \max\Big\{ k, \max_{s \in [\beta\tau, t_0]_{\mathbb{T}}} \phi(s) \Big\} \tag{3.4}$$

and

$$A(t) = f(t)g(t)\big[p(t) + a(t)\alpha^{\triangle}(t)\big] + \max_{\xi \in [\beta t, t]_{\mathbb{T}}} f^*(\xi) g^*(\xi) \big[q(t) + b(t)\alpha^{\triangle}(t)\big] \tag{3.5}$$

with functions $f^*(t)$ *and* $g^*(t)$ *defined by*

$$f^*(t) = \begin{cases} f(t), & t \in \mathbb{T}_0, \\ f(t_0), & t \in [\beta\tau, t_0]_{\mathbb{T}}, \end{cases} \tag{3.6}$$

and

$$g^*(t) = \begin{cases} g(t), & t \in \mathbb{T}_0, \\ g(t_0), & t \in [\beta\tau, t_0]_{\mathbb{T}}. \end{cases} \tag{3.7}$$

Proof From inequality (3.1), we have that

$$\begin{aligned} u(t) &\le f(t)g(t) \Big\{ k + \int_{t_0}^{t} \Big[p(s)u(s) + q(s) \max_{\xi \in [\beta s, s]_{\mathbb{T}}} u(\xi) \Big] \Delta s \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(\gamma)u(\gamma) + b(\gamma) \max_{\xi \in [\beta\gamma, \gamma]_{\mathbb{T}}} u(\xi) \Big] \circ \alpha^{-1}(s) \widetilde{\Delta} s \Big\}, \quad t \in \mathbb{T}_0. \end{aligned}$$

Define a function $v : [\beta\tau,\infty)_{\mathbb{T}} \to \mathbb{R}_+$ by

$$v(t) = \begin{cases} M + \int_{t_0}^{t} [p(s)u(s) + q(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} u(\xi)]\Delta s \\ \quad + \int_{\alpha(t_0)}^{\alpha(t)} [a(\gamma)u(\gamma) + b(\gamma)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} u(\xi)] \circ \alpha^{-1}(s)\widetilde{\Delta}s, & t \in \mathbb{T}_0, \\ M, & t \in [\beta\tau, t_0]_{\mathbb{T}}, \end{cases}$$

where M is defined by (3.4). Note that the function $v(t)$ is nondecreasing.

It follows that the inequality

$$u(t) \le f^*(t)g^*(t)v(t), \quad t \in [\beta\tau,\infty)_{\mathbb{T}},$$

holds. Therefore, for $t \in \mathbb{T}_0$ and $s \in [t_0, t]_{\mathbb{T}}$, we have

$$\max_{\xi\in[\beta s,s]_{\mathbb{T}}} u(\xi) \le \max_{\xi\in[\beta s,s]_{\mathbb{T}}} f^*(\xi)g^*(\xi)v(\xi) \le \max_{\xi\in[\beta s,s]_{\mathbb{T}}} f^*(\xi)g^*(\xi)v(s).$$

For $t \in \mathbb{T}_0$ and $s \in [\alpha(t_0), \alpha(t)]_{\widetilde{\mathbb{T}}}$, we have

$$\begin{aligned} \max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} u(\xi) \circ \alpha^{-1}(s) &= \max_{\xi\in[\beta\alpha^{-1}(s),\alpha^{-1}(s)]_{\mathbb{T}}} u(\xi) \\ &\le \max_{\xi\in[\beta\alpha^{-1}(s),\alpha^{-1}(s)]_{\mathbb{T}}} f^*(\xi)g^*(\xi)v(\xi) \\ &\le \max_{\xi\in[\beta\alpha^{-1}(s),\alpha^{-1}(s)]_{\mathbb{T}}} f^*(\xi)g^*(\xi) \max_{\xi\in[\beta\alpha^{-1}(s),\alpha^{-1}(s)]_{\mathbb{T}}} v(\xi) \\ &= \max_{\xi\in[\beta\alpha^{-1}(s),\alpha^{-1}(s)]_{\mathbb{T}}} f^*(\xi)g^*(\xi)v\bigl(\alpha^{-1}(s)\bigr) \\ &= \Bigl(\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} f^*(\xi)g^*(\xi)v(\gamma)\Bigr) \circ \alpha^{-1}(s). \end{aligned}$$

Then from the definition of $v(t)$ and the above analysis, we get for $t \in \mathbb{T}_0$ that

$$\begin{aligned} v(t) &\le M + \int_{t_0}^{t} \Bigl[p(s)f^*(s)g^*(s)v(s) + q(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} f^*(\xi)g^*(\xi)v(s)\Bigr]\Delta s \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} \Bigl[a(\gamma)f^*(\gamma)g^*(\gamma)v(\gamma) + b(\gamma)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} f^*(\xi)g^*(\xi)v(\gamma)\Bigr] \circ \alpha^{-1}(s)\widetilde{\Delta}s \\ &= M + \int_{t_0}^{t} \Bigl[p(s)f^*(s)g^*(s)v(s) + q(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} f^*(\xi)g^*(\xi)v(s)\Bigr]\Delta s \\ &\quad + \int_{t_0}^{t} \Bigl[a(s)f^*(s)g^*(s)v(s) + b(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} f^*(\xi)g^*(\xi)v(s)\Bigr]\alpha^{\Delta}(s)\Delta s \\ &= M + \int_{t_0}^{t} \Bigl\{f^*(s)g^*(s)\bigl[p(s) + a(s)\alpha^{\Delta}(s)\bigr] \\ &\quad + \max_{\xi\in[\beta s,s]_{\mathbb{T}}} f^*(\xi)g^*(\xi)\bigl[q(s) + b(s)\alpha^{\Delta}(s)\bigr]\Bigr\}v(s)\Delta s. \end{aligned} \tag{3.8}$$

Applying Gronwall's inequality for (3.8), we obtain

$$v(t) \le Me_A(t, t_0), \quad t \in \mathbb{T}_0,$$

which results in (3.3). This completes the proof. □

As a special case of Theorem 3.1, we obtain the following result.

Corollary 3.1 *Let the following conditions be fulfilled*:

(i) *The conditions* (i)-(iii) *of Theorem* 3.1 *are satisfied.*

(ii) *The function* $u \in C_{\mathrm{rd}}([\beta\tau,\infty)_{\mathbb{T}},\mathbb{R}_+)$ *and satisfies the inequalities*

$$\begin{aligned} u(t) \le{}& k + \psi \int_{t_0}^{t} \Big[p(s)u(s) + q(s) \max_{\xi\in[\beta s,s]_{\mathbb{T}}} u(\xi)\Big]\Delta s \\ &+ \omega \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(\gamma)u(\gamma) + b(\gamma) \max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} u(\xi)\Big] \circ \alpha^{-1}(s)\widetilde{\Delta}s, \quad t\in\mathbb{T}_0, \end{aligned} \tag{3.9}$$

$$u(t) \le \phi(t), \quad t\in[\beta\tau,t_0]_{\mathbb{T}}, \tag{3.10}$$

where constants $k \ge 0$ *and* $\psi,\omega \ge 1$.

Then

$$u(t) \le \psi\omega M e_B(t,t_0), \quad t\in\mathbb{T}_0, \tag{3.11}$$

holds, where M *is defined in* (3.4) *and*

$$B(t) = \psi\omega\big[p(t) + q(t) + a(t)\alpha^{\triangle}(t) + b(t)\alpha^{\triangle}(t)\big]. \tag{3.12}$$

Remark 3.1 If we take $\beta \to 1$, $\psi = \omega = 1$, $\alpha(t) = t$, then Corollary 3.1 reduces to Gronwall's inequality on time scales without 'maxima' as in Lemma 2.2.

In the case when in place of the constant k involved in Theorem 3.1 we have a function $k(t)$, we obtain the following result.

Theorem 3.2 *Let the following conditions be satisfied*:

(i) *The conditions* (i)-(iv) *of Theorem* 3.1 *are satisfied.*

(ii) *The function* $k \in C_{\mathrm{rd}}(\mathbb{T}_0,(0,\infty))$ *is nondecreasing.*

(iii) *The function* $u \in C_{\mathrm{rd}}([\beta\tau,\infty)_{\mathbb{T}},\mathbb{R}_+)$ *and satisfies the inequalities*

$$\begin{aligned} u(t) \le{}& k(t) + f(t) \int_{t_0}^{t} \Big[p(s)u(s) + q(s) \max_{\xi\in[\beta s,s]_{\mathbb{T}}} u(\xi)\Big]\Delta s \\ &+ g(t) \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(\gamma)u(\gamma) + b(\gamma) \max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} u(\xi)\Big] \circ \alpha^{-1}(s)\widetilde{\Delta}s, \quad t\in\mathbb{T}_0, \end{aligned} \tag{3.13}$$

$$u(t) \le \phi(t), \quad t\in[\beta\tau,t_0]_{\mathbb{T}}. \tag{3.14}$$

Then

$$u(t) \le Nk(t)f(t)g(t)e_A(t,t_0), \quad t\in\mathbb{T}_0, \tag{3.15}$$

holds, where $A(t)$ *is defined by* (3.5) *and*

$$N = \max\left\{1, \frac{\max_{s\in[\beta\tau,t_0]_{\mathbb{T}}}\phi(s)}{k(t_0)}\right\}. \tag{3.16}$$

Proof From inequality (3.13) we obtain, for $t \in \mathbb{T}_0$,

$$\begin{aligned}\frac{u(t)}{k(t)} &\leq 1 + f(t)\int_{t_0}^{t}\left[p(s)\frac{u(s)}{k(t)} + q(s)\frac{\max_{\xi\in[\beta s,s]_{\mathbb{T}}} u(\xi)}{k(t)}\right]\Delta s \\ &\quad + g(t)\int_{\alpha(t_0)}^{\alpha(t)}\left[a(\gamma)\frac{u(\gamma)}{k(t)} + b(\gamma)\frac{\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} u(\xi)}{k(t)}\right]\circ\alpha^{-1}(s)\widetilde{\Delta}s. \qquad (3.17)\end{aligned}$$

Let us define functions $k^* : [\beta\tau,\infty)_{\mathbb{T}} \to \mathbb{R}_+$ and $w : [\beta\tau,\infty)_{\mathbb{T}} \to \mathbb{R}_+$ by

$$k^*(t) = \begin{cases} k(t), & t \in \mathbb{T}_0, \\ k(t_0), & t \in [\beta\tau, t_0]_{\mathbb{T}}, \end{cases}$$

$$w(t) = \frac{u(t)}{k^*(t)}, \quad t \in [\beta\tau,\infty)_{\mathbb{T}}.$$

Note that the function $k^*(t)$ is nondecreasing on $t \in [\beta\tau,\infty)_{\mathbb{T}}$. From monotonicity of $k(t)$ and $\alpha(t)$ we get, for $t \in \mathbb{T}_0$ and $s \in [t_0, t]_{\mathbb{T}}$,

$$\frac{\max_{\xi\in[\beta s,s]_{\mathbb{T}}} u(\xi)}{k(t)} \leq \frac{\max_{\xi\in[\beta s,s]_{\mathbb{T}}} u(\xi)}{k^*(s)} = \max_{\xi\in[\beta s,s]_{\mathbb{T}}}\frac{u(\xi)}{k^*(s)} \leq \max_{\xi\in[\beta s,s]_{\mathbb{T}}}\frac{u(\xi)}{k^*(\xi)}. \qquad (3.18)$$

For $t \in \mathbb{T}_0$ and $s \in [\alpha(t_0),\alpha(t)]_{\widetilde{\mathbb{T}}}$, we have

$$\begin{aligned}\frac{\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} u(\xi)\circ\alpha^{-1}(s)}{k(t)} &= \frac{\max_{\xi\in[\beta\alpha^{-1}(s),\alpha^{-1}(s)]_{\mathbb{T}}} u(\xi)}{k(t)} \\ &\leq \frac{\max_{\xi\in[\beta\alpha^{-1}(s),\alpha^{-1}(s)]_{\mathbb{T}}} u(\xi)}{k^*(\alpha^{-1}(s))} \\ &= \max_{\xi\in[\beta\alpha^{-1}(s),\alpha^{-1}(s)]_{\mathbb{T}}}\frac{u(\xi)}{k^*(\alpha^{-1}(s))} \\ &\leq \max_{\xi\in[\beta\alpha^{-1}(s),\alpha^{-1}(s)]_{\mathbb{T}}}\frac{u(\xi)}{k^*(\xi)} \\ &= \max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}}\frac{u(\xi)}{k^*(\xi)}\circ\alpha^{-1}(s). \qquad (3.19)\end{aligned}$$

From inequalities (3.17), (3.18) and (3.19) and the definition of $w(t)$, we have

$$\begin{aligned} w(t) &\leq 1 + f(t)\int_{t_0}^{t}\Bigl[p(s)w(s) + q(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} w(\xi)\Bigr]\Delta s \\ &\quad + g(t)\int_{\alpha(t_0)}^{\alpha(t)}\Bigl[a(\gamma)w(\gamma) + b(\gamma)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} w(\xi)\Bigr]\circ\alpha^{-1}(s)\widetilde{\Delta}s, \quad t \in \mathbb{T}_0, \qquad (3.20) \\ w(t) &\leq \frac{\phi(t)}{k(t_0)}, \quad t \in [\beta\tau, t_0]_{\mathbb{T}}. \qquad (3.21)\end{aligned}$$

Using Theorem 3.1 for (3.20) and (3.21), we get

$$w(t) \leq Nf(t)g(t)e_A(t,t_0), \quad t \in \mathbb{T}_0,$$

which results in (3.15). This completes the proof. $\square$

Corollary 3.2 *Let the following conditions be fulfilled*:

(i) *The conditions* (i)-(iii) *of Theorem* 3.1 *and the condition* (ii) *of Theorem* 3.2 *are satisfied*.

(ii) *The function* $u \in C_{\mathrm{rd}}([\beta\tau,\infty)_{\mathbb{T}},\mathbb{R}_+)$ *and satisfies the inequalities*

$$u(t) \le k(t) + \psi \int_{t_0}^{t} \Big[p(s)u(s) + q(s) \max_{\xi\in[\beta s,s]_{\mathbb{T}}} u(\xi)\Big]\Delta s + \omega \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(\gamma)u(\gamma) + b(\gamma) \max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} u(\xi)\Big] \circ \alpha^{-1}(s)\widetilde{\Delta}s, \quad t \in \mathbb{T}_0, \tag{3.22}$$

$$u(t) \le \phi(t), \quad t \in [\beta\tau, t_0]_{\mathbb{T}}, \tag{3.23}$$

where constants $\psi, \omega \ge 1$.

Then

$$u(t) \le \psi\omega N k(t) e_B(t,t_0), \quad t \in \mathbb{T}_0, \tag{3.24}$$

holds, where N *and* $B(t)$ *are defined in* (3.16) *and* (3.12), *respectively*.

Remark 3.2 As a special case of Corollary 3.2, we have a result for dynamic Gronwall's inequality without 'maxima' ([22], Theorem 6.4 p.256).

In the case when the function involved in the right part of inequality (3.13) is not a monotonic function, we obtain the following result.

Theorem 3.3 *Let the following conditions be satisfied*:

(i) *The conditions* (i), (ii), (iv) *of Theorem* 3.1 *are satisfied*.

(ii) *The function* $\phi \in C_{\mathrm{rd}}([\beta\tau,\infty)_{\mathbb{T}},\mathbb{R}_+)$ *with* $\max_{s\in[\beta\tau,t_0]_{\mathbb{T}}} \phi(s) > 0$, *where* $0 < \beta < 1$ *and* $\tau = \min\{t_0, \alpha(t_0)\}$.

(iii) *The function* $u \in C_{\mathrm{rd}}([\beta\tau,\infty)_{\mathbb{T}},\mathbb{R}_+)$ *and satisfies the inequalities*

$$u(t) \le \phi(t) + f(t) \int_{t_0}^{t} \Big[p(s)u(s) + q(s) \max_{\xi\in[\beta s,s]_{\mathbb{T}}} u(\xi)\Big]\Delta s + g(t) \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(\gamma)u(\gamma) + b(\gamma) \max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} u(\xi)\Big] \circ \alpha^{-1}(s)\widetilde{\Delta}s, \quad t \in \mathbb{T}_0, \tag{3.25}$$

$$u(t) \le \phi(t), \quad t \in [\beta\tau, t_0]_{\mathbb{T}}. \tag{3.26}$$

Then

$$u(t) \le \phi(t) + f(t)g(t)h(t)e_A(t,t_0), \quad t \in \mathbb{T}_0, \tag{3.27}$$

holds, where $A(t)$ *is defined by* (3.5) *and*

$$h(t) = \max_{s\in[\beta\tau,t_0]_{\mathbb{T}}} \phi(s) + \int_{t_0}^{t} \Big[p(s)\phi(s) + q(s) \max_{\xi\in[\beta s,s]_{\mathbb{T}}} \phi(\xi)\Big]\Delta s + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(\gamma)\phi(\gamma) + b(\gamma) \max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} \phi(\xi)\Big] \circ \alpha^{-1}(s)\widetilde{\Delta}s, \quad t \in \mathbb{T}_0. \tag{3.28}$$

Proof From inequality (3.25), we have

$$u(t) \le \phi(t) + f(t)g(t)\left\{\int_{t_0}^{t}\Big[p(s)u(s) + q(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} u(\xi)\Big]\Delta s + \int_{\alpha(t_0)}^{\alpha(t)}\Big[a(\gamma)u(\gamma) + b(\gamma)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} u(\xi)\Big]\circ\alpha^{-1}(s)\widetilde{\Delta}s\right\}, \quad t\in\mathbb{T}_0.$$

Let us define a function $z : [\beta\tau,\infty)_{\mathbb{T}} \to \mathbb{R}_+$ by

$$z(t) = \begin{cases} \int_{t_0}^{t}[p(s)u(s) + q(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} u(\xi)]\Delta s \\ \quad + \int_{\alpha(t_0)}^{\alpha(t)}[a(\gamma)u(\gamma) + b(\gamma)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} u(\xi)]\circ\alpha^{-1}(s)\widetilde{\Delta}s, & t\in\mathbb{T}_0, \\ 0, & t\in[\beta\tau,t_0]_{\mathbb{T}}. \end{cases} \tag{3.29}$$

Therefore,

$$u(t) \le \phi(t) + f^*(t)g^*(t)z(t), \quad t\in[\beta\tau,\infty)_{\mathbb{T}}, \tag{3.30}$$

where $f^*(t)$, $g^*(t)$ are defined by (3.6) and (3.7), respectively.

From the definition of the function $z(t)$, it follows that

$$\begin{aligned} z(t) \le{} & \int_{t_0}^{t}\Big\{p(s)\big[\phi(s) + f^*(s)g^*(s)z(s)\big] \\ & + q(s)\Big[\max_{\xi\in[\beta s,s]_{\mathbb{T}}}\phi(\xi) + \max_{\xi\in[\beta s,s]_{\mathbb{T}}} f^*(\xi)g^*(\xi)z(\xi)\Big]\Big\}\Delta s \\ & + \int_{\alpha(t_0)}^{\alpha(t)}\Big\{a(\gamma)\big[\phi(\gamma) + f^*(\gamma)g^*(\gamma)z(\gamma)\big] \\ & + b(\gamma)\Big[\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}}\phi(\xi) + \max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} f^*(\xi)g^*(\xi)z(\xi)\Big]\Big\}\circ\alpha^{-1}(s)\widetilde{\Delta}s \\ \le{} & h(t) + \int_{t_0}^{t}\Big\{p(s)f^*(s)g^*(s)z(s) + q(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} f^*(\xi)g^*(\xi)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} z(\xi)\Big\}\Delta s \\ & + \int_{\alpha(t_0)}^{\alpha(t)}\Big\{a(\gamma)f^*(\gamma)g^*(\gamma)z(\gamma) + b(\gamma)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} f^*(\xi)g^*(\xi)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} z(\xi)\Big\} \\ & \circ\alpha^{-1}(s)\widetilde{\Delta}s, \quad t\in\mathbb{T}_0, \end{aligned} \tag{3.31}$$

$$z(t) \le \phi(t), \quad t\in[\beta\tau,t_0]_{\mathbb{T}}, \tag{3.32}$$

where a function $h(t)$ is defined in (3.28).

Since the function $h(t) : \mathbb{T}_0 \to (0,\infty)$ is nondecreasing and $h(t_0) = \max_{s\in[\beta\tau,t_0]_{\mathbb{T}}}\phi(s)$, by using Theorem 3.2 for (3.31) and (3.32), we get

$$z(t) \le h(t)e_A(t,t_0), \quad t\in\mathbb{T}_0,$$

which results in (3.27). This completes the proof. □

Now we will consider an inequality in which the unknown function into the left part is presented in a power.

Theorem 3.4 *Let the following conditions be fulfilled*:

(i) *The conditions* (i)-(iv) *of Theorem* 3.1 *are satisfied.*

(ii) *The function* $k \in C_{\mathrm{rd}}(\mathbb{T}_0, (0,\infty))$ *is nondecreasing and the inequality*

$$L := \max_{s\in[\beta\tau,t_0]_{\mathbb{T}}} \phi(s) \le \sqrt[n]{k(t_0)}, \quad n > 1, \tag{3.33}$$

holds.

(iii) *The function* $u \in C_{\mathrm{rd}}([\beta\tau,\infty)_{\mathbb{T}}, \mathbb{R}_+)$ *and satisfies the inequalities*

$$\begin{aligned} u^n(t) \le{}& k(t) + f(t)\int_{t_0}^{t}\Big[p(s)u(s) + q(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} u(\xi)\Big]\Delta s \\ &+ g(t)\int_{\alpha(t_0)}^{\alpha(t)}\Big[a(\gamma)u(\gamma) + b(\gamma)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} u(\xi)\Big]\circ\alpha^{-1}(s)\widetilde{\Delta}s, \\ & t\in\mathbb{T}_0, \end{aligned} \tag{3.34}$$

$$u(t) \le \phi(t), \quad t\in[\beta\tau,t_0]_{\mathbb{T}}. \tag{3.35}$$

Then

$$u(t) \le \frac{1}{n}c^{\frac{1-n}{n}}k(t) + \frac{n-1}{n}c^{\frac{1}{n}} + \left(\frac{1}{n}c^{\frac{1-n}{n}}\right)^3 f(t)g(t)\big(L + r(t)\big)e_D(t,t_0), \quad t\in\mathbb{T}_0, \tag{3.36}$$

holds, where

$$\begin{aligned} r(t) ={}& \int_{t_0}^{t}\Big[p(s)w(s) + q(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} w(\xi)\Big]\Delta s \\ &+ \int_{\alpha(t_0)}^{\alpha(t)}\Big[a(\gamma)w(\gamma) + b(\gamma)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} w(\xi)\Big]\circ\alpha^{-1}(s)\widetilde{\Delta}s, \end{aligned} \tag{3.37}$$

$$\begin{aligned} D(t) ={}& \left(\frac{1}{n}c^{\frac{1-n}{n}}\right)^2\Big\{f(t)g(t)\big[p(t) + a(t)\alpha^{\triangle}(t)\big] \\ &+ \max_{\xi\in[\beta t,t]_{\mathbb{T}}} f^*(\xi)g^*(\xi)\big[q(t) + b(t)\alpha^{\triangle}(t)\big]\Big\}, \end{aligned} \tag{3.38}$$

with

$$w(t) = \begin{cases} \frac{1}{n}c^{\frac{1-n}{n}}k(t) + \frac{n-1}{n}c^{\frac{1}{n}}, & t\in\mathbb{T}_0, \\ \frac{1}{n}c^{\frac{1-n}{n}}k(t_0) + \frac{n-1}{n}c^{\frac{1}{n}}, & t\in[\beta\tau,t_0]_{\mathbb{T}}, \end{cases} \tag{3.39}$$

for any constant $c > 0$ *and* $f^*(t)$, $g^*(t)$ *are defined by* (3.6) *and* (3.7), *respectively.*

Proof Firstly, from inequality (3.34), we have

$$\begin{aligned} u^n(t) \le{}& k(t) + f(t)g(t)\bigg(\int_{t_0}^{t}\Big[p(s)u(s) + q(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} u(\xi)\Big]\Delta s \\ &+ \int_{\alpha(t_0)}^{\alpha(t)}\Big[a(\gamma)u(\gamma) + b(\gamma)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} u(\xi)\Big]\circ\alpha^{-1}(s)\widetilde{\Delta}s\bigg), \quad t\in\mathbb{T}_0. \end{aligned}$$

Define a function $z : [\beta\tau, \infty)_{\mathbb{T}} \to \mathbb{R}_+$ by

$$z(t) = \begin{cases} \int_{t_0}^{t} [p(s)u(s) + q(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} u(\xi)]\Delta s \\ \quad + \int_{\alpha(t_0)}^{\alpha(t)} [a(\gamma)u(\gamma) + b(\gamma)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} u(\xi)] \circ \alpha^{-1}(s)\widetilde{\Delta}s, & t \in \mathbb{T}_0, \\ 0, & t \in [\beta\tau, t_0]_{\mathbb{T}}. \end{cases} \tag{3.40}$$

It follows from inequality (3.34) for $t \in \mathbb{T}_0$ that

$$u(t) \le \big[k(t) + f(t)g(t)z(t)\big]^{\frac{1}{n}}.$$

Using Lemma 2.3, for any $c > 0$, we obtain

$$\begin{aligned} u(t) &\le \frac{1}{n}c^{\frac{1-n}{n}}\big[k(t) + f(t)g(t)z(t)\big] + \frac{n-1}{n}c^{\frac{1}{n}} \\ &= \frac{1}{n}c^{\frac{1-n}{n}}k(t) + \frac{n-1}{n}c^{\frac{1}{n}} + \frac{1}{n}c^{\frac{1-n}{n}}f(t)g(t)z(t) \\ &= w(t) + \frac{1}{n}c^{\frac{1-n}{n}}f(t)g(t)z(t), \quad t \in \mathbb{T}_0. \end{aligned} \tag{3.41}$$

From inequality (3.33) and applying Lemma 2.3 , for any $c > 0$, we have

$$\sqrt[n]{k(t_0)} \le \frac{1}{n}c^{\frac{1-n}{n}}k(t_0) + \frac{n-1}{n}c^{\frac{1}{n}}. \tag{3.42}$$

Indeed, by using inequality (3.42), we have for $t \in [\beta\tau, t_0]_{\mathbb{T}}$

$$u(t) \le \phi(t) \le \phi(t) + \frac{1}{n}c^{\frac{1-n}{n}}f^*(t)g^*(t)z(t) \le w(t) + \frac{1}{n}c^{\frac{1-n}{n}}f^*(t)g^*(t)z(t), \tag{3.43}$$

where $w(t)$ is defined by (3.39).

Now, we define a nondecreasing function $v : \mathbb{T}_0 \to (0, \infty)$ by $v(t) = L + r(t)$, where L and $r(t)$ are defined by (3.33), (3.37), respectively.

From the definition of the function $z(t)$, it follows that

$$\begin{aligned} z(t) &\le \int_{t_0}^{t} \bigg\{ p(s)\bigg[w(s) + \frac{1}{n}c^{\frac{1-n}{n}}f^*(s)g^*(s)z(s)\bigg] \\ &\quad + q(s)\bigg[\max_{\xi\in[\beta s,s]_{\mathbb{T}}} w(\xi) + \frac{1}{n}c^{\frac{1-n}{n}} \max_{\xi\in[\beta s,s]_{\mathbb{T}}} f^*(\xi)g^*(\xi)z(\xi)\bigg]\bigg\}\Delta s \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} \bigg\{ a(\gamma)\bigg[w(\gamma) + \frac{1}{n}c^{\frac{1-n}{n}}f^*(\gamma)g^*(\gamma)z(\gamma)\bigg] \\ &\quad + b(\gamma)\bigg[\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} w(\xi) + \frac{1}{n}c^{\frac{1-n}{n}} \max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} f^*(\xi)g^*(\xi)z(\xi)\bigg]\bigg\} \circ \alpha^{-1}(s)\widetilde{\Delta}s \\ &\le v(t) + \frac{1}{n}c^{\frac{1-n}{n}} \int_{t_0}^{t} \Big[p(s)f^*(s)g^*(s)z(s) + q(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} f^*(\xi)g^*(\xi)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} z(\xi)\Big]\Delta s \\ &\quad + \frac{1}{n}c^{\frac{1-n}{n}} \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(\gamma)f^*(\gamma)g^*(\gamma)z(\gamma) + b(\gamma)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} f^*(\xi)g^*(\xi)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} z(\xi)\Big] \\ &\quad \circ \alpha^{-1}(s)\widetilde{\Delta}s, \quad t \in \mathbb{T}_0, \end{aligned} \tag{3.44}$$

$$z(t) \le \phi(t), \quad t \in [\beta\tau, t_0]_{\mathbb{T}}. \tag{3.45}$$

Applying Theorem 3.2 for (3.44) and (3.45), we obtain

$$z(t) \le \left(\frac{1}{n}c^{\frac{1-n}{n}}\right)^2 v(t)e_D(t,t_0), \quad t \in \mathbb{T}_0,$$

which results in (3.36). This completes the proof. □

The last result concerns inequalities which have powers on both sizes.

Theorem 3.5 *Let the following conditions be fulfilled*:

(i) *The conditions* (i)-(iv) *of Theorem* 3.1 *are satisfied.*

(ii) *The function* $k \in C_{\mathrm{rd}}(\mathbb{T}_0,(0,\infty))$ *is nondecreasing and the inequality*

$$K := \max_{s\in[\beta\tau,t_0]_{\mathbb{T}}}\left\{\phi^{\varepsilon}(s),\phi^{l}(s)\right\} \le \frac{m}{n}c^{\frac{m-n}{n}}k(t_0) + \frac{n-1}{n}c^{\frac{m}{n}} \tag{3.46}$$

holds, for any constant $c \ge 1$ *and* $n \ge m \ge l \ge \delta \ge \varepsilon > 1$.

(iii) *The function* $u \in C_{\mathrm{rd}}([\beta\tau,\infty)_{\mathbb{T}},\mathbb{R}_+)$ *and satisfies the inequalities*

$$\begin{aligned} u^n(t) \le{}& k(t) + f(t)\int_{t_0}^{t}\Big[p(s)u^m(s) + q(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}}u^l(\xi)\Big]\Delta s \\ &+ g(t)\int_{\alpha(t_0)}^{\alpha(t)}\Big[a(\gamma)u^{\delta}(\gamma) + b(\gamma)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}}u^{\varepsilon}(\xi)\Big]\circ\alpha^{-1}(s)\widetilde{\Delta}s, \\ & t\in\mathbb{T}_0, \end{aligned} \tag{3.47}$$

$$u(t) \le \phi(t), \quad t\in[\beta\tau,t_0]_{\mathbb{T}}. \tag{3.48}$$

Then

$$\begin{aligned} u(t) \le{}& \frac{1}{n}c^{\frac{1-n}{n}}k(t) + \frac{n-1}{n}c^{\frac{1}{n}} \\ &+ \frac{1}{n}c^{\frac{1-n}{n}}\left(\frac{m}{n}c^{\frac{m-n}{n}}\right)^2 f(t)g(t)\big(K+\lambda(t)\big)e_E(t,t_0), \quad t\in\mathbb{T}_0, \end{aligned} \tag{3.49}$$

holds, where

$$\begin{aligned} \lambda(t) ={}& \int_{t_0}^{t}\Big[p(s)\overline{w}(s) + q(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}}\overline{w}(\xi)\Big]\Delta s \\ &+ \int_{\alpha(t_0)}^{\alpha(t)}\Big[a(\gamma)\overline{w}(\gamma) + b(\gamma)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}}\overline{w}(\xi)\Big]\circ\alpha^{-1}(s)\widetilde{\Delta}s, \end{aligned} \tag{3.50}$$

$$\begin{aligned} E(t) ={}& \left(\frac{m}{n}c^{\frac{m-n}{n}}\right)^2\Big\{f(t)g(t)\big[p(t) + a(t)\alpha^{\triangle}(t)\big] \\ &+ \max_{\xi\in[\beta t,t]_{\mathbb{T}}}f^*(\xi)g^*(\xi)\big[q(t) + b(t)\alpha^{\triangle}(t)\big]\Big\}, \end{aligned} \tag{3.51}$$

with

$$\overline{w}(t) = \begin{cases} \frac{m}{n}c^{\frac{m-n}{n}}k(t) + \frac{n-1}{n}c^{\frac{m}{n}}, & t\in\mathbb{T}_0, \\ \frac{m}{n}c^{\frac{m-n}{n}}k(t_0) + \frac{n-1}{n}c^{\frac{m}{n}}, & t\in[\beta\tau,t_0]_{\mathbb{T}}, \end{cases} \tag{3.52}$$

and $f^*(t)$, $g^*(t)$ *are defined by* (3.6) *and* (3.7), *respectively.*

Proof From inequality (3.47), we have

$$u^n(t) \le k(t) + f(t)g(t)\biggl(\int_{t_0}^{t}\Bigl[p(s)u^m(s) + q(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} u^l(\xi)\Bigr]\Delta s + \int_{\alpha(t_0)}^{\alpha(t)}\Bigl[a(\gamma)u^\delta(\gamma) + b(\gamma)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} u^\varepsilon(\xi)\Bigr]\circ\alpha^{-1}(s)\widetilde{\Delta}s\biggr), \quad t\in\mathbb{T}_0.$$

We define a function $z : [\beta\tau,\infty)_{\mathbb{T}} \to \mathbb{R}_+$ by

$$z(t) = \begin{cases} \int_{t_0}^{t}[p(s)u^m(s) + q(s)\max_{\xi\in[\beta s,s]_{\mathbb{T}}} u^l(\xi)]\Delta s \\ \quad + \int_{\alpha(t_0)}^{\alpha(t)}[a(\gamma)u^\delta(\gamma) + b(\gamma)\max_{\xi\in[\beta\gamma,\gamma]_{\mathbb{T}}} u^\varepsilon(\xi)]\circ\alpha^{-1}(s)\widetilde{\Delta}s, & t\in\mathbb{T}_0, \\ 0, & t\in[\beta\tau,t_0]_{\mathbb{T}}. \end{cases} \tag{3.53}$$

From inequality (3.47) we have, for $t\in\mathbb{T}_0$,

$$u(t) \le \bigl[k(t) + f(t)g(t)z(t)\bigr]^{\frac{1}{n}},$$
$$u^l(t) \le \bigl[k(t) + f(t)g(t)z(t)\bigr]^{\frac{l}{n}},$$
$$u^m(t) \le \bigl[k(t) + f(t)g(t)z(t)\bigr]^{\frac{m}{n}},$$
$$u^\delta(t) \le \bigl[k(t) + f(t)g(t)z(t)\bigr]^{\frac{\delta}{n}},$$
$$u^\varepsilon(t) \le \bigl[k(t) + f(t)g(t)z(t)\bigr]^{\frac{\varepsilon}{n}}.$$

By using Lemma 2.3, for any $c\ge 1$, we obtain

$$u(t) \le \frac{1}{n}c^{\frac{1-n}{n}}k(t) + \frac{n-1}{n}c^{\frac{1}{n}} + \frac{1}{n}c^{\frac{1-n}{n}}f(t)g(t)z(t), \quad t\in\mathbb{T}_0, \tag{3.54}$$

$$\begin{aligned} u^\varepsilon(t) &\le \frac{\varepsilon}{n}c^{\frac{\varepsilon-n}{n}}k(t) + \frac{n-\varepsilon}{n}c^{\frac{\varepsilon}{n}} + \frac{\varepsilon}{n}c^{\frac{\varepsilon-n}{n}}f(t)g(t)z(t) \\ &\le \overline{w}(t) + \frac{m}{n}c^{\frac{m-n}{n}}f^*(t)g^*(t)z(t), \quad t\in\mathbb{T}_0, \end{aligned} \tag{3.55}$$

$$\begin{aligned} u^\delta(t) &\le \frac{\delta}{n}c^{\frac{\delta-n}{n}}k(t) + \frac{n-\delta}{n}c^{\frac{\delta}{n}} + \frac{\delta}{n}c^{\frac{\delta-n}{n}}f(t)g(t)z(t) \\ &\le \overline{w}(t) + \frac{m}{n}c^{\frac{m-n}{n}}f^*(t)g^*(t)z(t), \quad t\in\mathbb{T}_0, \end{aligned} \tag{3.56}$$

$$\begin{aligned} u^l(t) &\le \frac{l}{n}c^{\frac{l-n}{n}}k(t) + \frac{n-l}{n}c^{\frac{l}{n}} + \frac{l}{n}c^{\frac{l-n}{n}}f(t)g(t)z(t) \\ &\le \overline{w}(t) + \frac{m}{n}c^{\frac{m-n}{n}}f^*(t)g^*(t)z(t), \quad t\in\mathbb{T}_0, \end{aligned} \tag{3.57}$$

$$\begin{aligned} u^m(t) &\le \frac{m}{n}c^{\frac{m-n}{n}}k(t) + \frac{n-m}{n}c^{\frac{m}{n}} + \frac{m}{n}c^{\frac{m-n}{n}}f(t)g(t)z(t) \\ &\le \overline{w}(t) + \frac{m}{n}c^{\frac{m-n}{n}}f^*(t)g^*(t)z(t), \quad t\in\mathbb{T}_0. \end{aligned} \tag{3.58}$$

Moreover, we have

$$\begin{aligned} u^\varepsilon(t) &\le \phi^\varepsilon(t) \le \phi^\varepsilon(t) + \frac{m}{n}c^{\frac{m-n}{n}}f^*(t)g^*(t)z(t) \\ &\le \overline{w}(t) + \frac{m}{n}c^{\frac{m-n}{n}}f^*(t)g^*(t)z(t), \quad t\in[\beta\tau,t_0]_{\mathbb{T}}, \end{aligned} \tag{3.59}$$

and

$$\begin{aligned} u^l(t) &\le \phi^l(t) \le \phi^l(t) + \frac{m}{n} c^{\frac{m-n}{n}} f^*(t) g^*(t) z(t) \\ &\le \overline{w}(t) + \frac{m}{n} c^{\frac{m-n}{n}} f^*(t) g^*(t) z(t), \quad t \in [\beta\tau, t_0]_{\mathbb{T}}, \end{aligned} \tag{3.60}$$

where $\overline{w}(t)$ is defined by (3.52). From the definition of the function $z(t)$, it follows that

$$\begin{aligned} z(t) \le{}& \int_{t_0}^{t} \Big\{ p(s)\Big[\overline{w}(s) + \frac{m}{n} c^{\frac{m-n}{n}} f^*(s) g^*(s) z(s)\Big] \\ &+ q(s)\Big[\max_{\xi \in [\beta s, s]_{\mathbb{T}}} \overline{w}(\xi) + \frac{m}{n} c^{\frac{m-n}{n}} \max_{\xi \in [\beta s, s]_{\mathbb{T}}} f^*(\xi) g^*(\xi) z(\xi)\Big]\Big\} \Delta s \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} \Big\{ a(\gamma)\Big[\overline{w}(\gamma) + \frac{m}{n} c^{\frac{m-n}{n}} f^*(\gamma) g^*(\gamma) z(\gamma)\Big] \\ &+ b(\gamma)\Big[\max_{\xi \in [\beta\gamma, \gamma]_{\mathbb{T}}} \overline{w}(\xi) + \frac{m}{n} c^{\frac{m-n}{n}} \max_{\xi \in [\beta\gamma, \gamma]_{\mathbb{T}}} f^*(\xi) g^*(\xi) z(\xi)\Big]\Big\} \circ \alpha^{-1}(s) \widetilde{\Delta} s \\ \le{}& \rho(t) + \frac{m}{n} c^{\frac{m-n}{n}} \int_{t_0}^{t} \Big[p(s) f^*(s) g^*(s) z(s) + q(s) \max_{\xi \in [\beta s, s]_{\mathbb{T}}} f^*(\xi) g^*(\xi) \max_{\xi \in [\beta s, s]_{\mathbb{T}}} z(\xi)\Big] \Delta s \\ &+ \frac{m}{n} c^{\frac{m-n}{n}} \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(\gamma) f^*(\gamma) g^*(\gamma) z(\gamma) + b(\gamma) \max_{\xi \in [\beta\gamma, \gamma]_{\mathbb{T}}} f^*(\xi) g^*(\xi) \max_{\xi \in [\beta\gamma, \gamma]_{\mathbb{T}}} z(\xi)\Big] \\ &\circ \alpha^{-1}(s) \widetilde{\Delta} s, \quad t \in \mathbb{T}_0, \end{aligned} \tag{3.61}$$

$$z(t) \le \phi(t), \quad t \in [\beta\tau, t_0]_{\mathbb{T}}, \tag{3.62}$$

where a nondecreasing function $\rho(t) : \mathbb{T}_0 \to (0, \infty)$ is defined by $\rho(t) := K + \lambda(t)$ with K, $\lambda(t)$ defined in (3.46) and (3.50), respectively.

Applying Theorem 3.2 for (3.61) and (3.62), we obtain

$$z(t) \le \left(\frac{m}{n} c^{\frac{m-n}{n}}\right)^2 \rho(t) e_E(t, t_0), \quad t \in \mathbb{T}_0,$$

which results in (3.49). This completes the proof. □

4 An application

In this section, in order to illustrate our results, we consider the following first-order dynamic equation with 'maxima':

$$x^{\triangle}(t) = f\Big(t, x(t), \max_{s \in [\beta t, t]_{\mathbb{T}}} x(s)\Big), \quad t \in \mathbb{T}_0, \tag{4.1}$$

with the initial condition

$$x(t) = \phi(t), \quad t \in [\beta\tau, t_0]_{\mathbb{T}}, \tag{4.2}$$

where $f \in C_{\mathrm{rd}}(\mathbb{T}_0 \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\phi \in C_{\mathrm{rd}}([\beta\tau, t_0]_{\mathbb{T}}, \mathbb{R})$, $0 < \beta < 1$, τ is a constant such that $\beta\tau \le t_0$.

Corollary 4.1 *Assume that*:

(H_1) *There exists a strictly increasing function* $\alpha \in C_{\mathrm{rd}}(\mathbb{T}_0, \mathbb{R}_+)$ *such that* $\alpha(\mathbb{T}) = \widetilde{\mathbb{T}}$ *is a time scale and* $\min\{t_0, \alpha(t_0)\} = \tau$.

(H_2) *There exist functions* $a, b, c, d, \alpha^{\triangle} \in C_{\mathrm{rd}}(\mathbb{T}_0, \mathbb{R}_+)$ *such that for* $t \in \mathbb{T}_0$, $x_1, x_2 \in \mathbb{R}$,

$$\left|f(t, x_1, x_2)\right| \leq \left(a(t) + b(t)\alpha^{\triangle}(t)\right)|x_1| + \left(c(t) + d(t)\alpha^{\triangle}(t)\right)|x_2|. \tag{4.3}$$

Then the solution $x(t)$ *of IVP* (4.1)-(4.2) *satisfies the following inequality*:

$$\left|x(t)\right| \leq M e_A(t, t_0), \quad t \in \mathbb{T}_0, \tag{4.4}$$

where

$$M = \max_{s \in [\beta\tau, t_0]_{\mathbb{T}}} \left|\phi(s)\right|$$

and

$$A(t) = a(t) + c(t) + b(t)\alpha^{\triangle}(t) + d(t)\alpha^{\triangle}(t).$$

Proof It is easy to see that the solution $x(t)$ of IVP (4.1)-(4.2) satisfies the following equation:

$$x(t) = \phi(t_0) + \int_{t_0}^{t} f\Bigl(s, x(s), \max_{\xi \in [\beta s, s]_{\mathbb{T}}} x(\xi)\Bigr)\Delta s. \tag{4.5}$$

Using assumption (H_2), it follows from (4.5) that

$$\begin{aligned}
\left|x(t)\right| &\leq \left|\phi(t_0)\right| + \int_{t_0}^{t} \Bigl|f\Bigl(s, x(s), \max_{\xi \in [\beta s, s]_{\mathbb{T}}} x(\xi)\Bigr)\Bigr|\Delta s \\
&\leq \left|\phi(t_0)\right| + \int_{t_0}^{t} \Bigl[\left(a(s) + b(s)\alpha^{\triangle}(s)\right)\left|x(s)\right| + \left(c(s) + d(s)\alpha^{\triangle}(s)\right)\Bigl|\max_{\xi \in [\beta s, s]_{\mathbb{T}}} x(\xi)\Bigr|\Bigr]\Delta s \\
&\leq \left|\phi(t_0)\right| + \int_{t_0}^{t} \Bigl[a(s)\left|x(s)\right| + c(s)\max_{\xi \in [\beta s, s]_{\mathbb{T}}} \left|x(\xi)\right|\Bigr]\Delta s \\
&\quad + \int_{t_0}^{t} \Bigl[b(s)\left|x(s)\right| + d(s)\max_{\xi \in [\beta s, s]_{\mathbb{T}}} \left|x(\xi)\right|\Bigr]\alpha^{\triangle}(s)\Delta s \\
&= \left|\phi(t_0)\right| + \int_{t_0}^{t} \Bigl[a(s)\left|x(s)\right| + c(s)\max_{\xi \in [\beta s, s]_{\mathbb{T}}} \left|x(\xi)\right|\Bigr]\Delta s \\
&\quad + \int_{\alpha(t_0)}^{\alpha(t)} \Bigl[b(\gamma)\left|x(\gamma)\right| + d(\gamma)\max_{\xi \in [\beta\gamma, \gamma]_{\mathbb{T}}} \left|x(\xi)\right|\Bigr] \circ \alpha^{-1}(s)\widetilde{\Delta}s.
\end{aligned} \tag{4.6}$$

Hence Corollary 3.1 yields the estimate

$$\left|x(t)\right| \leq M e_A(t, t_0), \quad t \in \mathbb{T}_0. \tag{4.7}$$

Inequality (4.7) gives the bound on the solution $x(t)$ of IVP (4.1)-(4.2). □

Example 4.1 Consider the following first-order dynamic equation with 'maxima' on time scale $\mathbb{T} = \{2^n : n \in \mathbb{Z}\} \cup \{0\}$ ($\mathbb{Z}$ stands for the integer set):

$$\begin{cases} x^{\triangle}(t) = 2\sin((3+2t^2)x(t)) \\ \qquad\qquad + \tan^{-1}((e^t + 2\cos^2(\pi t)) \max_{s\in[\frac{1}{32}t,t]_{\mathbb{T}}} x(s)), & t \in \mathbb{T}_0, \\ x(t) = 1, & t \in [\frac{1}{128}, \frac{1}{4}]_{\mathbb{T}}, \end{cases} \tag{4.8}$$

where $\mathbb{T}_0 = [1/4, \infty) \cap \mathbb{T}$.

Here $\phi(t) = 1$, $\beta = 1/32$, $f(t, x(t), \max_{s\in[\beta t,t]_{\mathbb{T}}} x(s)) = 2\sin((3+2t^2)x(t)) + \tan^{-1}((e^t + 2\cos^2(\pi t)) \max_{s\in[\frac{1}{32}t,t]_{\mathbb{T}}} x(s))$, $t_0 = 1/4$, $\tau = 1/4$.

By choosing $\alpha(t) = 2t$, we can show that $\alpha(\mathbb{T}) = \widetilde{\mathbb{T}} \subseteq \mathbb{T}$ and $\min\{t_0, \alpha(t_0)\} = 1/4$. Clearly,

$$\begin{aligned} \left| f\left(t, x(t), \max_{s\in[\beta t,t]_{\mathbb{T}}} x(s)\right)\right| &= \left| 2\sin\left((3+2t^2)x(t)\right) + \tan^{-1}\left(\left(e^t + 2\cos^2(\pi t)\right) \max_{s\in[\frac{1}{32}t,t]_{\mathbb{T}}} x(s)\right)\right| \\ &\le \left(6+4t^2\right)\left|x(t)\right| + \left(e^t + 2\cos^2(\pi t)\right)\left|\max_{s\in[\frac{1}{32}t,t]_{\mathbb{T}}} x(s)\right|, \end{aligned}$$

and

$$\max_{s\in[(1/(128)),(1/(4))]_{\mathbb{T}}} \left|\phi(s)\right| = 1.$$

On the other hand, we have $\alpha^{\triangle}(t) = 2$. Set $a(t) = 6$, $b(t) = 2t^2$, $c(t) = e^t$ and $d(t) = \cos^2(\pi t)$. Hence, Corollary 4.1 yields the estimate

$$\left|x(t)\right| \le e_A(t, t_0), \quad t \in \mathbb{T}_0,$$

where

$$A(t) = 6 + e^t + 4t^2 + 2\cos^2(\pi t).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

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