

Quasi-exact thermodynamic properties of a relativistic spin-zero system under Cornell and generalized Morse potentials

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Received: 12.11.2012 • Accepted: 02.04.2013 • Published Online: 13.09.2013 • Printed: 07.10.2013

Abstract: The thermodynamic properties of a relativistic spin-zero system are investigated via the Klein–Gordon equation under the Cornell and generalized Morse potentials. In particular, the partition function and mean and free energies are calculated and their behavior versus variations of the potential coefficients is presented.

Key words: Thermodynamic properties, spin-zero system, Cornell potential, generalized Morse potential

PACS: 03.65.Fd, 03.65.pm, 0365.ca, 03.65.Ge

1. Introduction

Eigenfunctions and eigenvalues of wave equations give us all of the information required to describe a quantum system. The solutions, on the other hand, can be used to model and investigate various physical systems. In particular, the Klein–Gordon equation (KGE) has been a reliable basis to analyze spin-zero relativistic systems such as scalar mesons. The equation normally introduces 2 types of potential: the 4 vector potential $V(r)$ and the space-time scalar potential $S(r)$. In the case of $S(r) = V(r)$ and $S(r) = -V(r)$, the KGE and Dirac equation share the same energy spectrum [1,2]. As a result, our old methods of nonrelativistic quantum mechanics have been applied to the equation to solve it with various interaction terms. The list of used methodologies is rather lengthy and includes the quantization rules [3], Nikiforov–Uvarov technique [4], supersymmetry quantum mechanics [5,6], series expansion [7], and the ansatz method [8,9]. Moreover, we have not yet been able to construct a unified potential that works well in all fields. Consequently, various authors have considered different interaction terms such as linear, Coulomb, Eckhart, Yukawa, Pöschl–Teller, Morse, Woods–Saxon, and many others (see Refs. [10,11] and references therein). Among these potentials, the Cornell and generalized Morse potentials yield sound phenomenological consequences in particle and molecular physics, respectively. The Cornell potential consists of 2 terms; the inverse term (Coulomb term) arises from the 1-gluon exchange between the quark and its antiquark, and the linear term is included to take into account the confining effects. The purpose of this work is to study the thermodynamic properties of spin-zero systems under such 2 interactions. To obtain the thermodynamic properties, we use the partition function, which is the building block in statistical mechanics (or statistical thermodynamics), as is the wave function in quantum mechanics. It describes the statistical properties of a system in thermodynamic equilibrium. Having found the partition function, we can simply derive the other statistical quantities such as the total energy, free energy, entropy, and pressure.

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The organization of this paper is as follows. The KGE under the Cornell potential is presented in Section 2 and the corresponding thermodynamics properties are discussed. In Section 3 we do the same for work for the generalized Morse potential. Our conclusion appears in Section 4.

2. Thermodynamic properties under the Cornell potential

The s-wave KGE in the presence of vector and scalar potentials is written as [12]:

$$\left[\frac{d^2}{dr^2} + E_n^2 + V^2(r) - 2E_nV(r) - m^2 - S^2(r) - 2mS(r)\right]u_n(r) = 0, \tag{1}$$

where m is the mass of particle; E_n denotes the energy; $V(r)$ and $S(r)$ are the vector and scalar potentials, respectively; and n stands for the quantum number ($n = 0, 1, 2, \dots$). We consider both the scalar and vector interactions of the Cornell type (alternatively called Funnel):

$$V(r) = V_0r + \frac{V_1}{r}, \quad S(r) = S_0r + \frac{S_1}{r}, \tag{2}$$

where V_0, V_1, S_0 , and S_1 are the potential constants. By substituting Eq. (2) into Eq. (1), we arrive at

$$\left[\frac{d^2}{dr^2} - A_3 + \frac{A_1}{r^2} + \frac{A_2}{r} - A_4r - A_5r^2\right]u_n(r) = 0, \tag{3}$$

where

$$A_1 = V_1^2 - S_1^2, \quad A_2 = -2E_nV_1 - 2mS_1, \quad A_3 = -E_n^2 + m^2 + 2S_0S_1 - 2V_0V_1, \quad A_4 = 2E_nV_0 + 2mS_0, \quad A_5 = S_0^2 - V_0^2. \tag{4}$$

Let us now propose a series solution of the form

$$u_n(r) = \sum_{n=0}^{\infty} a_n r^{n+\mu} e^{pr + \frac{1}{2}qr^2}, \tag{5}$$

where μ , p , and q are the wave function parameters that we should determine. By substituting the latter solution in Eq. (3), we find

$$\begin{aligned} & \sum_{n=0}^{\infty} a_n(n+\mu)(n+\mu-1)r^{n+\mu-2} + \sum_{n=1}^{\infty} 2a_{n-1}p(n+\mu-1)r^{n+\mu-2} \\ & + \sum_{n=2}^{\infty} [2qa_{n-2}(n+\mu-2) + qa_{n-2} + p^2a_{n-2}]r^{n+\mu-2} \\ & + \sum_{n=3}^{\infty} 2qpa_{n-3}r^{n+\mu-2} + \sum_{n=4}^{\infty} q^2a_{n-4}r^{n+\mu-2} - A_3 \sum_{n=2}^{\infty} a_{n-2}r^{n+\mu-2} + A_1 \sum_{n=2}^{\infty} a_n r^{n+\mu-2} \\ & + A_2 \sum_{n=2}^{\infty} a_{n-1}r^{n+\mu-2} - A_4 \sum_{n=2}^{\infty} a_{n-3}r^{n+\mu-2} - A_5 \sum_{n=2}^{\infty} a_{n-4}r^{n+\mu-2} = 0, \end{aligned} \tag{6}$$

with

$$a_0[\mu(\mu-1) + A_1] = 0, \tag{7a}$$

$$a_1 = \frac{-(A_2 + 2p\mu)}{\mu(\mu + 1) + A_1}, \tag{7b}$$

$$a_2 = -\frac{2a_1p(\mu + 1) + (2a_0q\mu + a_0q + a_0p^2) - A_3a_0 + A_2a_1}{(\mu + 1)(\mu + 2) + A_1}, \tag{7c}$$

$$a_3 = -\frac{2a_2p(\mu + 2) + (2a_1q(\mu + 1) + a_1q + a_1p^2) + 2a_0qp - A_3a_1 + A_2a_2 - A_4a_0}{(\mu + 2)(\mu + 3) + A_1}, \tag{7d}$$

$$a_n = -\frac{(q^2 - A_5)a_{n-4} + (2pq - A_4)a_{n-3} + [2q(\mu + n - 2) + q + p^2 - A_3]a_{n-2} + [2p(\mu + n - 1) + A_2]a_{n-1}}{(\mu + n)(\mu + n - 1) + A_1} \tag{7e}$$

On the other hand, the series must be bounded for a typical $n = n_r$ to have a well-defined solution. This condition gives

$$\mu(\mu - 1) + A_1 = 0, \tag{8a}$$

$$q^2 - A_5 = 0, \tag{8b}$$

$$2pq - A_4 = 0, \tag{8c}$$

$$2q(\mu + n - 2) + q + p^2 - A_3 = 0, \tag{8d}$$

$$2p(\mu + n - 1) + A_2 = 0, \tag{8e}$$

or equivalently

$$\mu = \frac{1 \pm \sqrt{1 - 4A_1}}{2}, \tag{8f}$$

$$q = \pm \sqrt{S_0^2 - V_0^2}, \quad S_0 > V_0, \tag{8g}$$

$$p = \pm \frac{2E_n V_0 + 2mS_0}{2\sqrt{S_0^2 - V_0^2}}. \tag{8h}$$

The acceptable physical limit is the negative sign for p and q and the positive sign for μ . Therefore, the energy relation, from Eqs. (8a), (8c), (8d), and (4), takes the following form [12]:

$$E_n^2 + \frac{2V_0}{4(S_0^2 - V_0^2)} E_n + (-m^2 - 2S_0S_1 + 2V_0V_1 + 3\sqrt{S_0^2 - V_0^2} - 2n\sqrt{S_0^2 - V_0^2} - \sqrt{S_0^2 - V_0^2}(1 + \sqrt{1 + 4(S_1^2 - V_1^2)}) + \frac{mS_0}{2(S_0^2 - V_0^2)}) = 0, \tag{9}$$

Let us now continue with basic thermodynamic properties. The partition function is calculated via [13]

$$z = \sum_{n=0}^{\infty} e^{-\beta E_n}, \quad \beta = \frac{1}{kT}, \tag{10}$$

where k denotes the Boltzmann constant. The term $e^{-\beta E_n}$ is known as the Boltzmann factor, which is a weighting factor that determines the relative probability of a particle to be in a state n in a multistate system in thermodynamic equilibrium at temperature T . By solving Eq. (9) with respect to E_n and substituting E_n

into Eq. (10), the partition function of the system can be obtained. It is observed that the partition function depends on potential constants. In Figure 1, the behavior of Z vs. β is shown for different values of V_1 and the increasing behavior of Z for increasing V_1 is depicted. The behavior of Z versus S_1 and V_1 is represented for different values of β in Figure 2, which demonstrates the way the partition function (Z) depends on the potential parameters and consequently how the partition functions differs from one system to another. It is clear that the partition function has an increasing behavior as V_1 increases and decreases with increasing S_1 . The mean energy is simply calculated via

$$U = -\frac{\partial}{\partial \beta} \ln(z). \tag{11}$$

The behavior of U vs. β is shown in Figure 3 for different values of V_1 . It is seen that as V_1 decreases, the mean energy increases. In Figure 4, we have plotted the mean energy vs. S_1 and V_1 for different values of β . Figure 4 indicates that the mean energy depends on the potential parameters and it is increasing with increasing S_1 . Other thermodynamic quantities frequently present in physical studies are the free energy F , the entropy S , and the specific heat C , which are defined via

$$F = -\frac{1}{\beta} \ln(z), \tag{12}$$

$$C = \frac{\partial}{\partial T} U, \tag{13}$$

$$S = K \ln(z) + KT \frac{\partial \ln(z)}{\partial T} = K \ln(z) - K\beta \frac{\partial \ln(z)}{\partial \beta}, \tag{14}$$

respectively. The behavior of F vs. β is indicated in Figure 5 for different values of V_1 . It is clear that for decreasing V_1 , the free energy increases. In Figure 6, we have plotted the free energy vs. S_1 and V_1 for different

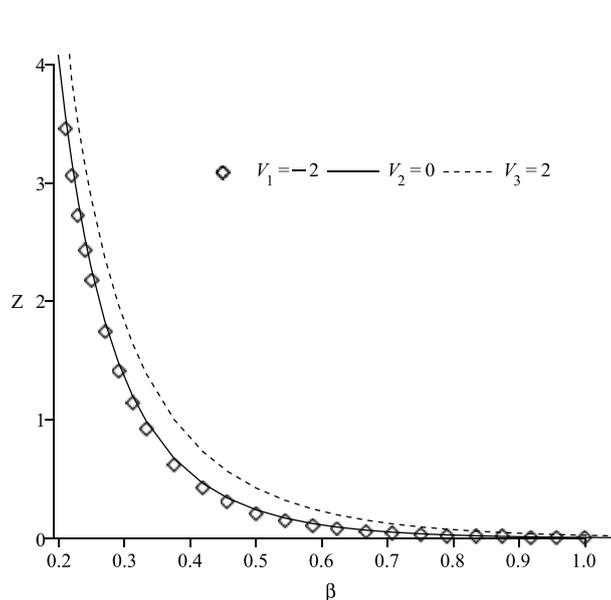


Figure 1. The behavior of Z versus β for $m = 1$, $S_0 = 2$, $V_0 = 1$, and $S_1 = 3$.

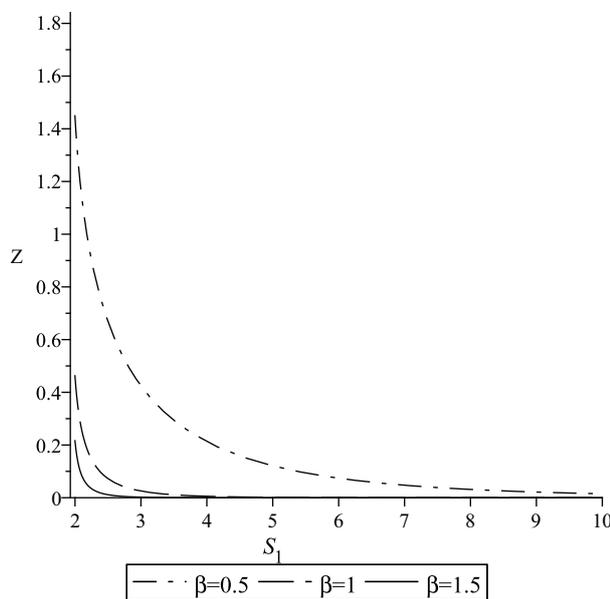


Figure 2. The behavior of Z versus $\begin{cases} a) S_1 \\ b) V_1 \end{cases}$ for $\begin{cases} m = 1, S_0 = 2, V_0 = 1 \\ m = 1, S_0 = 2, V_0 = 1 \end{cases}$ and $\begin{cases} V_1 = 2 \\ S_1 = 3. \end{cases}$

values of β . It is obvious that the free energy increases with increasing S_1 but decreases for increasing V_1 . Figures 7 and 8 depict the variation of entropy and specific heat vs. β . We see that when β increases, both the entropy and the specific heat increase.

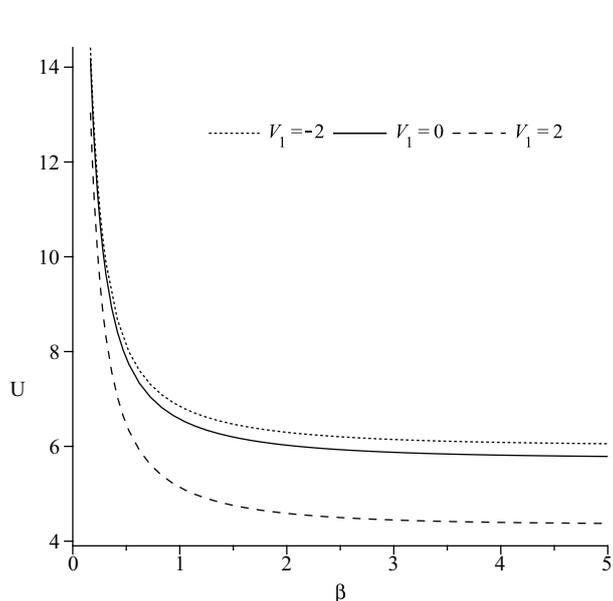


Figure 3. The behavior of U versus β for $m = 1$, $S_0 = 2, V_0 = 1$, and $S_1 = 3$.

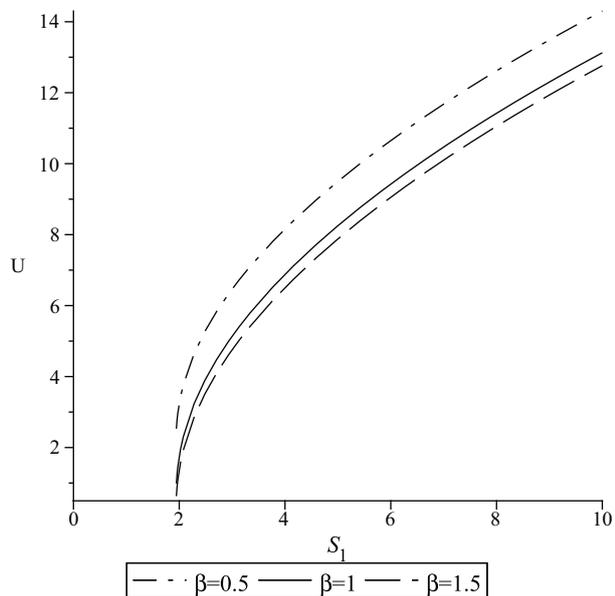


Figure 4. The behavior of U versus $\left\{ \begin{array}{l} a) S_1 \\ b) V_1 \end{array} \right.$ for $\left\{ \begin{array}{l} m = 1, S_0 = 2, V_0 = 1 \\ m = 1, S_0 = 2, V_0 = 1 \end{array} \right.$ and $\left\{ \begin{array}{l} V_1 = 2 \\ S_1 = 3. \end{array} \right.$

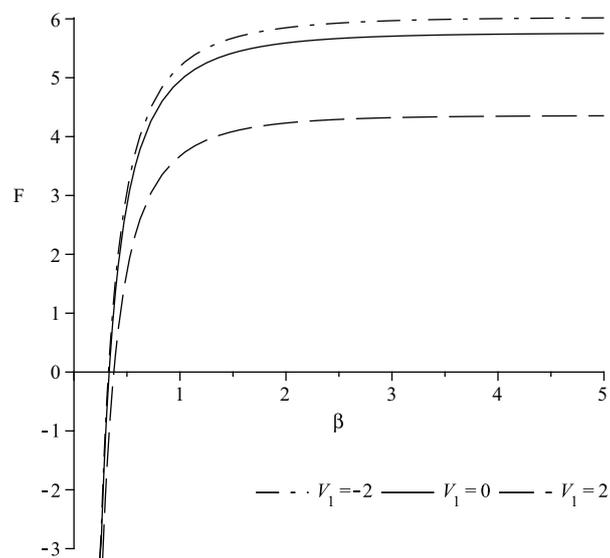


Figure 5. The behavior of F versus β for $m = 1$, $S_0 = 2, V_0 = 1$, and $S_1 = 3$.

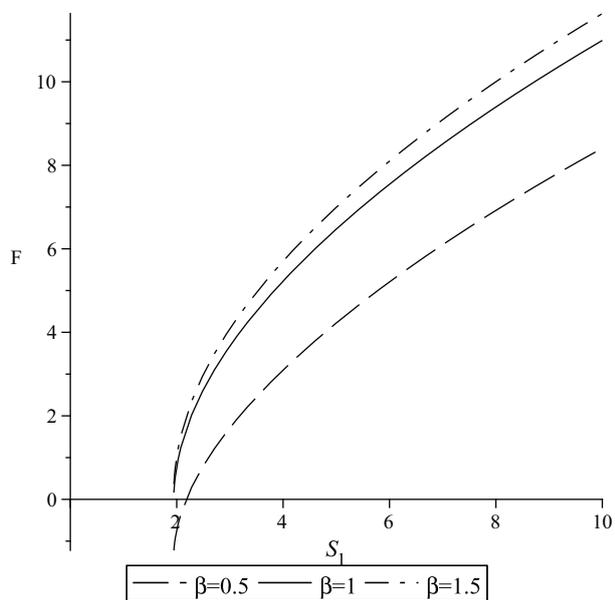


Figure 6. The behavior of F versus $\left\{ \begin{array}{l} a) S_1 \\ b) V_1 \end{array} \right.$ for $\left\{ \begin{array}{l} m = 1, S_0 = 2, V_0 = 1 \\ m = 1, S_0 = 2, V_0 = 1 \end{array} \right.$ and $\left\{ \begin{array}{l} V_1 = 2 \\ S_1 = 3. \end{array} \right.$

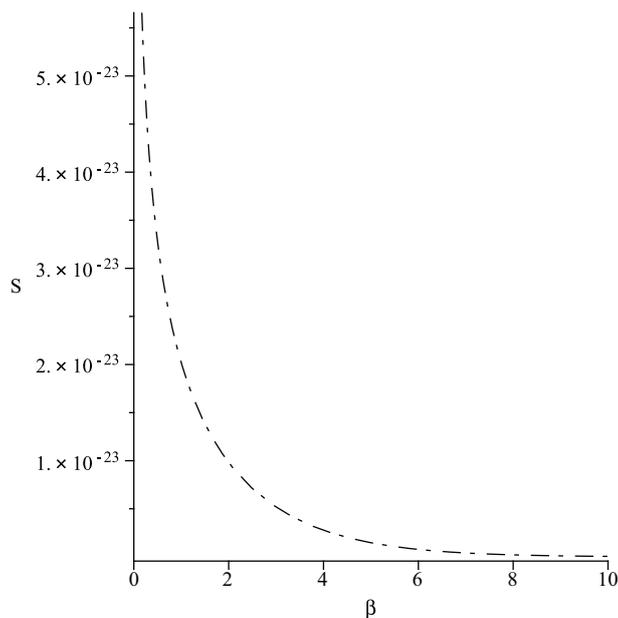


Figure 7. The behavior of S versus β for $m = 1$, $S_0 = 2, V_0 = 1, V_1 = 2$, and $S_1 = 3$.

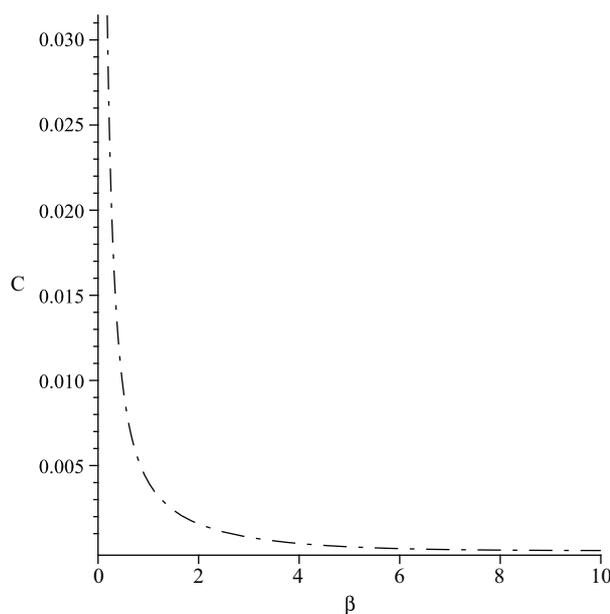


Figure 8. The behavior of C versus β for $T = 300$, $m = 1, S_0 = 2, V_0 = 1, S_1 = 3$, and $V_1 = 2$.

3. Thermodynamic properties under the generalized Morse potential

Now we consider the vector generalized Morse potential

$$V(r) = V'_1 e^{-2a(r-r_0)} - V'_2 e^{-a(r-r_0)}, \tag{15}$$

where V'_1 and V'_2 are potential constants, a is the potential width, and r_0 denotes the equilibrium distance. In this section, we just consider the vector generalized Morse potential and solve the KGE with vanishing scalar potential. Substitution of the generalized Morse potential into Eq. (1), using the change of variable $x = \frac{r-r_0}{r_0}, \alpha = ar_0$ as well as applying the transformations $z = e^{-\alpha x}$ and $u_n(z) = \frac{1}{\sqrt{z}} \phi_n(z)$, we arrive at

$$\left[\frac{d^2}{dz^2} - A_3 + \frac{A_1}{z^2} + \frac{A_2}{z} - A_4 z - A_5 z^2 \right] \phi_n(z) = 0, \tag{16}$$

with

$$A_1 = Q^2(E^2 - m^2) + \frac{1}{4}, A_2 = 2Q^2V'_2, A_3 = Q^2(V'^2_2 + 2V'_1m), A_4 = 2Q^2V'_1V'_2, A_5 = Q^2V'^2_1. \tag{17}$$

By applying the same approach as in Section 2, the corresponding energy after algebra is found as [14]:

$$E_n = \pm \left\{ m^2c^4 + \frac{1}{Q^2} (2n + 1 + Q\tilde{V}_1)^2 \right\}^{\frac{1}{2}}, \tag{18}$$

where

$$\tilde{V}_1 = \frac{V'^2_2}{2V'_1} + mc^2, \quad Q^2 = \frac{1}{\hbar^2 c^2 \alpha^2}. \tag{19}$$

By the same token, from Eqs. (6) through (10), the thermodynamic properties of the system can be obtained. The behavior of the partition function (z), mean energy (U), free energy (F), entropy (S), and specific heat (C) of the system are represented in Figures 9–13, respectively. In Figures 14–16, the behavior of the partition function (z), mean energy (U), and free energy (F) vs. V'_1 and V'_2 are shown for different values of β . Figures 9, 10, 12, and 13 show the decreasing behavior of the partition function, mean energy, entropy, and specific heat for increasing β . Figure 11 indicates that as β increases, the free energy tends to a constant value. We understand from Figures 15 and 16 that for positive values of V'_1 , the mean energy (U) and free energy (F) of the system decrease, and for negative values of V'_1 these quantities increase.

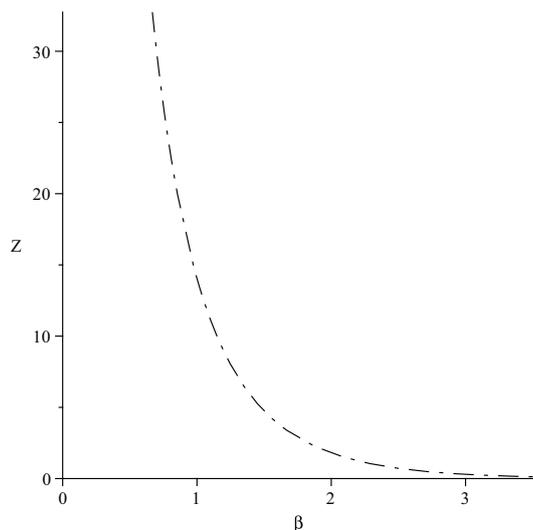


Figure 9. The behavior of Z versus β for $c = 1, \hbar = 1, m = 1, \alpha = 0.01, V'_1 = 0.01,$ and $V'_2 = 0.01$.

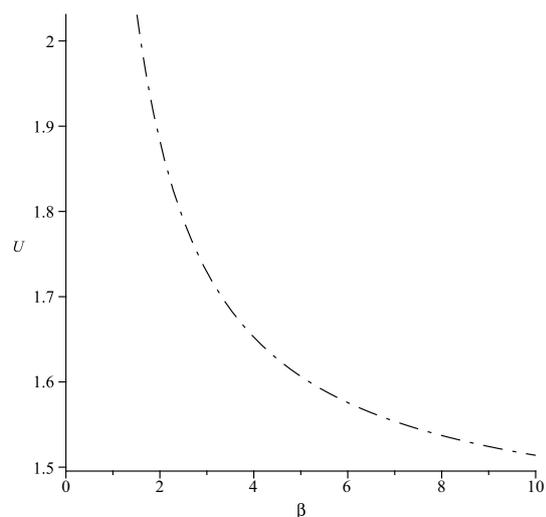


Figure 10. The behavior of U versus β for $c = 1, \hbar = 1, m = 1, \alpha = 0.01, V'_1 = 0.01,$ and $V'_2 = 0.01$.

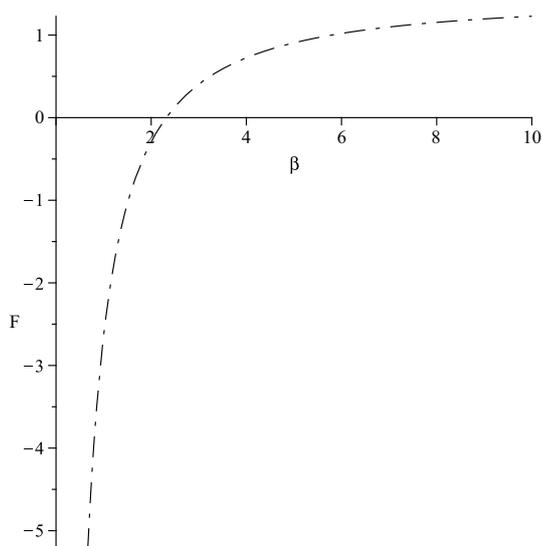


Figure 11. The behavior of F versus β for $c = 1, \hbar = 1, m = 1, \alpha = 0.01, V'_1 = 0.01,$ and $V'_2 = 0.01$.

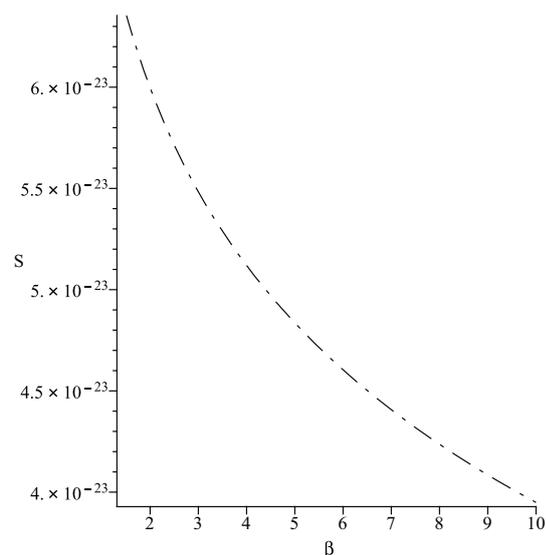


Figure 12. The behavior of S versus β for $c = 1, \hbar = 1, m = 1, \alpha = 0.01, V'_1 = 0.01,$ and $V'_2 = 0.01$.

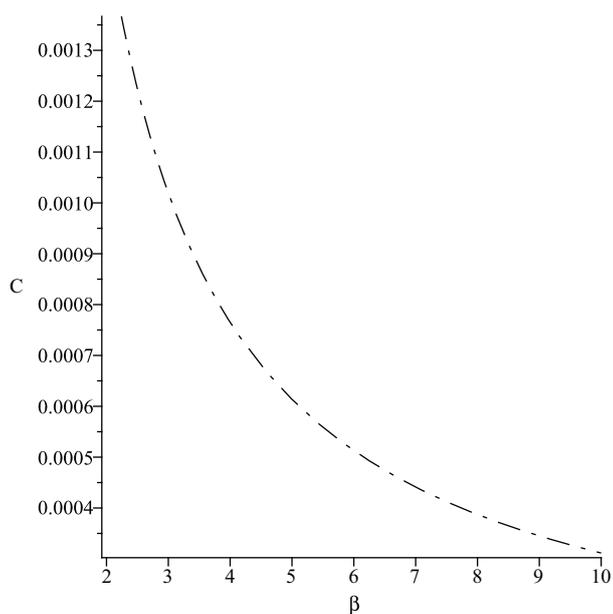


Figure 13. The behavior of C versus β for $T = 300$, $c = 1, \hbar = 1, m = 1, \alpha = 0.01, V_1' = 0.01$, and $V_2' = 0.01$.

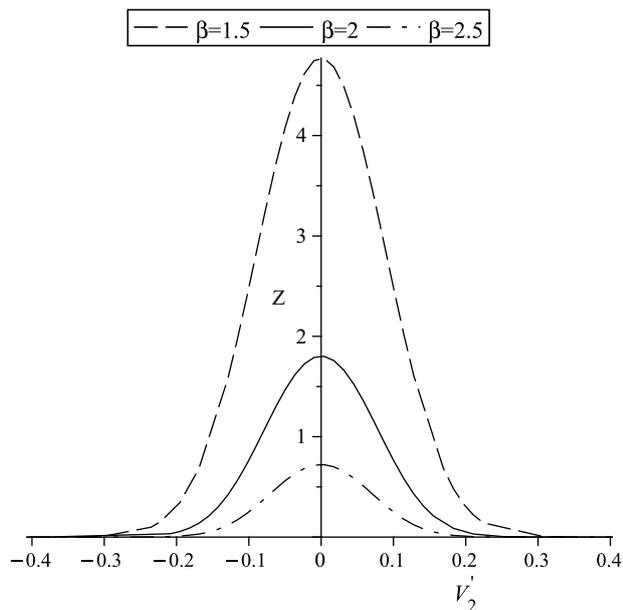


Figure 14. The behavior of Z versus $\begin{cases} a) V_2' \\ b) V_1' \end{cases}$ for $\begin{cases} m = c = \hbar = 1, \alpha = 0.01 \\ m = c = \hbar = 1, \alpha = 0.01 \end{cases}$ and $\begin{cases} V_1' = 0.01 \\ V_2' = -0.1. \end{cases}$

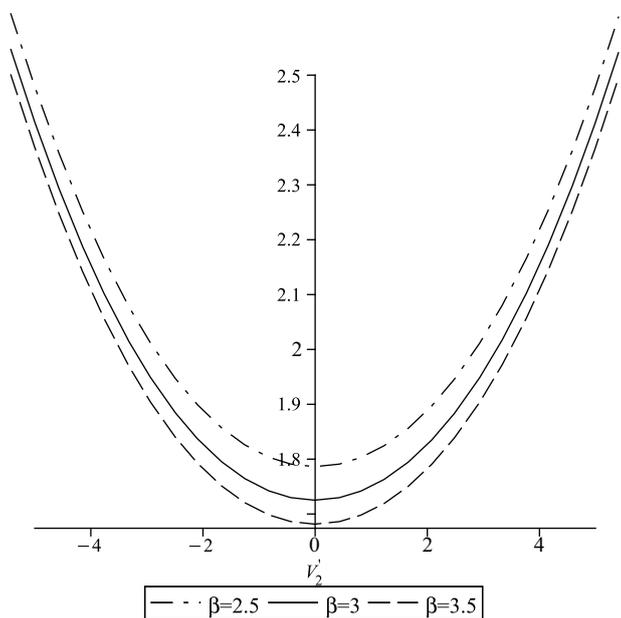


Figure 15. The behavior of U versus $\begin{cases} a) V_2' \\ b) V_1' \end{cases}$ for $\begin{cases} m = c = \hbar = 1, \alpha = 0.01 \\ m = c = \hbar = 1, \alpha = 0.01 \end{cases}$ and $\begin{cases} V_1' = 15 \\ V_2' = 0.01. \end{cases}$

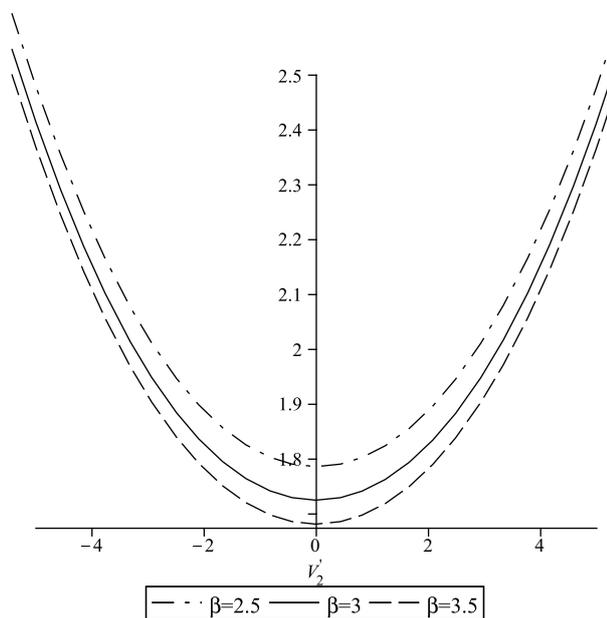


Figure 16. The behavior of F versus $\begin{cases} a) V_2' \\ b) V_1' \end{cases}$ for $\begin{cases} m = c = \hbar = 1, \alpha = 0.01 \\ m = c = \hbar = 1, \alpha = 0.01 \end{cases}$ and $\begin{cases} V_1' = 15 \\ V_2' = -0.1. \end{cases}$

4. Conclusions

The wide application of Cornell and generalized Morse potentials, as well as the presence of many relativistic spin-zero systems in various physical and chemical sciences, motivated us to consider the corresponding thermodynamics properties of such systems when adopting the KGE. In our study the basic thermodynamics quantities, including the partition function, mean energy, free energy, entropy, and specific heat, have been depicted as the function of various potential parameters and quantum numbers. On the basis of the fits obtained by us, our results can be used for several systems.

Acknowledgment

The authors wish to give their sincere gratitude to the referees for their comments and critiques, which improved the manuscript.

References

- [1] P. Alberto, A. S. de Castro and M. Malheiro, *Phys. Rev. C* **75**, (2007), 047303.
- [2] A. I. Ahiezer and V. B. Berestetskii, *Kvantovaya Elektrodinamika (Quantum Electrodynamics)*, Nauka (1981) [in Russian].
- [3] X. Y. Gu and S. H. Dong, *J. Math. Chem.* **49**, (2011), 2053.
- [4] H. Hassanabadi, S. Zarrinkamar, M. Hamzavi and A. A. Rajabi, *Few-Body Syst.* **51**, (2011), 69.
- [5] G. F. Wei and S. H. Dong, *Can. J. Phys.* **89**, (2011), 1225.
- [6] A. A. Andrianov and M. V. Ioffe, *J. Phys. A* **45**, (2012), 503001.
- [7] S. H. Dong and G. H. Sun, *Phys. Lett. A* **314**, (2003), 261.
- [8] H. Hassanabadi, B. H. Yazarloo, S. Zarrinkamar and H. Rahimov, *Commun. Theor. Phys.* **57**, (2012), 339.
- [9] S. H. Dong, *Factorization Method in Quantum Mechanics*, Springer (2007).
- [10] W. C. Qiang and S. H. Dong, *Phys. Lett. A* **363**, (2007), 169.
- [11] A. D. Alhaidari, H. Bahlouli and A. Al-Hasan, *Phys. Lett. A* **349**, (2006), 87.
- [12] H. Hassanabadi, H. Rahimov and S. Zarrinkamar, *Ann. Phys. (Berlin)* **523**, (2011), 566.
- [13] S. H. Dong, M. Lozada-Cassou, J. Yu, F. Jimenez-Ángeles and A. L. Rivera, *Int. J. Quantum Chem.* **107**, (2007), 366.
- [14] A. Arda and R. Sever, *Phys. Scr.* **82**, (2010), 065007.